Direct evaluation of boson dynamics via finite-temperature time-dependent variation with multiple Davydov states

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Recent advances in quantum optics allow for exploration of boson dynamics in dissipative many-body systems. However, the traditional descriptions of quantum dissipation using reduced density matrices are unable to capture explicit information of bath dynamics. In this work, efficient evaluation of boson dynamics is demonstrated by combining the multiple Davydov Ansatz with finite-temperature time-dependent variation, going beyond what state-of-the-art density matrix approaches are capable to offer for coupled electron-boson systems. To this end, applications are made to excitation energy transfer in photosynthetic systems, singlet fission in organic thin films, and circuit quantum electrodynamics in superconducting devices. Thanks to the multiple Davydov Ansatz, our analysis of boson dynamics leads to clear revelation of boson modes strongly coupled to electronic states, as well as in-depth description of polaron creation and destruction in the presence of thermal fluctuations.

I. INTRODUCTION

Central to the exciton dynamics in condensed phase systems is the interaction between the electronic and the bath degrees of freedom (DOFs) such as molecular vibration and photon.1,2 In the weak coupling regime, this interaction has been usually considered as being responsible for energy dissipation. On the other hand, in the strong coupling and non-Markovian regime, this interaction serves to create mixture of exciton with other quasiparticles, and significantly modify exciton dynamics and spectroscopic signatures of such molecular systems. Developments of quantum optical technology2 and ultrafast nonlinear spectroscopy3–9 have shed light on various properties of quantum mixing between these DOFs. The concept of the polaroic states (also referred to in the literature as vibronic excitons10–12) created by resonance between electronic excitation and intramolecular vibrational state10–27 was accepted as a plausible explanation for long-lived quantum beating in two-dimensional (2D) electronic spectra of photosynthetic light harvesting complexes.28–33 The concept of polaron is also applied to the interpretation of oscillatory behavior in 2D electronic spectra of singlet fission in organic thin films.34–38 Recent experiments in cavity quantum electrodynamics (QED) provides a realistic physical setup where the light-matter interaction is much larger than the atomic and cavity frequencies.39–41 In this strong light-matter coupling regime, the description of atom and photon as distinct entities breaks down, and a description in terms of quantum mechanic Vol. 1.55 mixed atomic and photon states, i.e., polaritons is required.42

In general, descriptions of quantum dissipative system involving interactions between electronic and bath DOFs can be divided into two main classes: the reduced density matrix approach and the wave function propagation method. In the former, interplay between electronic states and the bath DOFs can be characterized by the two-body correlation function of the collective energy gap coordinate or the corresponding spectral density.3 This approach provides insight into the influence of thermal fluctuations on the exciton-boson mixture in a natural manner. In the reduced density matrix formalism, all explicit information of the bath DOFs is lost, and the coupling between the electronic and the bath DOFs is only reflected in the electronic populations. However, recent progress in the cavity QED makes it possible to address the dynamics of photonic DOFs.43,44 Moreover, ultrafast spectroscopic studies have pointed to participation of intramolecular vibrations in singlet fission.34,45,46 Thus, it becomes increasingly important to evaluate explicit boson dynamics of a many-body boson-electronic system, and better understandings of electronic-vibrational interactions may help control dynamics and design promising new photovoltaic materials. Alternatively, the wave function propagation method gives access to the dynamics of all bath DOFs explicitly.1 One of the established wave function propagation approach is to employ the Dirac-Frenkel time-dependent variational principle with the Davydov Ansätze48–51 or their extension in the form of superpositions of the Davydov Ansätze.52–56 In an early effort, the explicit boson dynamics was investigated by the variational approach with the single Davydov Ansätze in the Holstein molecular crystal model in order to capture the temporal interplay between electronic and vibrational DOFs.51 Recently, the multiple Davydov Ansätze have been utilized, which offered significant improvements in the resulting dynamics.54 This approach was extended to the boson dynamics in the Holstein molecular crystal model under an external electronic field to analyze effects of exciton-phonon coupling on Bloch oscillations.57 In addition, Kühn and coworkers have investigated impacts of the polaronic states on EET dynamics in the Fenna-Matthews-Olson (FMO) protein complex by tracking time evolution of bath DOFs based on the multilayer multiconfigurational time-dependent Hartree (ML-MCTDH) approach.24,25 Their calculations clearly showed the importance of phonons of local electronic ground states in the polaronic states. However, the zero temperature assumption may

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lead to unreliable prediction for their role at finite temperature because the electronic and vibrational quantum mixtures are fragile against thermal fluctuations.\textsuperscript{22} In a preceding paper,\textsuperscript{58} we extended this variational approach to finite temperature dynamics of the spin-boson model by adopting a Monte Carlo importance sampling method. The time-dependent variational approach with multiple Davydov states remains surprisingly efficient even at an elevated temperature, and thus can explore effects of thermal fluctuations on the exciton-boson quasiparticle by tracking dynamics of bath DOFs.

In this work, we are concerned with explicit boson dynamics at finite temperatures that can be uncovered by the multiple Davydov Ansätze. We will demonstrate that our methodology can reveal interesting bath information that is previously unavailable to state-of-the-art density matrix approaches. To this end, applications are made to three examples, i.e., photosynthetic excitation energy transfer (EET), singlet fission process in organic materials, and circuits QED. The EET dynamics and single fission processes have been of great interest because of their potential importance to effective energy conversion. Recently, impacts of vibration-induced fluctuations on fission dynamics have been investigated with the multi-level Redfield theory.\textsuperscript{35} The singlet population as functions of the time and the frequency of the vibrational mode has been probed, but how the vibrational modes evolve remains unknown. Meanwhile, the circuits QED attracts great interest in the community of quantum optics for its importance to implementing qubits. The quantized electromagnetic (EM) field, i.e., photons, can be used to manipulate qubits. The photon-number distribution as functions of the frequency and the coupling strength in the stationary state has been recently investigated via a semi-analytical variational method\textsuperscript{42}, but questions remain on how the distribution approaches to the stationary state. Thanks to advances in experimental techniques, the coupling between the qubit and the EM field reaches the ultrastrong coupling regime\textsuperscript{39–41} where the rotating wave approximation (RWA) breaks down.\textsuperscript{30–32} Existing methods such as those of the quasiadiabatic propagator path integral (QUAPI) and the hierarchy equations of motion (HEOM) are valid only for weak and intermediate coupling\textsuperscript{47,55} but fail in the strong coupling regime, because the demand of memory size increases exponentially at strong coupling.\textsuperscript{47} The method of the multi-D\textsubscript{2} Ansatz, on the other hand, can effectively and accurately treat both weak and strong coupling regimes in a unified manner. Results from the multi-D\textsubscript{2} Ansatz were found to agree well with those from other methods such as HEOM in the weak to intermediate coupling regime.\textsuperscript{59} Moreover, the demand of the memory size of the multi-D\textsubscript{2} Ansatz is insensitive to the coupling strength. In this work, it will be demonstrated using the aforementioned three examples that information on the system and its boson environment can be obtained simultaneously in efficient dynamics simulation.

The rest of the paper is organized as follows. In Sec. II, we introduce the Hamiltonian and the Davydov Ansätze with Monte Carlo importance sampling method. In Secs. IIIA, IIIB, and IIIC, we apply our approach to explicit boson dynamics in photosynthetic EET, singlet fission, and circuits QED. Finally, Sec. IV is devoted to concluding remarks.

