1 Infinitesimal Generators

1.1 Let $\{T(t) : t \geq 0\}$ is a semi-group of bounded operators in Banach space $\mathscr{X}$, i.e., it satisfies that $T(t)T(s) = T(t+s)$ for all $s, t > 0$ and $T(0) = I$. Let $f(t) = \ln \|T(t)\|$. Suppose that $f(t)$ is bounded on $[0, a]$, show that

(1) $f(t)$ is sub-additive, i.e., $f(t+s) \leq f(t) + f(s)$ for all $t, s > 0$.

(2) $\lim_{t\to\infty} \frac{1}{t} f(t) = \inf_{t>0} \frac{1}{t} f(t)$.

Proof. (1) $f(t+s) = \ln \|T(t+s)\| = \ln \|T(t)T(s)\| \leq \ln(\|T(t)\| \|T(s)\|) = \ln \|T(t)\| + \ln \|T(s)\| = f(t) + f(s)$;

(2) It is not difficult to see that $f(t)$ is bounded on any finite interval $[0, s]$. Suppose that $f(t)$ is bounded by M_s on $[0, s]$. Fix s. Any t can be written as $t = ns + r$, where n is an integer and $0 \leq r < a$. Then we have from subadditivity of f that

$$f(t) \leq nf(s) + f(r) \leq nf(s) + M_s.$$

Divide it by t,

$$\frac{f(t)}{t} \leq \frac{n}{ns+r} f(s) + \frac{M_s}{t} \leq \frac{f(s)}{s} + \frac{M_s}{t}$$

Let $t \to \infty$,

$$\limsup_{t\to\infty} \frac{f(t)}{t} \leq \frac{f(s)}{s}.$$

Take infimum on the right-hand side,

$$\limsup_{t\to\infty} \frac{f(t)}{t} \leq \inf_{s>0} \frac{f(s)}{s}.$$

Notice that it holds trivially

$$\liminf_{t\to\infty} \frac{f(t)}{t} \geq \inf_{s>0} \frac{f(s)}{s}.$$

The proof is now complete. □

1.2 Let $\{T(t) : t \geq 0\}$ is a semigroup of bounded operators, such that $T(0) = I$ and strong continuity at $t = 0$, i.e., $s\text{-}\lim_{t\to 0^+} T(t) = I$. Show that the semi-group is strongly continuous.

Proof. We shall show that $t \mapsto T(t)x$ is continuous for all $x \in \mathscr{X}$. It is easy to show right strong continuity.

$$\lim_{t\to t_0^+} \|T(t)x - T(t_0)x\| = \lim_{t\to t_0^+} \|T(t_0)T(t-t_0)x - x\| \leq \|T(t_0)\| \lim_{t\to t_0^+} \|T(t-t_0)x - x\| = 0.$$

To prove the left strong continuity, it suffices to show that $\|T(t)\|$ to be uniformly bounded when t is near t_0. In fact, it holds that $\|T(t)\| \leq Me^{\omega t}$ for some M and ω. Refer to the text before Lemma 7.1.6. □

1.3 Let $\{T(t) : t \geq 0\}$ is a semigroup of bounded operators on Hilbert space $\mathscr{H}$ and satisfies $T(0) = I$ and weak continuity at $t = 0$. Show that the semigroup is strongly continuous.

Proof. Since $T(t)x \rightharpoonup x$, the uniform boundedness theorem tells us that $\|T(t)x\|$ is uniformly bounded in a neighbourhood of $t = 0$. Again by the uniform boundedness principle, it holds that $\|T(t)\|$ is uniformly bounded near $t = 0$. It is then easy to see that for a fixed $x_0 \in \mathscr{X}$, $x(t) = T(t)x_0$ is bounded on any compact interval of t.

Suppose that $0 \leq a < t < b < \xi - \epsilon < \xi$, where $\epsilon > 0$. Since $x(\xi) = T(\xi)x_0 = T(t)T(\xi - t)x_0 = T(t)x(\xi - t)$, we have that

$$(b-a)x(\xi) = \int_a^b x(\xi)dt = \int_a^b T(t)x(\xi - t)d\eta$$

and so, by $\sup_{a\leq t\leq b} \|T(t)\| < \infty$, we obtain

$$(b-a)\|x(\xi \pm \epsilon) - x(\xi)\| = \left\| \int_a^b T(t)(x(\xi \pm \epsilon - t) - x(\xi - t))dt \right\|$$

$$\leq \sup_{a\leq t\leq b} \|T(t)\| \cdot \int_{\xi-b}^{\xi-a} \|x(t\pm\epsilon) - x(t)\| dt$$

The right hand side tends to zero as $\epsilon \to 0^+$, as may be seen by approximating $x(t)$ by finite-valued functions.

So far we have proved that $x(t)$ is strongly continuous at $t > 0$. Now we prove the strong continuity at $t = 0$. For any positive rational number r we have $T(t)x(r) = x(t+r)$, and thus $s\text{-}\lim_{t\to 0^+} T(t)x(r) = x(r)$. Let M denote the set consisting of all finite linear combinations with rational coeffcients of $x(r)$'s, then $s\text{-}\lim_{t\to 0^+} T(t)x = x$ for all $x \in M$. On the other hand, for any $t \in [0,1]$

$$\|x(t) - x_0\| \leq \|T(t)x - x\| + \|x - x_0\| + \|T(t)(x_0 - x)\| \leq \|T(t)x - x\| + \left(\sup_{t\in[0,1]} \|T(t)\| + 1\right) \|x_0 - x\|$$

and thus

$$\limsup_{t\to 0^+} \|x(t) - x_0\| \leq \left(\sup_{t\in[0,1]} \|T(t)\| + 1\right) \|x_0 - x\| \tag{1}$$

for any $x \in M$. It is clear that $\{x(t) : t \geq 0\} \subset \overline{M}$ from the weak closedness of $\overline{M}$ and weak right-continuity of $\{x(t)\}$. Therefore the right side of (1) can be made arbitrarily close to 0, concluding that $x(t) \to x_0$ strongly as $t \to 0^+$. □

1.4 Let $\{T(t) : t \geq 0\}$ be a strongly continuous semi-group of operators on $\mathscr{X}$ and A its infinitesimal generator. Show that the following three conditions are equivalent:

(1) $D(A) = \mathscr{X}$;
(2) $\lim_{t\to 0^+} \|T(t) - I\| = 0$;
(3) $A \in L(\mathscr{X})$ and $T(t) = \exp(tA)$.

Proof. (3)⇒(1): It follows easily from series manipulation that $A_t x \to Ax$ for all x as $t \to 0$.

(1)⇒(2): Since $A_t x \to x$ as $t \to 0$, by uniform boundedness principle, A_t is uniformly bounded, say by M, in a small neighbourhood of $t = 0$. Then $\|T(t) - I\| \leq Mt \to 0$ as $t \to 0$.

(2)⇒(3): It is easy to verify that

$$\lim_{s\to 0} \frac{1}{s} \int_r^{s+r} T(t)dt = T(r)$$

In particular, there exists δ such that

$$\left\| \frac{1}{t} \int_0^t T(s)ds - I \right\| < 1$$

for all $0 < t < \delta$, then $\frac{1}{t}\int_0^t T(s)ds$ is invertible for $t \in (0, \delta)$. Now,

$$\frac{1}{s} \int_r^{r+s} T(t)dt - \frac{1}{s} \int_0^s T(t)dt = \frac{1}{s}(T(s) - I) \int_0^r T(t)dt \tag{2}$$

(because $\int_r^{r+s} - \int_0^s = \int_s^{r+s} - \int_0^r$). It follows that for $t \in (0, \delta)$,

$$\frac{1}{s}(T(s) - I) = \left(\frac{1}{s} \int_r^{r+s} T(t)dt - \frac{1}{s} \int_0^s T(t)dt\right) \left(\int_0^r T(t)dt\right)^{-1}$$

The right-hand side tends to $(T(r) - I)(\int_0^r T(t)dt)^{-1}$, so the left-hand side A_s converges to some bounded linear operator when $s \to 0$. It is obvious that this limit operator must be A. Taking limit $s \to 0$ in (2), we obtain that

$$T(r) - I = A \int_0^r T(s)ds$$

Iterated substitution gives

$$T(r) = I + A + \frac{A^2}{2} + \cdots + \frac{A^n}{n!} + \frac{A^{n+1}}{n!} \int_0^r (r-t)^n T(t) dt$$

Let $n \to \infty$, we see that $T(r) = \exp(rA)$ for all $r \geq 0$. □

1.5 Let $\mathscr{X} = C_0[0, \infty) = \{f \in C[0, \infty) : \lim_{x \to +\infty} f(x) = 0\}$, $\|f\| = \sup |f(s)|$. Define on $\mathscr{X}$ a linear operator

$$T(t) : a(\cdot) \mapsto a(t + \cdot).$$

Show that $\{T(t) : t \geq 0\}$ is a strongly contraction semigroup on $\mathscr{X}$.

Proof. It is obvious that $T(t+s) = T(t) + T(s)$ and $T(0) = I$. Now we show that $\|T(t)a - T(t_0)a\| \to 0$. This is because $\|T(t)a - T(t_0)a\| = \sup_s |a(t+s) - a(t_0+s)|$ and a is uniformly continuous. Finally, it is obvious that $\|T(t)\| \leq 1$ for all $t \geq 0$. □

1.6 Let $\mathscr{X} = L^2(\mathbb{R})$, for $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$, define

$$((T(y)f)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} f(\xi) d\xi, \quad y > 0,$$
$$T(0)f = f.$$

Show that $\{T(y) : y \geq 0\}$ is a strongly continuous semigroup on $\mathscr{X}$ and $\|T(y)\| = 1$. (*Remark.* The integral gives a harmonic function on the upper plane with boundary value f)

Proof. First we show that $T(y)f \in L^2(\mathbb{R})$. Indeed, by Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} f(\xi) d\xi \right|^2 dx \\
\leq &\frac{1}{\pi^2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} d\xi \right) \left(\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} |f(\xi)|^2 d\xi \right) dx \\
= &\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} |f(\xi)|^2 dx d\xi \\
= &\|f\|_2^2
\end{aligned}$$

Hence $\|T(y)\| \leq 1$. On the other hand,

$$\left(T(y)\chi_{[-R,R]}\right)(x) = \frac{1}{\pi} \left(\arctan\left(\frac{R-x}{y}\right) + \arctan\left(\frac{R+x}{y}\right) \right)$$

and thus

$$\begin{aligned}
\frac{\|T(y)\chi_{[-R,R]}\|_2^2}{\chi_{[-R,R]}\|_2^2} &\geq \frac{1}{R} \int_{-R/2}^{R/2} \frac{1}{\pi^2} \left(\arctan\left(\frac{R-x}{y}\right) + \arctan\left(\frac{R+x}{y}\right) \right)^2 dx \\
&\geq \frac{1}{R} \int_{-R/2}^{R/2} \left(\frac{2}{\pi} \arctan \frac{R}{2y} \right)^2 dx \\
&= \left(\frac{2}{\pi} \arctan \frac{R}{2y} \right)^2 \to 1
\end{aligned}$$

as $R \to \infty$, which implies that $\|T(y)\| = 1$ for $y > 0$. It is trivial that $T(0) = I$ and $\|T(0)\| = 1$.