II. FORMULATION

In this section, we present a brief introduction to finite temperature time-dependent variation with the multiple Davydov Ansätze, and define some operator for investigating boson dynamics. For this purpose, we consider the EET Hamiltonian in a dimer as an example. The Hamiltonian of a light harvesting system consisting of two pigments is expressed as\textsuperscript{1,4}

\[
\hat{H} = \frac{\epsilon}{2}\sigma_z + \Delta \sigma_x + \sum_{m=1}^{2} \langle m | \sum_{k} g_{m,k} (b_{m,k}^\dagger + b_{m,k}) \rangle + \hat{H}_b
\]

(2.1)

where \(\sigma_i (i = x, z)\) are the Pauli operators defined as \(\sigma_x = |1\rangle \langle 2| + |2\rangle \langle 1|\) and \(\sigma_z = |1\rangle \langle 1| - |2\rangle \langle 2|\) with \(|n\rangle (n = 1, 2)\) representing electronic localized states. \(\epsilon\) is the energetic difference between two electronic states. \(\Delta\) is the electronic coupling strength between two electronic states. The bath Hamiltonian is \(\hat{H}_b = \sum_{m=1}^{2} \sum_{k} \omega_{m,k} b_{m,k}^\dagger b_{m,k}\). The dynamics of \(k\)-th mode of \(m\)-th electronic state is described by the creation operator, \(b_{m,k}^\dagger\) and the annihilation operator, \(b_{m,k}\). The system-bath coupling is usually characterized by the spectral density. The spectral density, \(J_m(\omega)\), given in terms of the displacement \(g_{m,k}\) as

\[
J_m(\omega) = \sum_{k} g_{m,k}^2 \delta (\omega - \omega_{m,k}).
\]

The Schrödinger equation for Hamiltonian (2.1) is solved by employing the time-dependent variational approach with the multi-D\textsubscript{2} Ansatz\textsuperscript{23,56,57,59} in order to simulate quantum dynamics of both electronic and bath DOFs in the non-Markovian regime. The corresponding trial wave functions take the form:

\[
|D_2^M(t)\rangle = \sum_{i=1}^{M} \sum_{n=1}^{2} c_{i,n}(t) |n\rangle \\
\times e^{i \sum_{m} \sum_{k} (f_{i,m,k}(t)b_{m,k}^\dagger - f_{i,m,k}^*(t)b_{m,k}) |0\rangle_b},
\]

(2.3)

where \(|0\rangle_b\) is the vacuum state of the boson bath. The variational parameter \(c_{i,n}(t)\) is the amplitude in the electronic state \(|n\rangle\), \(f_{i,m,k}(t)\) represents the phonon displacement with \(i\) and \(k\) denoting the \(i\)-th coherent state and \(l\)-th bath mode, and \(M\) is the Ansatz multiplicity, which labels the numbers of single Davydov states included. For \(M = 1\), the multi-D\textsubscript{2} Ansatz reduces to the traditional single D\textsubscript{2} Ansatz. The time-dependent variational parameters, \(c_{i,n}(t)\) and \(f_{i,m,k}(t)\), are determined by adopting the Lagrangian formalism of the Dirac-Frenkel time-dependent variational principle.\textsuperscript{1,60,61} Detailed derivations of the equations of motion for the variational parameters can be found in Refs.\textsuperscript{54 and 55.}

In order to include temperature effects in dynamics described by the multi-D\textsubscript{2} Ansatz, initial conditions for variational parameters are chosen with Monte Carlo importance sampling in accordance with the equilibrium distribution.\textsuperscript{58,59,62} The equilibrium density matrix of the bath...
at a finite temperature is a diagonal matrix, and can be expressed as:
\[
\hat{\rho}_{\text{vib}}^{\text{eq}} = \int d^2\alpha p(\alpha; \beta)|\alpha\rangle\langle \alpha|,
\]
where $|\alpha\rangle$ denotes a direct product of coherent states $(\alpha_{1,1}, \alpha_{1,2}, \cdots, \alpha_{1,N_b}, \alpha_{2,1}, \alpha_{2,2}, \cdots, \alpha_{2,N_b})$ for the $2N_b$ discrete bath modes, and is expressed as $|\alpha\rangle = \exp(\sum_i \alpha_i b_i^\dagger - H.c.)|0\rangle_{\text{vib}}$. Each $\alpha_i$ runs over all of the feasible coherent states. The element of area $d^2\alpha$, on the complex plane of $\alpha_i$ denotes $\text{dRe}(\alpha_i) \cdot \text{dIm}(\alpha_i)$, which $\text{Re}(\alpha_i)$ and $\text{Im}(\alpha_i)$ are the real and imaginary part of $\alpha_i$, respectively. $p(\alpha; \beta)$ represents the diagonal elements of the density matrix in the coherent state representation and can be expressed as:
\[
p(\alpha; \beta) = \prod_k \left[ e^{\beta \omega_k - 1} \pi \exp\left(-|\alpha_k|^2(e^{\beta \omega_k} - 1)\right) \right].
\]
As shown in Eq. (2.5), $p(\alpha; \beta)$ is a positive definite function of $\alpha$ and can be seen as a probability density. The observables $\langle \hat{O}(t) \rangle$ at finite temperature can be calculated by the technique of Monte Carlo importance sampling as:
\[
\langle \hat{O}(t) \rangle = \frac{1}{N_s} \sum_{i} \langle D^M (t; \alpha_i) | \hat{O} | D^M (t; \alpha_i) \rangle,
\]
where $N_s$ is the sampling number, and the configuration $\alpha_i$ for the bath is numerically generated according to $p(\alpha; \beta)$ by the importance sampling, where $p(\alpha; \beta)$ is the Boltzmann distribution used as the weighting function in the importance sampling procedures. Initial condition parameters are $c_{1,1}(0) = 1$, $c_{1,2}(0) = 0$, $c_{i,n}(0) = 0$ for $i \neq 1$ and $f_{i,m,k}(0) = \alpha_{m,k}$ for all $i, m$ and $k$.

The reduced density matrix element of the diabatic basis is given by:
\[
\rho_{nm}(t) = \frac{1}{N_s} \sum_{i} \langle D^M (t; \alpha_i) | n \rangle \langle m | D^M (t; \alpha_i) \rangle = \sum_{i,j} \langle c^*_{i,n}(t)c_{j,m}(t) S_{i,j}(t) \rangle_{\text{avg}},
\]
where $\langle \cdots \rangle_{\text{avg}}$ denotes averaging over realizations, and $S_{i,j}(t)$ is known as the Debye-Waller factor:
\[
S_{i,j}(t) = e^{\sum_{m,k}(f_{i,m,k}(t)f_{j,m,k}(t)-|f_{i,m,k}(t)||f_{j,m,k}(t)|^2)/2}.
\]

To investigate impacts of thermal fluctuations on mixed electronic and vibrational states, we explicitly track the boson DOFs. For this purpose, the phonon number is calculated by the expectation value of the Hamiltonian operator:
\[
H_{\text{me},k} = \omega_k b^\dagger_{m,k} b_{m,k} (1 - |m\rangle \langle m|),
\]
where the expectation value gives the energy of the vibrational level on the electronic ground state for $m$th pigment, regardless of the electronic excitation of the other pigments ($m \neq n$). The phonon displacement can be obtained by the Hamiltonian:
\[
H_{\text{me},k} = [\omega_k b^\dagger_{m,k} b_{m,k} + g_{m,k}(b^\dagger_{m,k} + b_{m,k})]|m\rangle \langle m|,
\]
where the expectation value of this operator characterizes the energy of the vibrational level on the electronic excited state for $m$th pigment. By using the multiple Davydov Ansätze, the expectation values of the phonon number and the phonon displacement for time $t$ can be evaluated as follows:
\[
\chi_{m,g,k}(t) = \frac{1}{N_s} \sum_{i} \langle D^M (t; \alpha_i) | H_{\text{me},k} | D^M (t; \alpha_i) \rangle = \sum_{i,j} \langle c^*_{i,n}(t)c_{j,m}(t)f^*_{i,m,k}(t)f_{j,m,k}(t) S_{i,j}(t) \rangle_{\text{avg}} \times (1 - \delta_{n,m}),
\]
and
\[
\chi_{m,e,k}(t) = \frac{1}{N_s} \sum_{i} \langle D^M (t; \alpha_i) | H_{\text{me},k} | D^M (t; \alpha_i) \rangle = \sum_{i,j} \langle c^*_{i,m}(t)c_{j,m}(t)f^*_{i,m,k}(t)f_{j,m,k}(t) S_{i,j}(t) \rangle_{\text{avg}}.
\]

### III. APPLICATION TO THREE MODELS AND RESULTS

In this section, we present results of our boson dynamics analysis based on the multi-D$_2$ Ansatz as it is applied to several models. The reliability of the variational approach with the multi-D$_2$ Ansatz at finite temperature is established through a comparison with the benchmark results obtained from the iterative QUAPI technique or the reaction coordinate mapping master equation (RCME) approach, which leads to numerically exact dynamics in this theoretical model. Subsection IIIA treats the photysynthetic EET involving one single vibrational mode. In subsection IIIB, we focus on the model of singlet fission process, which a singlet excited state is converted into two triplet excited states. In subsection IIC, then shows results for a model of the circuit QED in superconducting devices.