When $f \in \mathscr{S}(\mathbb{R})$, Notice that $T(y)f$ is exactly $u(\cdot, y)$ that satisfies

$$\Delta u = 0, \quad y > 0$$

$$u(x,0) = f(x,0)$$

It is then obvious that $T(t+s) = T(t) + T(s)$ for $f \in \mathscr{S}(\mathbb{R})$, which can be extended to the entire $L^2(\mathbb{R})$ easily because $\mathscr{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and $\|T(y)\| \leq 1$.

Now we show $\|T(y)f - f\| \to 0$ as $y \to 0^+$. It suffices to show this for $f \in C_0^\infty(\mathbb{R})$ and density of test functions allows us to extend this result to $L^2(\mathbb{R})$. First we show that $T(y)f \to f$ uniformly pointwise as $y \to 0^+$. Let $\epsilon > 0$ be given. Since f is uniformly continuous, there exists δ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Then

$$\begin{aligned}
|(T(y)f)(x) - f(x)| &= \left| \int_{\mathbb{R}} \frac{y}{(x-\xi)^2 + y^2} (f(\xi) - f(x)) d\xi \right| \\
&\leq \int_{|x-\xi|<\delta} \frac{y|f(\xi) - f(x)|}{(x-\xi)^2 + y^2} d\xi + \int_{|x-\xi|>\delta} \frac{y|f(\xi) - f(x)|}{(x-\xi)^2 + y^2} d\xi \\
&=: I + J,
\end{aligned}$$

where

$$I \leq \epsilon \int_{\mathbb{R}} \frac{y}{(x-\xi)^2 + y^2} d\xi = \epsilon$$

and

$$\begin{aligned}
J &\leq 2\|f\|_\infty \int_{|x-\xi|>\delta} \frac{y}{(x-\xi)^2 + y^2} d\xi \\
&= 2\|f\|_\infty \left(\pi - 2 \arctan \frac{\delta}{y} \right) \to 0
\end{aligned}$$

uniformly w.r.t. x as $y \to 0$. Hence $T(y)f \to f$ uniformly w.r.t. x. Suppose that supp $f \in [-K, K]$. Now,

$$\begin{aligned}
\|T(y)f - f\|^2 &\leq \int_{|x|\leq K} |(T(y)f)(x) - f(x)|^2 dx + \int_{|x|\geq K} |(T(y)f)(x)|^2 dx \\
&\leq \|(T(y))f - f\|_\infty \cdot 2K + \int_{|x|\geq K} |(T(y)f)(x)|^2 dx
\end{aligned}$$

The first term goes to 0 as $y \to 0^+$. For the second term, we have

$$\begin{aligned}
\int_{|x|\geq K} |(T(y)f)(x)|^2 dx &\leq \frac{1}{\pi} \int_{|\xi|\leq K} |f(\xi)|^2 \int_{|x|\geq K} \frac{y}{(x-\xi)^2 + y^2} dx d\xi \\
&\leq \int_{|\xi|\leq K} |f(\xi)|^2 \left(\pi - \arctan \frac{K - \xi}{y} - \arctan \frac{K + \xi}{y} \right) d\xi \\
&\to 0 \text{ as } y \to 0^+ \text{ by Dominated Convergence Theorem.}
\end{aligned}$$

Therefore we conclude that $(T(y) - I)f \to 0$ as $y \to 0^+$, whence the strong continuity condition is satisfied by Problem 1.2. □

1.7 Let $\{T(t) : t \geq 0\}$ be a strongly continuous semi-group on $\mathscr{X}$. Suppose that $x \in X$, $w\text{-}\lim_{t\to 0^+} \frac{1}{t}(T(t) - I)x = y$, show that $x \in D(A)$ and $y = Ax$.

Proof. Let $f \in X^*$ and $\lambda > 0$ large enough. It is easy to see that $e^{-\lambda t} f(T(t)x)$ has right derivative, which equals to $e^{-\lambda t} f(T(t)(y - \lambda x))$. The derivative is continuous in t, we have on integration

$$f(x) = -\int_0^\infty e^{-\lambda t} f(T(t)(y - \lambda x)) dt,$$

for all $f \in X^*$. Therefore,

$$x = -\int_0^\infty e^{-\lambda t} T(t)(y - \lambda x) = -(\lambda I - A)^{-1}(y - \lambda x)$$

The conclusion follows immediately by multiplying $(\lambda I - A)$ on both sides. □

1.8 Let $\{T(t) : t \geq 0\}$ be a strongly continuous semi-group on a Hilbert space $\mathscr{H}$. Suppose that A is its generator and $T(t)$ is a normal operator for all $t \geq 0$. Show that A is normal using Gelfand transform.

1.9 Prove Hille-Yosida-Phillips Theorem (Theorem 7.1.7): A densely-defined closed linear operator A is an infinitesimal generator of some strongly continuous semigroup $\{T(t) : t \geq 0\}$ if and only if

(1) $\exists \omega_0 > 0$ such that $(\omega_0, \infty) \subset \rho(A)$;

(2) $\exists M > 0$ such that

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}, \quad n = 1, 2, \ldots$$

whenever $\lambda > \omega > \omega_0$.

Proof. The necessity has been proved in Lemma 7.1.6. The proof of sufficiency follows the same outline as in Theorem 7.1.5 by defining

$$B_\lambda = \lambda^2(\omega_0 + \lambda - A)^{-1} - \lambda I$$

for $\lambda > 0$. □

1.10 Let $\{T(t) : t \geq 0\}$ be a strongly continuous semi-group and A its infinitesimal generator. Suppose that $\omega_0 \in \mathbb{R}$ satisfies $\{\lambda : \Re\lambda > \omega_0\} \subset \rho(A)\}$. Show that

(1) The set $\{R_\lambda(A)x : x \in D(A)\}$ is dense in $D(A)$, where $\Re\lambda > \omega_0$;

(2) The range of $R_\lambda(A)^n$ is dense for all $n \geq 1$, where $\Re\lambda > \omega_0$;

(3) $D(A^n)$ is dense for all $n \geq 1$.

Proof. (1) Let $x \in D(A^2)$. It follows from $R_\lambda(A)(\lambda - A)x = x$ that $x \in R(R_\lambda(A)|_{D(A)})$. Hence $D(A^2) \subseteq R(R_\lambda(A)|_{D(A)})$. The conclusion follows immediately from the density of $D(A^2)$.

(2) It follows from $R_\lambda(A)(\lambda - A)x = x$ ($x \in D(A)$) that $x \in R(R_\lambda(A))$, that is, $D(A) \subset R(R_\lambda(A))$ and $R(R_\lambda(A))$ is therefore dense. Now, it $R_\lambda(A)^n(\lambda - A)^n x = x$ for all $x \in D(A^n)$, whence it follows that $D(A^n) \subset R(R_\lambda(A)^n)$. Part (3) shows that $D(A^n)$ is dense, and hence $R(R_\lambda(A)^n)$ is dense.

(3) This statement actually hold for *any* strongly continuous semi-group with no further assumptions. To see this, let ϕ be any function in $C_0^\infty[0, 1]$ and define for any x

$$x_\phi = \int_0^1 \phi(t)T(t)x dt.$$

Then

$$\begin{aligned}
\frac{T(h) - 1}{h} x_\phi &= \frac{1}{h}\int_0^1 \left(\int_0^t \phi'(s)ds\right)(T(t+h) - T(t))x dt \\
&= \frac{1}{h}\int_0^1 \phi'(s)\left(\int_s^1 (T(t+h)x - T(t)x)dt\right) ds \\
&= \frac{1}{h}\int_0^1 \phi'(s)\left(\int_1^{1+h} T(t)x dt - \int_s^{s+h} T(t)x dt\right) ds \\
&= \frac{1}{h}\int_0^1 \phi'(s)ds \int_1^{1+h} T(t)x dt - \frac{1}{h}\int_0^1 \phi'(s)\int_s^{s+h} T(t)x dt ds \\
&= -\frac{1}{h}\int_0^1 \phi'(s)\int_s^{s+h} T(t)x dt ds
\end{aligned}$$

Since $\frac{1}{h}\int_s^{s+h} T(t)x ds \to T(s)x$ as $h \to 0^+$ and ϕ' is bounded, the limit of the right-hand side exists as $h \to 0^+$ and equals to

$$-\int_0^1 \phi'(s)T(s)x ds.$$

Hence $x_\phi \in D(A)$ and

$$Ax_\phi = -\int_0^1 \phi'(s)T(s)xds.$$

Now it is clear that $x_\phi \in D(A^n)$ for all n. We can choose a sequence $\{\phi_j\} \subset C_0^\infty[0,1]$ such that $\phi_j \geq 0$, $\int_0^1 \phi_j = 1$ and $\operatorname{supp}\phi_j$ tends to 0. It is not difficult to see that $x_{\phi_j} \to x$. Hence $D(A^n)$ is dense. $\square$

1.11 Let $\{T(t) : t \geq 0\}$ be a strongly continuous semi-group and A its infinitesimal generator. Suppose that $f \in C^1([0,\infty); \mathscr{X})$. Show that the differential equation of operators

$$\frac{dx(t)}{dt} = Ax(t) + f(t), \tag{3}$$

$$x(0) = x_0 \in D(A) \tag{4}$$

has a unique solution in $C(\mathbb{R}^1_+; D(A)) \cap C^1(\mathbb{R}^1_+, \mathscr{X})$, which is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds. \tag{5}$$

Proof. The first term $T(t)x_0$ on the right of (5) satisfies the homogeneous differential equation and the initial condition, it suffices to show that the second term satisfies 3 with initial value 0.

$$\begin{aligned}\int_0^t T(t-s)f(s)ds &= \int_0^t T(t-s)\left(f(0) + \int_0^s f'(r)dr\right)ds \\ &= \left(\int_0^t T(t-s)ds\right)f(0) + \int_0^t f'(r)\left(\int_r^t T(t-s)ds\right)dr\end{aligned}$$