#### A. Photosynthetic excitation energy transfer

In this subsection, we apply our variational approach to the phonon dynamics in photosynthetic EET. We consider EET dynamics in a hetero-dimer. To describe EET, we restrict the electronic state of the pigments in a molecular dimer system to the ground state, and the first excited state. Thus, the Hamiltonian of the light harvesting system consisting of two pigments is expressed as Eq. (2.1). For simplicity,
we consider a single intramolecular vibration on each of the pigments, with frequency \( \omega_{\text{vib}} \) and the Huang-Rhys factor \( S \). We model the spectral density using the Brownian oscillator model\(^1\) with the vibrational relaxation rate, \( \gamma_{\text{vib}} \), such that

\[
J_m(\omega) = \frac{4\lambda^m_{\text{vib}} \gamma_{\text{vib}} \omega^2 \omega_m}{(\omega^2 - \omega^2_{\text{vib}})^2 + 4\gamma^2_{\text{vib}} \omega^2},
\]

where \( \lambda^m_{\text{vib}} = \omega_{\text{vib}} S_m \) has been introduced. To obtain numerical solutions to the equations of motion of the variational parameters, the continuum spectral density of Eq. (3.1) needs to be discretized. In this study, the method of linear discretization is employed, in which one divides the frequency domain \([\omega_{\text{min}}, \omega_{\text{max}}]\) into \( N_\delta \) equal intervals \( \Delta \omega \), where \( N_\delta \) is the number of discrete vibrational modes. We restrict the frequency of the discrete modes to narrow range around the value of \( \omega_{\text{vib}} \) in order to reduce the computational cost. The reduction of the low-frequency discrete modes does not give large deviations from the exact dynamics.\(^{24}\) This is also demonstrated through a comparison with the benchmark results in this section. The \( k \)th frequency is given by \( \omega_k = k \Delta \omega = k(\omega_{\text{max}} - \omega_{\text{min}})/N_\delta \). From Eq. (2.2), the displacement of \( g_{m,k} \) for each \( \omega_k \) is then given by \( g^2_{m,k} = J_m(\omega_k) \Delta \omega \).

We focus on a dimer which produces no beating of electronic origin in the absence of polaronic contributions. The dimer is inspired by bacteriochlorophyll (BChl) 3 and 4 (pigments 1 and 2, respectively) in the FMO complex of *Chlorobaculum tepidum*,\(^{22,69-71}\) which serve as the two lowest energy exciton states in the single-excitation manifold. For simplicity, the Franck-Condon transition energy of each pigment and their electronic coupling are set to \( \epsilon = 150 \text{ cm}^{-1} \) and \( \Delta = -50 \text{ cm}^{-1} \), respectively. The molecular dynamics simulation of light-harvesting complex II of *Rhodospirillum molischianum* predicts a large value of the Huang-Rhys factors in BChls \( (S = 0.5)\).\(^{72}\) Accurate vibrational structure calculations of excited states can be difficult and so we also consider estimates based on experimental measurements. In this study we choose \( S_1 = S_2 = S = 0.025,\)\(^{17}\) which is within the range of the experimentally measured values.\(^{73,74}\)

Firstly, we focus on the dynamics of intramolecular vibrations of pigments 1 and 2 at 77 K, as shown in Fig. 1. The low-frequency vibrational modes of BChl in solutions or protein environments have been investigated in a number of experiments, revealing the most strongly coupled vibrational mode at approximately \( 180 \text{ cm}^{-1} \).\(^{73,74}\) Thus, to describe the effects of the Franck-Condon active vibrational modes, we consider a single vibrational mode with frequency \( \omega_{\text{vib}} = 180 \text{ cm}^{-1} \) in this subsection. In this situation, the gap between the two electronic energy eigenstates resonates with the vibrational frequency, \( |\epsilon^2 + 4\Delta^2|^{1/2} \approx \omega_{\text{vib}} \), and hence, the effects of the vibrational mode are expected to be maximized under given conditions.\(^{17}\) The vibrational relaxation rate is set to \( \gamma_{\text{vib}} = 2 \text{ ps} \).

Figure 2(a) presents time evolution of the donor (pigment 1) population affected by intramolecular vibrations of pigments 1 and 2. Our results by the multi-D\(_2\) Ansatz is quantitatively consistent with numerically exact results of the RCME ap-
FIG. 3. Time evolution of donor population dynamics in the molecular dimer with intramolecular vibration of pigment 2 in the case of (a) $\gamma_{ vib}^{-1} = 2$ ps, (c) $\gamma_{ vib}^{-1} = 1$ ps and (e) $\gamma_{ vib}^{-1} = 500$ fs. The dashed line indicates the benchmark result by RCME. Panels (b), (d) and (f) present phonon energy of intramolecular vibration of pigment 2 in the case of $\gamma_{ vib}^{-1} = 2$, $\gamma_{ vib}^{-1} = 1$ ps and $\gamma_{ vib}^{-1} = 500$ fs, respectively. The spectral density is discretized into 50 equally spaced modes in the interval [120:220] cm$^{-1}$. The calculations were performed using the same parameters except $\gamma_{ vib}$ as in Fig. 2.

The population dynamics in Fig. 2(a) involves two oscillating components: a faster oscillation with a small amplitude and a slow oscillation with a large amplitude. The faster oscillating component is due to EET between 0-0 vibronic transition, $|\phi_{1g}\rangle|\chi_{g0}\rangle \leftrightarrow |\phi_{1e}\rangle|\chi_{e0}\rangle$, and the 0-0 vibronic transition, $|\phi_{2g}\rangle|\chi_{g0}\rangle \leftrightarrow |\phi_{2e}\rangle|\chi_{e0}\rangle$, which is induced by the coupling strength, $\Delta \langle \chi_{e0}|\chi_{g0}\rangle|\chi_{g0}|\chi_{e0}\rangle = \Delta e^{-S/2} = 0.987 \Delta$ for $S = 0.025$. The slower oscillating component arises from the two interactions. As shown in red arrows in Fig. 1, one is the interaction between the 0-0 vibronic transition, $|\phi_{1g}\rangle|\chi_{g0}\rangle \leftrightarrow |\phi_{1e}\rangle|\chi_{e0}\rangle$, and the 0-1 vibronic transition, $|\phi_{2g}\rangle|\chi_{g0}\rangle \leftrightarrow |\phi_{2e}\rangle|\chi_{e1}\rangle$, which is a result of the coupling strength, $\Delta \langle \chi_{e1}|\chi_{g0}\rangle|\chi_{g0}|\chi_{e1}\rangle = -\Delta e^{-S/2} \sqrt{3} = -0.156 \Delta$. The other is the interaction between the 0-1 vibronic transition, $|\phi_{1g}\rangle|\chi_{g1}\rangle \leftrightarrow |\phi_{1e}\rangle|\chi_{e0}\rangle$, and the 0-0 vibronic transition, $|\phi_{2g}\rangle|\chi_{g0}\rangle \leftrightarrow |\phi_{2e}\rangle|\chi_{e0}\rangle$, and participates the 1-st vibrational level on electronic ground state, as shown in blue arrows in Fig. 1.