Note that

$$T(t) - T(r) = \int_r^t AT(s)ds,$$

it follows that

$$A\int_r^t T(t-s)ds = T(t-r) - I,$$

and then

$$\begin{aligned}A\int_0^t T(t-s)f(s)ds = (T(t)-I)f(0) + \int_0^t f'(r)(T(t-r)-I)dr \\ = T(t)f(0) - f(t) + \int_0^t f'(r)T(t-r)dr \quad (6)\end{aligned}$$

On the other hand,

$$\frac{d}{dt}\int_0^t T(t-s)f(s)ds = T(t)f(0) + \int_0^t T(s)f'(t-s)ds. \tag{7}$$

Comparing (6) and (7), we see that

$$\frac{d}{dt}\int_0^t T(t-s)f(s)ds = A\int_0^t T(t-s)f(s)ds + f(t).$$

It is clear that the initial value of the second term of (5) is 0. Hence $x(t)$ given in (5) is a solution to the differential equation indeed. The continuity of $x'(t)$ can be easily concluded from (7) using the continuity of f'. It is clear that $x(t) \in D(A)$.

In fact, if $x(t) \in C(\mathbb{R}^1_+; D(A)) \cap C^1(\mathbb{R}^1_+, \mathscr{X})$ is a solution to (3) and (4), then

$$\frac{d}{ds} T(t-s)x(s) = -T'(t-s)x(s) + T(t-s)x'(s) = -T(t-s)Ax(s) + T(t-s)x'(s) = T(t-s)f(s).$$

Integrate on both sides, we obtain that

$$x(t) - T(t)x(0) = \int_0^t T(t-s)f(s)ds,$$

which is exactly (5). The uniqueness of the solution is proved. □

2 Examples of Infinitesimal Generators

2.1 Let $\mathscr{X} = \{f : \mathbb{D} \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} c_n z^n, \|f\|^2 = \sum |c_n|^2 < \infty\}$, where $\mathbb{D}$ is the open disc in the complex plane. Define on $\mathscr{X}$

$$(T(t)f)(z) = \sum_{n=0}^{\infty} (n+1)^{-t} c_n z^n.$$

Show that $\{T(t) : t \geq 0\}$ is a strongly continuous semi-group of positive self-adjoint operators. Find its infinitesimal generator A and show that $\ln \frac{1}{n+1}$ $(n \geq 1)$ are eigenvalues of A.

Proof. It is obvious that $\|T(t)\| \leq 1$, $T(t+s) = T(t)T(s)$ and $T(0) = I$. Now we show that $T(t)f \to f$ strongly for all $f \in \mathscr{X}$. Given $\epsilon > 0$ and $f = \sum_{n=0}^{\infty} c_n z^n$, choose N big enough such that $\sum_{n>N} |c_n|^2 < \frac{\epsilon^2}{2}$. Choose t small enough such that $1 - (\frac{1}{N+1})^t < \frac{\epsilon}{\sqrt{2}\|f\|}$. Then

$$\begin{aligned}
\|f - T(t)f\|^2 &= \sum_{n=0}^{N} \left(1 - \frac{1}{(n+1)^t}\right)^2 |c_n|^2 + \sum_{n=N+1}^{\infty} \left(1 - \frac{1}{(n+1)^t}\right)^2 |c_n|^2 \\
&\leq \frac{\epsilon^2}{2\|f\|^2} \cdot \|f\|^2 + \sum_{n=N+1}^{\infty} |c_n|^2 \\
&\leq \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2,
\end{aligned}$$

that is, $\|f - T(t)f\| \leq \epsilon$ when t is sufficiently small. Therefore $\{T(t) : t \geq 0\}$ is a strongly continuous semi-group. It is straightforward to verify that $T(t)$ is positive and self-adjoint (note that the inner product $(\sum c_n z^n, \sum d_n z^n) = \sum c_n \overline{d_n}$).

Define a linear operator A as

$$Af = \sum_{n=0}^{\infty} c_n \ln \frac{1}{n+1} \cdot z^n$$

on

$$D(A) = \left\{ f = \sum c_n z^n \in \mathscr{X} : \sum |c_n|^2 \ln^2(n+1) < \infty \right\}.$$

It is clear that $D(A)$ is dense because $x^n \in D(A)$ for all n. We claim that A is closed. Suppose that $f_n \to f$, $Af_n \to g$, $f_n = \sum c_{nk} z^k$, $f = \sum c_n z^n$ and $g = \sum d_n z^n$. Since $Af_n \to g$,

$$\sum_{k=0}^{\infty} \left| c_{nk} \ln \frac{1}{k+1} - d_k \right|^2 \to 0,$$

or,

$$\sum_{k=1}^{\infty} \left| c_{nk} - \frac{d_k}{\ln \frac{1}{k+1}} \right|^2 \ln^2 \frac{1}{k+1} \to 0,$$

(because d_0 must be 0) which implies that

$$\sum_{k=1}^{\infty}\left|c_{nk}-\frac{d_k}{\ln\frac{1}{k+1}}\right|^2\to 0.$$

Comparing with $f_n\to f$, or, equivalently,

$$\sum_{k=0}^{\infty}|c_{nk}-c_k|^2\to 0.$$

we obtain that

$$d_n=c_n\ln\frac{1}{n+1}$$

for all $n\geq 0$, that is, $f\in D(A)$ and $g=Af$. Therefore A is a densely-defined closed operator.