In order to elucidate the relation between the population oscillation and the phonon excitations, Figs. 2(b), (c) and (d) exhibits time evolution of $\chi_{2e,k}(t)$, $\chi_{1g,k}(t)$ and $\chi_{1e,k}(t)$, respectively. As shown in Fig. 2(b), the discrete modes of the pigment 2 around $\omega_{ vib} = 180$ cm$^{-1}$ are excited just after the photoexcitation. The peak position of phonon excitation on the state $|\phi_{2e}\rangle$ consequently split into two peaks: $\omega_1 \approx 170$ cm$^{-1}$ and $\omega_2 \approx 190$ cm$^{-1}$. This peak splitting indicates the creation of the polaronic states, $|e_1^+\rangle$ and $|e_1^-\rangle$, by the resonance between $|\phi_{1e}\rangle|\chi_{e0}\rangle$ and $|\phi_{2e}\rangle|\chi_{e1}\rangle$ (red arrows in Fig. 1). This is a novel feature of explicit boson dynamics calculations because the reduced density matrix approach cannot reveal the formation of polaronic states directly. The polaronic states survive for at least 5 ps due to the long vibrational relaxation time, $\gamma_{ vib}^{-1} = 2$ ps. In Fig. 2(c), the phonon excitation of discrete modes of the pigment 1 around 180 cm$^{-1}$ is revealed, indicating for EET driven by resonance between the 1-0 vibronic transition, $|\phi_{1g}\rangle|\chi_{g1}\rangle \leftrightarrow |\phi_{1e}\rangle|\chi_{e0}\rangle$, and the 0-0 vibronic transition, $|\phi_{2g}\rangle|\chi_{g0}\rangle \leftrightarrow |\phi_{2e}\rangle|\chi_{e0}\rangle$ (blue arrows in Fig. 1). This clearly shows the importance of the phonon dynamics on the electronic ground state. This observation is consistent with the ML-MCTDH results by Refs. 24 and 25, and corroborates the argument by Ishizaki and coworkers. After 1 ps, the peak splitting due to the creation of the polaronic states can be observed, as with Fig. 2(b). The phonon displacement $\chi_{1e,k}(t)$ in Fig. 2(d) corresponds to vibrational excitation on the electronic excited state of pigment 1. This vibrational excitation leads to EET accelerated by resonance between the 1-1 vibronic transition, $|\phi_{1g}\rangle|\chi_{g1}\rangle \leftrightarrow |\phi_{1e}\rangle|\chi_{e1}\rangle$, and the 0-1 vibronic transition, $|\phi_{2g}\rangle|\chi_{g0}\rangle \leftrightarrow |\phi_{2e}\rangle|\chi_{e1}\rangle$, as shown in magenta arrow in Fig. 1. The phonon number, $\chi_{2g,k}(t)$, of pigment 2 is negligibly small (not shown) as the state $|\phi_{1g}\rangle|\chi_{g1}\rangle$ does not participate in the resonant vibronic transitions in the model dimer studied here.

Next, we investigate how thermal fluctuations affect the polaronic states. For simplicity, we consider a molecular dimer affected by only intramolecular vibration of pigment 2, and focus on EET between 0-0 vibronic transition $|\phi_{1g}\rangle|\chi_{g0}\rangle \leftrightarrow |\phi_{1e}\rangle|\chi_{e0}\rangle$ and the 0-1 vibronic transition $|\phi_{2g}\rangle|\chi_{g0}\rangle \leftrightarrow |\phi_{2e}\rangle|\chi_{e1}\rangle$ as shown in red arrows in Fig. 1. Therefore, we plot only the phonon excitation on the electronic state $|\phi_{2e}\rangle$. The EET dynamics is simulated for various dephasing times, $\gamma_{ vib}$, in Fig. 3. Figure 3(a) presents time evolution of the electronic donor population affected by only intramolecular vibrations of pigment 2 in the case of $\gamma_{ vib}^{-1} = 2$ ps. Calculations in Fig. 3 were performed using the same parameters as in Fig. 2 (except $\gamma_{ vib}$). Figure 3(b) shows the phonon excitation on the electronic state $|\phi_{2e}\rangle$ corresponding to electronic donor population dynamics in Fig. 3(a), and behaves in a similar way as the case of two vibrational modes in Fig. 2(b). In the case of $\gamma_{ vib}^{-1} = 1$ ps, the oscillatory behavior of donor population dynamics in Fig. 3(c) quickly vanishes compared with Fig. 3(a). The peak of lower frequency ($\omega_k \approx 170$ cm$^{-1}$) in $\chi_{2e,k}(t)$ in the case of $\gamma_{ vib}^{-1} = 1$ ps disappears after 3 ps, suggest-
ing that the fast thermal fluctuations eradicate the electronic-vibrational mixture. This tendency becomes with increasing dephasing rate, $\gamma_{\text{vib}}$, as shown in Figs. 3(e) and (f).

In order to explore impacts of the phonon frequency on polaron formation, we calculate time evolution of electronic donor population dynamics for the values of $\omega_{\text{vib}} = 140 \text{ cm}^{-1}$ and $160 \text{ cm}^{-1}$ with results shown in Figs. 4(a) and (c), respectively. The calculations are performed using the same parameters as in Fig. 2 (except $\omega_{\text{vib}}$). The phonon frequency in Fig. 4 is far from the resonance condition, and the population dynamics shows no oscillatory component due to the vibronic mixing between the vibronic transitions shown in the red arrows of Fig. 1, and these off-resonant vibration does not promote EET dynamics compared with the resonant case in Fig. 3(a). The discrete modes around $\omega_{\text{vib}}$ are excited after the photoexcitation, as shown in Figs. 4(b) and (d). The excitation of the discrete modes on the state $|\varphi_{2e}\rangle$ diffuse over a wide range of phonon frequencies, and does not clearly show electronic-vibrational mixture unlike Figs. 3(b). Recent analyses of beating amplitude in 2D electronic spectra have demonstrated that at the cryogenic temperature, the polaronic states are robust over a wide phonon frequency range around the vibronic resonance even under the influence of environmentally induced fluctuations. Our results suggest that the electronic-vibrational mixing is sensitive to the frequency of the intramolecular vibration, contrary to speculations based on 2D electronic spectra.

B. Singlet fission

In this subsection, we apply our boson dynamics approach to singlet fission processes. In this study, we adopt a model dimer for singlet fission process based on the four-electron four-orbital picture. A simple scheme of the singlet fission process is consider as $g \rightarrow S_1 \rightarrow TT$, which involves the electronic ground state ($g$), the photoexcited singlet excited state ($S_1$, $|1\rangle$), and the triplet pair state (TT, $|2\rangle$). The Hamiltonian of the singlet fission process consisting of two electronic states is expressed as

$$\hat{H} = \frac{\epsilon}{2} \sigma_z + \Delta \sigma_x + \sum_{m}^{2} |m\rangle \langle m| \sum_{k} g_{m,k} b_k^b + b_k^\dagger$$

$$+ \sum_{k} \omega_k b_k^b b_k^\dagger, \quad (3.2)$$

where $b_k$ ($b_k^\dagger$) represents the annihilation (creation) operator of $k$-th phonon mode with frequency $\omega_k$. The functional form of spectral density $J_m(\omega)$ are same as Eq. (3.1) in the subsection IIIA. The multi-D$_2$ Ansatz is adopted

$$|D_2^M(t)\rangle = \sum_{i,j}^M \sum_{n}^{2} c_{i,n}(t) |n\rangle \times e^{\sum_{k} f_{i,k}(t) b_k^b + f_{i,k}(t) b_k} |0\rangle_b,$$

where $f_{i,k}(t)$ represents the phonon displacement with $i$ and $k$ denoting the $i$th coherent sate and the $k$-th boson mode. The expectation values of the phonon number and the phonon displacement for time $t$ can be evaluated as follows:

$$\chi_{gg}(t) = \langle D_2^M(t) |b_k^b b_k^\dagger |D_2^M(t)\rangle = \sum_{i,j}^M \sum_{n}^{2} c_{i,n}(t) \langle c_{j,m}(t) f_{i,k}^+(t) f_{j,k}(t) S_{i,j}(t) \rangle,$$

$$\chi_{ek}(t) = \langle D_2^M(t) |g_{m,k} \omega_k (b_k^b + b_k) |D_2^M(t)\rangle = \sum_{i,j}^M c_{i,m}(t) \langle c_{j,m}(t) \times [g_{m,k} f_{i,k}^+(t) + f_{j,k}(t)] S_{i,j}(t) \rangle,$$

where

$$S_{i,j}(t) = e^{\sum_{k} (f_{i,k}(t) f_{j,k}(t) - f_{i,k}(t) f_{j,k}(t)) \frac{1}{2} - \frac{1}{2} |f_{j,k}(t)|^2}.$$

In the Hamiltonian of singlet fission in our study, it is assumed that the fluctuations of the electronic state induced by phonons are correlated. Both the phonon number and the phonon displacement display same information, and thus we present only time evolution of the phonon displacement in this subsection.