Next we show that A generates a contraction semigroup. For $\lambda>0$, it is easy to verify that $\lambda I-A$ is injective, and $\|\lambda f-Af\|\geq\lambda\|f\|$, thus $\lambda I-A$ is invertible and $\|R_\lambda(A)\|\leq\lambda^{-1}$. By Hille-Yosida Theorem we know that A generates a contraction semigroup.

Now, to show that A is the infinitesimal generator of $\{T(t)\}$, it suffices to show that $A_tf\to Af$ on $D(A)$. Let $f\in D(A)$, $f=\sum c_nz^n$, then

$$\begin{aligned}\|A_tf-Af\|^2&=\sum|c_n|^2\left|\frac{(n+1)^{-t}-1}{t}-\ln\frac{1}{n+1}\right|^2\\&=\sum|c_n|^2\left|\frac{e^{t\ln\frac{1}{n+1}}-1}{t}-\ln\frac{1}{n+1}\right|^2\\&\leq t\sum|c_n|^2\ln^2\frac{1}{n+1}\to 0\end{aligned}$$

as $t\to 0^+$, where we used $e^x-1\leq x+x^2$ for all $x\leq 1$. □

2.2 Let $\mathscr{X}=L^2(-\pi,\pi)$. Define

$$\begin{gathered}(T(t)f)(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}G(\theta-\xi,t)f(\xi)d\xi,\quad t>0\\T(0)f=f,\end{gathered}$$

where the integral kernel $G(\theta,t)=1+2\sum_{n=1}^{\infty}e^{-n^2t}\cos n\theta$. Show that $\{T(t):t\geq 0\}$ is a strongly continuous semi-group. Is it a contraction semi-group?

Proof. The procedure is similar to Exercise 7.1.6, based on the following properties of $G(\theta,t)$:

(1) $G(\theta,t)$ is continuous;

(2) $G(\theta,t)\geq 0$ for all t;

(3) For every δ, $K(x,t)\to 0$ uniformly on $\delta<|x|<\pi$.

We shall prove the properties later. For now let us assume their correctness. For simplicity let the kernel $G(\theta,t)$ absorb the normalisation coefficient $\frac{1}{2\pi}$. Using the same trick as in 7.1.6, we obtain that $\|T(t)\|\leq 1$ for all $t>0$. It is easy to verify that $T(t+s)=T(t)T(s)$ (because $e^{-n^2(t+s)}=e^{-n^2t}e^{-n^2}s$) and $T(0)=I$. To show the right strong convergence at $t=0$, we can assume that $f\in C_0^\infty(-\pi,\pi)$. Using uniform continuity of f and Property (3) of the kernel $G(\theta,t)$, it is easy to show that $\|T(y)f-f\|\to 0$. We therefore conclude that $\{T(t)\}$ is a contraction semi-group.

To see Property (1), just notice that the sum of continuous functions is uniformly convergent. To see (2) and (3), apply Poisson's Summation formula

$$\sum_{k=-\infty}^{\infty} g(x+2k\pi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{g}(n)e^{inx}$$

to

$$g(x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$$

(note that $\hat{g}(\xi) = e^{-\xi^2 t}$), we obtain that

$$G(x,t) = \frac{1}{\sqrt{4\pi t}}\sum_{k=-\infty}^{\infty} e^{-\frac{(x+2\pi k)^2}{4t}},$$

whence Property (2) becomes obvious. Property (3) follows from integral approximation of $G(x,t)$,

$$G(\delta,t) \le \frac{1}{\sqrt{\pi t}}\int_{\delta}^{\infty} e^{\frac{-x^2}{4t}}\,dx = 1-\Phi\Big(\frac{\delta}{2\sqrt{t}}\Big) \to 0. \qquad \square$$

Remark. In fact, $T(t)f$ gives the solution $u(\cdot,t)$ to the heat equation on a circle S^1

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2}, \quad t>0,\ \theta\in(-\pi,\pi)$$
$$u(\pi,t) = u(-\pi,t)$$
$$u(\theta,0) = f(\theta), \quad \theta\in(-\pi,\pi)$$

when $f\in H^2(S^1)$ such that $f(-\pi)=f(\pi)$.

2.3 Let $\mathscr{X} = C(-\infty,\infty)$, the space of bounded uniformly continuous functions on $(-\infty,\infty)$. Define the linear operator $T(t)$ by

$$(T(t)u)(s) = \begin{cases} u(s), & t=0; \\ e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}u(s-n\mu), & t>0. \end{cases},$$

where $\lambda,\mu>0$. Show that $\{T(t): t\ge 0\}$ is a strongly continuous contraction semi-group, and its infinitesimal generator is the *difference operator* A:

$$(Au)(s) = \lambda(u(s-\mu)-u(s)).$$

Proof. It is easy to see that $T(t)u$ is uniformly continuous (using the uniform continuity of f) and $\|T(t)\|\le 1$. We have

$$\begin{aligned} T_w(T_t(u))(s) &= e^{-\lambda w}\sum_{n=0}^{\infty}\frac{(\lambda w)^n}{n!}\left(e^{-\lambda t}\sum_{m=0}^{\infty}\frac{(\lambda t)^m}{m!}u(s-m\mu-n\mu)\right) \\ &= e^{-\lambda(w+t)}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(\lambda w)^l}{l!}\frac{(\lambda t)^{k-l}}{(k-l)!}u(s-k\mu) \\ &= e^{-\lambda(w+t)}\sum_{k=0}^{\infty}\frac{1}{k!}\lambda^k(w+t)^k u(s-k\mu) \\ &= T_{t+w}(u)(s) \end{aligned}$$