Figure 5(a) presents time evolution of the singlet state population in the case of $\gamma_{\text{vib}} = 500 \text{ fs}$. The population dynamics...
under the fluctuations due to a slow vibrational dephasing time \( \gamma_{\text{vib}} = 1 \) ps. The excited levels of the discrete modes around \( \omega_{\text{vib}} \) behave as stiff vibrational levels, which create a resonance between the singlet and the correlated triplet pair states by matching their energy difference. These results show that the discrete modes except those around \( \omega_{\text{vib}} \) are not important for population dynamics of \( S_1 \) and TT states, and these vibrational levels do not drive fission dynamics when the vibrational mode does not satisfy the energy matching condition. As shown in Fig. 5(c), at \( \gamma_{\text{vib}}^{-1} = 500 \) fs, the width of the excited levels of the discrete modes broadens around \( \omega_{\text{vib}} \), but the excited levels are still distributed mainly in the vicinity of \( \omega_{\text{vib}} \). If one continues decreasing the vibrational relaxation rate to \( \gamma_{\text{vib}}^{-1} = 250 \) fs, the boson dynamics shows the excitation of the discrete modes in the broad range of \( \omega_k \) as displayed in Fig. 5(b). The discrete modes away from \( \omega_{\text{vib}} \) affect the superposition of \( S_1 \) and TT, and thus the quantum mixture created by the resonance between the vibronic transitions is quickly eroded.

In this subsection, we considered the singlet fission dynamics with single intramolecular vibration. Previous theoretical study in singlet fission dynamics demonstrated that several vibrational modes with relatively small Huang-Rhys factors may not enhance fission process by themselves but can dramatically affect the interplay between fission dynamics and other dominant vibrational modes.\(^{35}\) The fitting of measured absorption spectra of tetracene derivatives in cyclohexane with spectroscopic model based on quantum chemical calculations reveals that several intramolecular vibrational modes in tetracene derivatives have Huang-Rhys factors beyond 0.1.\(^{86}\) Investigation of the boson dynamics based the realistic spectral density model extracted by spectroscopies or molecular dynamics simulation may identify functional vibrational modes for promoting singlet fission dynamics, and better understandings of interplay between electronic and vibrational states provide insights into how to control dynamics and help design promising new organic solar materials.

C. Circuit QED

In this subsection, the multi-D\(_2\) Ansatz is employed to investigate interactions between a qubit and an EM field with focus on explicit boson dynamics. Figure 6 is the schematics of a circuit with a Josephson junction coupled with an EM field supplied by a coplanar transmission line resonator, in which the QED is investigated.\(^{87,88}\) Figure 6(a) shows an experimental setup of a system that is equivalent to a qubit implemented by a real atom placed in a cavity of EM wave. The corresponding circuit of Fig. 6(a) is plotted in Fig. 6(b). The energy of the qubit can be tuned by changing the external magnetic field, which controls over the interaction between the qubit and the EM field. The energy separation of the qubit can be set through appropriate circuit design easily.\(^{88}\) These properties makes the circuit a powerful tool to investigate a quantum system interacting with the EM field. For qubits interacting with a transmission line, the whole system can be modeled as spins coupled to one-dimensional EM field.\(^{82,89-94}\) The system

\[ \begin{align*}
\chi_k(t) & = \text{multi-D}_2 \text{ Ansatz} \\
\gamma_{\text{vib}}^{-1} & = \text{RCME}^{35,66}
\end{align*} \]
Hamiltonian can be formulated as

\[
H = \frac{1}{2} \sum_{k} \left( \omega_{k} b_{k}^{\dagger} b_{k} + \frac{\sigma_{x}}{2} \sum_{k} (g_{k}^{*} b_{k} + g_{k} b_{k}^{\dagger}) \right),
\]

where \( g_{k} \) is the strength of qubit-EM coupling, \( b_{k}^{\dagger} (b_{k}) \) is the creation (annihilation) operator of the quantized EM field with frequency \( \omega_{k} \). The emitter is modeled as a two-level atom, also referred to as the qubit or the spin, and \( \epsilon \) is the splitting of the two level system typically set by the Josephson energy associated with the superconducting qubit. A detailed derivation of the system Hamiltonian is given by Appendix A. The transmission line of length \( L \) is discretized into \( N \) parts equally, and the length of each part is \( \delta x = L/N \). The coupling strength (see Appendix A) is

\[
g_{k} = \begin{cases} 
    ig \sqrt{\frac{2\omega_{k}}{\pi v_{EM}}} e^{i\delta x k} \cos \left( \frac{\delta x k}{2} \right) & k > 0 \\
    -ig \sqrt{\frac{2\omega_{k}}{\pi v_{EM}}} e^{i\delta x k} \cos \left( \frac{\delta x k}{2} \right) & k < 0,
\end{cases}
\]

in which \( g \) is determined by the parameters of the qubit-line system, and \( l_{0} \) is the inductance per unit length. \( g_{k} \) is a complex number and can be expressed as \( g_{k} = |g_{k}| e^{i\theta_{k}} \). The phase \( \theta_{k} \) can be seen as the arbitrary extra phase of the EM field and does not change the value of observables such as \( \langle \sigma_{x} \rangle \) and \( \langle b_{k}^{\dagger} b_{k} \rangle \). The factor \( \cos(\delta x k/2) \) is approximately 1 for low frequencies. Letting \( \delta x \to 0 \), for the modes with sufficiently low frequencies, Eq. (3.8) yields the Ohmic spectral density

\[
J(\omega) = \sum_{k} |g_{k}|^{2} \delta(\omega - \omega_{k}) = 2\alpha \omega,
\]

in which

\[
\alpha = \frac{g^{2}}{\pi v_{EM}},
\]

and \( v_{EM} \) is the speed of the EM field. In the literature, an exponential decay is usually employed as the cutoff of the spectral density, \( J(\omega) = 2\alpha \omega e^{-\omega/\omega_{c}} \).

In the weak coupling limit, it is customary to invoke the RWA, where the Hamiltonian in Eq. (3.7) is truncated such that only qubit levels dressed by adjacent Fock states are included. However, this approximation breaks down for large coupling case, and so for our purposes the model must be addressed in its full complexity. A particular defect of the RWA is the lack of many-body renormalization, that is, the strong reduction of the bare tunneling energy \( \epsilon \) to a smaller value. This renormalization effect is, however, well described by the multiple Davydov trial states, where the dressing of the qubit levels occurs via coherent states. The multi-Davydov ansatz and the phonon displacement \( \chi_{k}(t) \) given by Eq. (3.4) and (3.6) are used here, respectively.

From the discussion above, the coupling \( g_{k} \) can be obtained in two procedures. One is from a continuous spectral density in Eq. (3.11), and the other is from a discrete form of the coupling in Eq. (3.8). The continuous one includes the conditions that \( \delta x \to 0 \) and the frequencies are sufficiently low. It is necessary to know whether the two procedures would lead to the same dynamics of the qubit. As shown in Fig. 7, the dynamics of the population difference \( P_{z}(t) = \langle \sigma_{z} \rangle(t) \) from the two procedures are compared for various coupling strengths \( \alpha \). At weak coupling (\( \alpha = 0.05 \)), the two procedures produce quite similar dynamics (both in amplitude and phase). At strong coupling (\( \alpha = 0.2 \)), however, the two procedures generate different dynamics, which shows that the conditions under which the Ohmic spectral density is valid are met only for weak coupling. High frequency modes of the EM field are not excited at weak coupling, and therefore the two procedures lead to similar dynamical behaviors, while same is not true at strong coupling with the excitation of those modes. It is inferred that for strong coupling, excitation of the high frequency modes must be avoided if the Ohmic spectral density can be employed to model the EM field. Finally, to demonstrate the reliability of our simulations, the QUAPI approach is employed to provide benchmark calculations. As shown in Fig. 7(c)-(e) the two methods are in quantitative agreement from the very weak coupling \( \alpha = 0.02 \) to the strong coupling \( \alpha = 0.5 \). The deviation is observed at \( \alpha = 0.7 \). At strong coupling, the QUAPI method needs a larger memory time \( \tau_{mem} = K \delta t \) to achieve convergence, where \( K \) is the memory parameter and \( \delta t \) is the time step size. The memory size for QUAPI convergence is found to increase exponentially with \( K \).
Recently, in Ref. 42, the photon number $\langle b_k^\dagger b_k \rangle$ at stationary state is investigated as a function of the coupling strength $\alpha$ and the frequency of the modes $\omega_k$. It is found that a peak in the distribution exists around the renormalized qubit frequency. While how the distribution approaches to the stationary state is still unknown. Thus, simulations are performed to clarify the evolution of the EM field coupling with the qubit. Figure 8(a) presents $P_\alpha(t)$ for the various values of $\alpha$. The multi-D2 Ansatz reveals incoherent dynamics for $\alpha > 0.5$, whereas the dynamics is coherent for $\alpha = 0.2$. This result is consistent with that of the non-interacting blip approximation method, which showed that a coherent-incoherent crossover takes place at $\alpha = 0.5$.\textsuperscript{95} In order to investigate the interplay between the spin and the photon, Figs. 8(b), 8(c) and 8(d) show time evolution of the photon number $\chi_k(t)$ for the cases of $\alpha = 0.2, 0.5$ and $0.7$, respectively. After electronic excitation at $t = 0$, the peak of the photon number is located in the vicinity of the cut-off frequency $\omega_c$, and the frequency decreases over time. In the case of $\alpha = 0.2$, the peak position finally converges to $\omega_k = 0.1$, which is comparable to $\epsilon$ and resonates the energy gap between two spin states. In the case of the incoherent phase, the oscillation amplitude of photon modes become large due to increase of $\alpha$, and the peak position converges to $\omega_k = 0$.