It is trivial that $T(0)=I$. Now we verify that $\|T(t)u-u\|_\infty \to 0$ strongly as $t\to 0^+$. Write

$$(T(t)u-u)(x) = e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}u(x-n\mu) + (e^{-\lambda t}-1)u(x)$$

The second term goes to 0 uniformly because u is bounded. So does the first term, since

$$\left| e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} u(x - n\mu) \right| \leq \|s\|_\infty e^{-\lambda t}(e^{-\lambda t} - 1) = \|u\|_\infty (1 - e^{-\lambda t}).$$

Therefore $\{T(t)\}$ is a strongly continuous contraction semi-group. Now,

$$(A_t u)(x) = e^{-\lambda t} \sum_{n=2}^{\infty} \frac{\lambda^n t^{n-1}}{n!} u(x - n\mu) + \lambda e^{-\lambda t} u(x - \mu) + \frac{e^{\lambda t} - 1}{t} u(x).$$

The second term goes to $\lambda s(x - \mu)$ and the third term $\lambda u(x)$, and both convergences are uniform. The first term goes to 0 uniformly, as

$$\left| e^{-\lambda t} \sum_{n=2}^{\infty} \frac{\lambda^n t^{n-1}}{n!} u(x - n\mu) \right| \leq \|s\|_\infty e^{-\lambda t} \left(\frac{e^{\lambda t} - 1}{t} - \lambda \right) = \|u\|_\infty \left(\frac{1 - e^{-\lambda t}}{t} - \lambda e^{-\lambda t} \right) \to 0.$$

We conclude that

$$s - \lim_{t \to 0^+} (A_t u)(s) = \lambda(u(s - \mu) - u(s))$$

for all $u \in C(-\infty, \infty)$. The limit function is in $C(-\infty, \infty)$, too. The proof is now complete. □

2.4 Let $\mathscr{X} = C(\mathbb{R}^n, \mathbb{R})$ and $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Consider the following system of ODEs

$$\frac{dx(t)}{dt} = b(x(t)), \quad x(0) = \xi,$$

which is an autonomous system. For every $\xi \in \mathbb{R}^n$ there exists a solution $x(t, \xi)$, $t \in \mathbb{R}$ such that $x(t) \in C^1(\mathbb{R}, \mathbb{R}^n)$. Define the linear operator $T(t)$ on $\mathscr{X}$ as

$$[T(t)f](\xi) = f(x(t, \xi)), \quad t \geq 0.$$

Show that $\{T(t) : t \geq 0\}$ is a strongly continuous semi-group. Let A be its generator, then $C_0^1(\mathbb{R}^n; \mathbb{R}) \in D(A)$ and whenever $f \in C_0^1(\mathbb{R}^n; \mathbb{R})$ it holds that

$$(Af)(x) = \sum_{i=1}^{n} b^i(x) \frac{\partial f(x)}{\partial x_i}. \tag{8}$$

Proof. It is clear that each $T(t)$ is a continuous linear operator. It follows from the properties of the linear system that $T(t + s) = T(t)T(s)$ and that $x(t, \xi) \to x(0, \xi)$ uniformly as $t \to 0^+$, the latter of which implies that $T(t)f \to f$ as $t \to 0^+$. It is also trivial that $T(0) = I$. Hence $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup.

Note that by the chain rule,

$$\frac{df(x(t))}{dt} = \langle f'(x(t)), x'(t) \rangle = \langle f'(x(t)), b(x(t)) \rangle = \sum_{i=1}^{n} b^i(x(t)) \frac{\partial f(x(t))}{\partial x_i}.$$

It is clear that $C^1(\mathbb{R}^n, \mathbb{R}) \subset D(A)$ for the infinitesimal generator A and that Af takes the form (8). □

2.5 Let A be the infinitesimal generator of a contraction semi-group. Suppose that B is a dissipative operator, with $D(B) \supset D(A)$, and

$$\|Bu\| \leq a\|Au\| + b\|u\|,$$

for some $0 < a < 1/2$, $b > 0$, and all $u \in D(A)$. Show that $A + B$ is a closed dissipative operator (defined on $D(A)$) and generates a contraction semi-group.

Proof. Since A is the generator of a contraction semi-group, according to Hille-Yosida Theorem, $(\lambda - A)^{-1}$ exists for all $\lambda > 0$ and $\|(\lambda - A)^{-1}\| \leq 1/\lambda$. Thus $\|A(\lambda - A)^{-1}\| = \|\lambda(\lambda - A)^{-1} - I\| \leq 2$. For $u \in D(A)$,

$$\|B(\lambda - A)^{-1}u\| \leq a\|A(\lambda - A)^{-1}u\| + b\|(\lambda - A)^{-1}u\| \leq \left(2a + \frac{b}{\lambda}\right)\|u\|.$$

Thus for λ sufficiently large, $\|B(\lambda - A)^{-1}\| < 1$ and $I - B(\lambda - A)^{-1}$ is invertible. Since $R(\lambda - A) = \mathscr{X}$ and

$$\lambda - A - B = (I - B(\lambda - A)^{-1})(\lambda - A),$$

we see that $R(\lambda - A - B) = \mathscr{X}$. It is easy to verify that $A + B$ is closed (Corollary 6.5.3) and dissipative (both A and B are dissipative). Hence $A + B$ generates a contraction semigroup. □

Remark. The conclusion still holds if we know that $A + B$ is closed and dissipative, without assuming that B is dissipative. This observation will be used in the proof of the next problem.