To further investigate the evolution of the EM field, we also extract the peak of the photon number at each time from Figs. 8 (b)-(d), and plot the evolution of the peak in the same figure with a log-log scale. As shown in Fig. 8(e), the three curves coincide before reaching their stationary states. $\alpha$ only changes the steady-state frequency, as shown in the case of $\alpha = 0.2$. Thus, the evolution of the peak is insensitive with
the coupling strength. Interestingly, the decrease of the curves satisfies a power law with a slope of roughly 1. To the best of our knowledge, the power law behavior of the peak is found for the first time, and further study is needed to elucidate its underlying mechanism. The peak position in the cases of $\alpha = 0.5$ and $\alpha = 0.7$ shows that the oscillation of the photon number survives even at very long times. This long-lived oscillation is due to a finite value of $P_c(t)$ at long times, as discussed using an analytical formula by Schröder and Chin.\textsuperscript{96}

To probe the relation between the peak of the photon distribution and the number of modes $N_b$ (i.e., the system size), simulations are performed for the system with the Ohmic bath of $N_b = 120$ modes. The power-law behavior is still found to exist, and the slopes of the curves in a log-log plot are approximately 1. Results show that the power-law behavior does not depend on the system size. Simulations are also carried out for the peak of the photon distribution using a discrete form of coupling $g_k$. The coupling strengths $\alpha = 0.2, 0.5$ and 0.7 are chosen, similar to the case of the Ohmic spectral density. The number of modes is $N_b = 180$. While the three curves decrease in time with similar rates as the ones in the Ohmic spectral density case, absent in the plot is the power-law behavior, which is likely attributed to the Ohmic spectral density.

This peak behavior in the steady state can be understood in the framework of polaron unitary transformation.\textsuperscript{32} In the polaron transformed frame, the transition frequency $\epsilon$ is renormalized due to fluctuations induced by the displaced photon modes. For large coupling strengths, the renormalized transition frequency is reduced to zero. Therefore, the frequency of the photon modes that resonate the energy gap between two spin states decrease with increasing coupling strength $\alpha$, and the peak position finally reach zero in the limit of strong coupling.

The multi-$D_2$ Ansatz is an efficient method for dynamics simulation of open quantum systems. Here we discuss the performance of the multi-$D_2$ Ansatz using the circuit QED as an example. The trial wave function of the multi-$D_2$ Ansatz for the spin-boson model is

$$ |D_2^M\rangle = \sum_n \left[ A_n(t) |1\rangle + B_n(t) |2\rangle \right] e^{\sum_i f_{ni}(t)b_i - f^\dagger_{ni}(t)b^\dagger_i} |0\rangle_b $$

(3.12)

where $|1\rangle$ and $|2\rangle$ represent the up and down states, respectively. The variational parameters $A_n$ and $B_n$ are the amplitudes of the up and down states, respectively.

To estimate the memory size needed, the equations of motion are expressed in the matrix form (see Appendix B for details),

$$ G\dot{x}^T = b. $$

(3.13)

It follows that the memory size used by the multi-$D_2$ Ansatz is approximately proportional to $N_b^2 \cdot M^2$ as estimated below. The expression of the vector $\dot{x}$ includes time derivatives of the variational parameters, $A_n(t)$, $B_n(t)$ and $f_{ni}(t)$, and their complex conjugates. There are $M$ elements for $A_n$ and $B_n$, and $M \cdot N_b$ elements for $f_{ni}$. Thus, the number of elements of $\dot{x}$ is $2M \cdot (N_b + 2)$. Because $G$ is a full rank square matrix, $G$ has $4M^2 \cdot (N_b + 2)^2$ elements. Thus, the memory needed for the multi-$D_2$ Ansatz with the multiplicity $M$ and the number of modes $N_b$ is roughly $kM^2 \cdot N_b^2$ and $k \approx 4$.

We can also estimate how the computational time grows with the increase of the number of modes. Since $N_b$ is usually much larger than $M$, and is the main factor that determines the computational time, the multiplicity is fixed at $M = 6$. For $N_b = 180, 230$ and $300$, the computational time needed is $120h, 236h$ and $355h$, respectively. The computational time is found to increase linearly with the number of modes. One may be under the impression that the computational time would grow rapidly with the increase of the size of $G$. However, as $G$ is a block matrix, it is found that the computation time increases only linearly. The parameters employed above are for cases of moderate to strong coupling. For the ultra-strong coupling, one can use the single Davydov Ansatz, and computation is much faster. For the QUAPI method, however, a sufficiently large memory time $\tau_{\text{mem}} = K \cdot \delta t$ is needed to obtain convergence. Memory needed for the QUAPI method to reach convergence is proportional to $D^{2K+2}$, where $D$ is the dimension of the system Hilbert space, and $D = 2$ for the spin-boson model.\textsuperscript{97} The needed memory time is long when the coupling is strong or the long-lasting bath memory effect exists. For these cases, it is difficult or even impossible to achieve QUAPI convergence because of the exponential increase of the memory size with $K$. The situation worsens as the dimension $D$ increases. Thus, for cases with strong memory effects, the multi-$D_2$ Ansatz is an efficient approach in comparison with the QUAPI method. Meanwhile, with larger $D$, the memory size needed increases slower for the multi-$D_2$ Ansatz than for the QUAPI method.

In this subsection, we focus on interplay between the qubit and photon modes in a circuit QED system. There is much recent interest in hybrid light-matter systems on the delocalization-localization transition of polaritons in a coupled-cavity array.\textsuperscript{99,100} In addition, beyond the picture of an ideal cavity-QED system, coupling to phonons has been shown to play a crucial role. A non-Markovian description of bath phonon dissipation in the phonon is essential to explain various phenomena involving phonons in these systems. For strong qubit-phonon coupling, phonon dynamics may destroy the delocalized polariton states despite phonon assisted transfer between the polariton states. Our boson dynamics analysis based on the multiple Davydov Ansatz is expected to capture the dynamical transition of the polariton in the QED coupled to phonon, which will be investigated in the future study.

IV. CONCLUDING REMARKS

In this work, we have presented the methodology of exploring boson dynamics of coupled electron-boson systems by using a finite-temperature method of wave function propagation with the multiple Davydov Ansatz, going beyond what state-of-the-art density matrix approaches are capable to offer. In the finite-temperature time-dependent variational approach, a large number of boson modes are treated explicitly, and individual trajectories of the excitations with an initial thermal
equilibrium state of the boson modes can be traced at a given temperature. Thus, the effect of thermal fluctuations on the electronic-boson dynamics can be probed with great detail. The Dirac-Frenkel time-dependent variational principle is employed to derive the equations of motion of the variational parameters. Our method is non-perturbative, which allows for exploration of boson dynamics of realistic systems where the electronic resonant coupling and the boson reorganization are of the similar strength, without having to consider one of them as a small perturbation. Also, due to the explicit treatment of boson modes, an arbitrary form of spectral density function can be assumed, including both continuous spectral density functions and specific prominent molecular vibrations.

In order to demonstrate applicability of the boson dynamics analysis, we have deployed our method to the boson dynamics in three scenarios, i.e., the photosynthetic EET, the singlet fission process, and the circuit QED. In these physical systems, the exciton often couples strongly to the boson DOFs, and treatments of the boson bath based on traditional reduced density matrix approaches wipe out a wealth of information on coupled electron-boson dynamics. Our analysis of boson dynamics revealed the peak position of boson excitation in the excited state manifold splitting into two peaks in photosynthetic EET. This peak splitting indicates the creation of polaronic states by the resonance between two vibronic transitions on different pigments. This is a novel feature from explicit calculations of boson dynamics as the reduced density matrix approach cannot unveil the formation of polaronic states directly. Our analysis of boson dynamics also revealed the role of bosons in the ground state manifold on photosynthetic EET dynamics. Meanwhile, for singlet fission processes, it is shown in the analysis of boson dynamics that a large vibrational relaxation rate $\gamma_{vib}^{-1}$ induces excitation of the discrete modes for a broad range of the phonon frequency, which leads to quick erosion of quantum mixture created by the resonance between the vibronic transitions. Finally, our analysis also uncovers new information on the circuit QED systems whose environment is composed of photons. As the Davydov Ansätze can treat different forms of the bath spectral density function, one can compare in great detail boson dynamics from a discrete form of the coupling $g_k$ with that from the Ohmic spectral density, leading to the conclusion that the Ohmic spectral density is a good approximation in the presence of weak coupling. More interestingly, a power law behavior of the photon-distribution peak is discovered for the first time, opening up a new venue for further study of its underlying mechanism. Successful application to photosynthetic EET, singlet fission and QED demonstrates the versatility of our approach to boson dynamics. It has been convincingly shown that analysis of boson dynamics based on the wave function propagation technique leads to clear revelation of boson modes strongly coupled to electronic states, as well as in-depth description of polaron creation and destruction in the presence of thermal fluctuations.

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Appendix A: Derivation of Hamiltonian in circuit QED system

The transmission line can be seen as inductors connecting in series whose inductance is $l_0$ per unit length, as shown in Fig. 6. Meanwhile, the capacitance per unit length is $c_0$. The artificial atom can be implemented with various circuits, such as the Cooper-pair box\textsuperscript{37}, persistent current qubit and radio frequency superconducting quantum interference device (RF SQUID)\textsuperscript{38}. In this section, the RF SQUID is employed as an example. The flux qubit is a two level system at low temperature, and the Hamiltonian is

$$H_q = \frac{1}{2} (-\varepsilon_0 \sigma_z - \delta \sigma_x),$$

in which $\varepsilon_0 = (2I_p)(\Phi_x - \Phi_0)/2$ is the bias, and $\delta$ is the level repulsion. $I_p$ is the persistence current and is roughly equal to half the critical current $I_c$ of the Josephson junction. If the qubit works at or near $\Phi_x = \Phi_0/2$, the Hamiltonian can be written in a simpler form

$$H_q = \frac{\varepsilon}{2} (|0\rangle \langle 0| - |1\rangle \langle 1|) = \frac{\varepsilon}{2} \sigma_z,$$

where $\varepsilon = \sqrt{\varepsilon_0^2 + \delta^2}$. $|0\rangle$ and $|1\rangle$ are the symmetric and antisymmetric superpositions of clockwise and anti-clockwise persistent currents, respectively.

The time dependent current density $j(x, t)$ and charge density $q(x, t)$ in the transmission line generate the EM field. The Lagrangian of the 1D EM field is

$$L_{EM} = \int_0^L d\xi \left[ \frac{1}{2c_0} q^2(\xi, t) - \frac{l_0}{2} J^2(\xi, t) \right],$$

in which $l_0$ and $c_0$ are the inductance per unit length and capacitance per unit length, respectively.\textsuperscript{99} With the flux,

$$\Phi(x, t) = \int_0^t d\tau V(x, \tau),$$

the Lagrangian is expressed as

$$L_{EM} = \int_0^L d\xi \left\{ \frac{c_0}{2} \dot{\Phi}^2(\xi, t) - \frac{1}{2l_0} \left[ \frac{d\Phi(\xi, t)}{d\xi} \right]^2 \right\},$$

where $V(x, t)$ is the local voltage on the transmission line. To simulate the system with the variational method, we discretize the transmission line with the length of $L$ into $N$ parts equally,
and the length of each part is \( \delta x = L/N \). The discretized Lagrangian is

\[
L_{EM} = \sum_n \left[ \frac{c_0 \delta x}{2} \Phi_n^2 - \frac{1}{2 \omega_0 \delta x} (\Phi_{n+1} - \Phi_n)^2 \right]. \tag{A.6}
\]

With the creation and annihilation operators \( b_k^\dagger \) and \( b_k \) satisfying

\[
b_k = \sqrt{\frac{c_0 \delta x \omega_k}{2}} (Q_k + \frac{i}{c_0 \delta x \omega_k} P_k),
\]

\[
b_k^\dagger = \sqrt{\frac{c_0 \delta x \omega_k}{2}} (Q_k - \frac{i}{c_0 \delta x \omega_k} P_k), \tag{A.7}
\]

and

\[
b_k^\dagger b_k = b_{-k}, k = \frac{2\pi}{L} m (m = 0, \pm 1, \pm 2, \ldots, \pm N/2), \tag{A.8}
\]

the Hamiltonian of the EM field can be expressed as

\[
H_{EM} = \sum_k \omega_k (b_k^\dagger b_k + \frac{1}{2}). \tag{A.9}
\]

Here \( Q_k \) is the norm mode of \( \Phi_n \), satisfying \( \Phi_n = 1/\sqrt{N} \sum_k Q_k e^{-i \delta x k} \). \( P_k \) is the conjugate momentum of \( Q_k \), and

\[
\omega_k = \omega_c \sqrt{2 - 2 \cos(\delta x k)}, \tag{A.10}
\]

is the frequency of the \( k \)-th mode. In Eq. (A.10), \( \omega_c = \nu_{EM}/\delta x \) is the cutoff frequency and \( \nu_{EM} = 1/\sqrt{\mu_0 c_0} \) is the speed of the EM field.

Then the interaction between the EM field and the qubit can be estimated as follows. The circuit of the qubit is a closed loop and has a magnetic dipole moment \( \mu \). The transmission line excites oscillating magnetic field \( B \), and the interaction energy between the qubit and the EM field is \( H = -\mu \cdot B \). When the frequency of EM field is not sufficiently low, \( B = \mu_0 \sigma z j_n/(2\pi r) \), in which \( j_n \) is the local current on the \( n \)-th segment of the transmission line, \( \mu_0 \) is the permeability of vacuum, \( r \) is the distance between the qubit and the transmission line. With Eq. (A.4),

\[
j_n = -\frac{\Phi_{n+1} - \Phi_{n-1}}{2 \omega_0 \delta x} = -\sum_{k<0} \sqrt{\frac{\omega_k}{2 \omega_0}} (ie^{-i \delta x k} b_k - ie^{i \delta x k} b_k^\dagger) \cos \left( \frac{\delta x k}{2} \right) + \sum_{k>0} \sqrt{\frac{\omega_k}{2 \omega_0}} (ie^{-i \delta x k} b_k - ie^{i \delta x k} b_k^\dagger) \cos \left( \frac{\delta x k}{2} \right). \tag{A.11}
\]

Thus the interaction Hamiltonian is

\[
H_i = \frac{I_b A \mu_0}{2 \pi r} \sigma_z j_n = -\sqrt{\frac{2 \omega_0}{\pi \mu_0}} \sum_{k<0} \left( ie^{-i \delta x k} b_k - ie^{i \delta x k} b_k^\dagger \right) \cos \left( \frac{\delta x k}{2} \right) + \sqrt{\frac{2 \omega_0}{\pi \mu_0}} \sum_{k>0} \left( ie^{-i \delta x k} b_k - ie^{i \delta x k} b_k^\dagger \right) \cos \left( \frac{\delta x k}{2} \right) = \sum_k \sigma_z (g_k^b b_k + g_k b_k^\dagger), \tag{A.12}
\]

in which

\[
g_k = \begin{cases} 
  ig \sqrt{\frac{2 \omega_0}{\pi \mu_0}} e^{i \delta x k} \cos \left( \frac{\delta x k}{2} \right) & k > 0 \\
  -ig \sqrt{\frac{2 \omega_0}{\pi \mu_0}} e^{-i \delta x k} \cos \left( \frac{\delta x k}{2} \right) & k < 0 
\end{cases}, \tag{A.13}
\]

and

\[
g = \frac{I_b A \mu_0}{\pi r}. \tag{A.14}
\]

The advantage of the circuit system of the transmission line is that the zero-point energy distributes over a very small effective volume which yield much stronger interaction between the EM field and the qubit. The spectral density function

\[
J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k), \tag{A.15}
\]

fully determines the influence of the bath on the qubit. Letting \( \delta x \to 0 \), for the modes with sufficiently low frequencies, the spectral density is Ohmic,

\[
J(\omega) = 2\alpha \omega, \tag{A.16}
\]

where the coupling strength \( \alpha = |g|^2/(\pi \omega_0 \nu_{EM}) \) is the dimensionless coupling strength.

Appendix B: The multiple Davydov Ansatz for the spin-boson model

To apply the Dirac-Frenkel time-dependent variational principle, the Lagrangian \( L \) is defined as

\[
L = \frac{i}{2} \left[ D_2^M(t) \left| \frac{\partial}{\partial t} \right| D_2^M(t) \right] - \frac{i}{2} \left[ D_2^M(t) \left| \frac{\partial}{\partial t} \right| D_2^M(t) \right] - \langle D_2^M(t) \hat{H} D_2^M(t) \rangle. \tag{A.1}
\]
The explicit form of the Lagrangian is,

\[
L = \frac{i}{2} \sum_n \sum_{n} \left[ A_n^* \dot{A}_n - \dot{A}_n^* A_n \right] + \frac{1}{2} A_n^* A_n \sum_k \left( \dot{f}_{nk}^* f_{nk} + f_{nk} \dot{f}_{nk}^* - \dot{f}_{nk} f_{nk}^* \right)
- f_{nk} \ddot{f}_{nk}^* + 2 \ddot{f}_{nk} f_{nk} - 2 \dot{f}_{nk}^* \dot{f}_{nk} 
+ \frac{i}{2} \sum_n \sum_{n} \left[ B_n^* \dot{B}_n - \dot{B}_n^* B_n \right] + \frac{1}{2} B_n^* B_n \sum_k \left( \dot{f}_{nk}^* f_{nk} + f_{nk} \dot{f}_{nk}^* - \dot{f}_{nk} f_{nk}^* \right)
- f_{nk} \ddot{f}_{nk}^* + 2 \ddot{f}_{nk} f_{nk} - 2 \dot{f}_{nk}^* \dot{f}_{nk} 
\right] R(f_{n^*}, f_n) + \frac{\varepsilon}{2} \sum_n A_n R(f_{n^*}, f_n),
\tag{A.2}
\]

in which

\[
\left\langle D_2^{M}(t) \left| \hat{H} \right| D_2^{M}(t) \right\rangle = \sum_n \sum_{n} \left\{ A_n^* A_n \left[ \sum_l \omega_l f_{nl}^* f_{nl} \right] \right. 
+ \sum_l \frac{1}{2} \left( g_l f_{nl}^* + g_l^* f_{nl} \right) \right] R(f_{n^*}, f_n) 
+ B_n^* B_n \left[ \sum_l \omega_l f_{nl}^* f_{nl} - \sum_l \frac{1}{2} \left( g_l f_{nl}^* + g_l^* f_{nl} \right) \right] R(f_{n^*}, f_n) 
\left. - \frac{\varepsilon}{2} \left( A_n^* B_n + B_n^* A_n \right) R(f_{n^*}, f_n) \right\}.
\tag{A.3}
\]

\( R(f_{n^*}, f_n) \equiv \exp \left[ \sum_l \left( f_{nl}^* f_{nl} - \frac{1}{2} \left( |f_{nl}|^2 + |f_{nl}|^2 \right) \right) \right] \) denotes the Debye-Waller factor. Adopting the Lagrangian formalism of the Dirac-Frenkel time-dependent variational principle, the derive equations of motion for the variational parameters can be calculated with,

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial u_n^*} \right) - \frac{\partial L}{\partial u_n^*} = 0, \tag{A.4}
\]

where \( u_n^* \) denotes the complex conjugate of the variational parameters \( u_n \), which can be \( A_n, B_n \) or \( f_{nl} \). And the equations of motion are

\[
\sum_n \left[ -i \dot{A}_n + \frac{i}{2} A_n \sum_k \left( \dot{f}_{nk} f_{nk}^* + f_{nk} \dot{f}_{nk}^* - 2 f_{nk} \dot{f}_{nk} \right) \right] \times R(f_{n^*}, f_n) 
= \sum_n A_n \left[ - \sum_k \omega_k f_{mk}^* f_{nk} - \sum_k \frac{1}{2} \left( g_k f_{mk}^* + g_k^* f_{mk} \right) \right] \times R(f_{n^*}, f_n) 
+ \frac{\varepsilon}{2} \sum_n B_n R(f_{n^*}, f_n), \tag{A.5}
\]

and

\[
\sum_n \left[ -i \dot{B}_n + \frac{i}{2} B_n \sum_k \left( \dot{f}_{nk} f_{nk}^* + f_{nk} \dot{f}_{nk}^* - 2 f_{nk} \dot{f}_{nk} \right) \right] \times R(f_{n^*}, f_n) 
= \sum_n B_n \left[ - \sum_k \omega_k f_{mk}^* f_{nk} - \sum_k \frac{1}{2} \left( g_k f_{mk}^* + g_k^* f_{mk} \right) \right] \times R(f_{n^*}, f_n) 
+ \frac{\varepsilon}{2} \sum_n A_n R(f_{n^*}, f_n). \tag{A.6}
\]

\[
\frac{i}{2} \sum_n \left[ -2 A_n^* A_n f_{nl} - 2 A_n^* A_n \dot{f}_{nl} \right] 
+ A_n^* A_n \sum_k \sum_i \left( \dot{f}_{nk} f_{nk} + f_{nk} \dot{f}_{nk} - 2 f_{nk} \dot{f}_{nk} \right) \times R(f_{n^*}, f_n) 
+ \frac{i}{2} \sum_n \left[ -2 B_n^* B_n f_{nl} - 2 B_n^* B_n \dot{f}_{nl} \right] 
+ B_n^* B_n \sum_k \sum_i \left( \dot{f}_{nk} f_{nk} + f_{nk} \dot{f}_{nk} - 2 f_{nk} \dot{f}_{nk} \right) \times R(f_{n^*}, f_n) 
= \sum_n f_{nl} A_n^* A_n \left[ - \sum_k \sum_i \omega_k f_{mk}^* f_{nk} \right] 
- \sum_k \frac{1}{2} \left( g_k f_{mk}^* + g_k^* f_{mk} \right) \right] R(f_{n^*}, f_n) 
+ \sum_n f_{nl} B_n^* B_n \left[ - \sum_k \sum_i \omega_k f_{mk}^* f_{nk} \right] 
+ \sum_k \frac{1}{2} \left( g_k f_{mk}^* + g_k^* f_{mk} \right) \right] R(f_{n^*}, f_n) 
+ \sum_n A_n^* A_n \left( - \omega_l f_{nl} - \frac{g_l}{2} \right) R(f_{n^*}, f_n) 
+ \sum_n B_n^* B_n \left( - \omega_l f_{nl} + \frac{g_l}{2} \right) R(f_{n^*}, f_n) 
+ \sum_n \frac{\varepsilon}{2} f_{nl} \left( A_n^* B_n + B_n^* A_n \right) R(f_{n^*}, f_n). \tag{A.7}
\]

Form the complex conjugate of the above equations, one can simply get the equations of motions of \( A_n^*, B_n^* \) and \( f_{nl}^* \). The left hand of each equation is a linear combination of \( \dot{A}_n, \dot{B}_n, \dot{f}_{nl} \) and their complex conjugates. And the right hand of each equation is a function whose arguments are the variational parameters. Thus, the equations can be expresses in a matrix form

\[
G \dot{x}^T = b, \tag{A.8}
\]
in which \( x \) is comprised with the variational parameters \( A_n \), \( B_n \), \( f_{nl} \) and their complex conjugates.