2.6 Let A and C be dissipative operators on Banach space $\mathscr{X}$. Suppose that there is a dense set D, $D \subset D(A)$, $D \subset D(C)$ so that

$$\|(A - C)u\| \leq a(\|Au\| + \|Cu\|) + b\|u\|$$

for some $0 < a < 1$, $b > 0$ and all $u \in D$. Show that

(1) $\bar{A}$ generates a contraction semigroup if and only if $\bar{C}$ does.

(2) $D(\overline{A|_D}) = D(\overline{C|_D})$.

Proof. (1) It suffices to show that $R(\lambda - A)$ is dense for some $\lambda > 0$ if and only if $R(\mu - C)$ is dense for some $\mu > 0$. The proof is similar to Exercise 6.5.7. Let $B = C - A$ with $D(B) = D$ and define $T_\lambda = A + \lambda B$. Then $T_0 = A$, $T_1 = C$, $Au = T_\lambda u - \lambda Bu$ and $Cu = T_\lambda u + (1 - \lambda)Bu$. The inequality in the problem implies that

$$\|Bu\| \leq a(\|T_\lambda u - \lambda Bu\| + \|T_\lambda u + (1 - \lambda)Bu\|) + b\|u\| \leq 2a\|T_\lambda u\| + a\|Bu\| + b\|u\|,$$

or

$$\|Bu\| \leq \frac{2a}{1 - a}\|T_\lambda u\| + \frac{b}{1 - a}\|u\| \tag{9}$$

Let $0 \leq \lambda' \leq 1$. If $\frac{2a\lambda'}{1-a} < \frac{1}{2}$, the preceding problem implies that $R(\lambda - T_{\lambda+\lambda'}) = R(\lambda - T_\lambda - \lambda' B)$ is dense for some $\lambda > 0$ if and only if $R(\mu - T_\lambda)$ is dense for some $\mu > 0$. Thus starting from $\overline{T_0} = \bar{A}$ and applying this result finitely many times, we obtain the conclusion.

(2) It can be proved similarly by propagating the property from T_0 to T_1 in finitely many steps. Note that the inequality (9) implies the equivalence of the graph norms with respect to T_λ and $T_{\lambda+\lambda'}$. □

3 One-Parameter Unitary Groups and Stone's Theorem

3.1 Let $\{U(t) : t \in \mathbb{R}\}$ is a strongly continuous unitary group and D a dense set in $\mathscr{X}$ such that $U(t)D \subset D$ for all $t \in \mathbb{R}$. Suppose that $U(t)$ is strongly differentiable on D, i.e., $U(t)x$ is differentiable with respect to t for all $x \in D$. Show that $-i\frac{dU(t)}{dt}|_{t=0}$ is essentially self-adjoint on D, and its closure is the infinitesimal generator of the unitary group.

Proof. Let $B = -i\frac{dU(t)}{dt}|_{t=0}$. Then for $x, y \in D$,

$$(Bx, y) = \lim_{t\to 0} -i\left(\frac{U(t) - I}{t}x, y\right) = \lim_{t\to 0} -i\left(x, \frac{U(-t) - I}{t}y\right) = \lim_{t\to 0}\left(x, -i\frac{U(-t) - I}{-t}y\right) = (x, By),$$

which implies that B is symmetric. We can also establish that $\frac{dU}{dt}(t)x = iAU(t)x = iU(t)Ax$. The rest follows similarly to part (3) and (4) of the next problem. □

3.2 (Another proof of Stone's Theorem) Let $\{U(t) : t \in \mathbb{R}\}$ is a strongly continuous unitary group.

(1) $\forall f \in C_0^\infty(\mathbb{R})$, $\forall x \in X$, define
$$x_f = \int_{-\infty}^{\infty} f(t)U(t)x dt,$$
under Riemann sense. Let D be the set of all possible bounded linear combinations of x_f's. Show that D is dense.

(2) For $x \in D$, $U(t)x$ is differentiable. Find
$$\frac{dU(t)x}{dt}\bigg|_{t=0}$$

(3) Define the operator A on D as
$$Ax = -iU'(0)x,$$
show that A is essentially self-adjoint.

(4) Let $V(t) = e^{it\bar{A}}$, show that $V(t) = U(t)$.

Proof. (1) The integral exists because $\|f(t)U(t)x\| \le \|f\|_\infty \|x\|$, and the integral is over a compact set. Choose $g \in C_0^\infty(\mathbb{R})$ with support in $[-1, 1]$ such that $\int g = 1$ and let $g_\epsilon(x) = \epsilon^{-1} g(x/\epsilon)$, then
$$\|x - x_{g_\epsilon}\| \le \int_{\mathbb{R}} g_\epsilon(t)\|(U(t) - I)x\| \le \sup_{|t|\le\epsilon} \|(U(t) - I)x\| \to 0$$
as $t \to 0^+$ by strong continuity of $\{U(t)\}$. Therefore D is dense.

(2) Let $f \in C_0^\infty(\mathbb{R})$, we have
$$\begin{aligned}\frac{U(t)x_f - U(t_0)x_f}{t - t_0} &= \frac{1}{t - t_0}\left(\int_{-\infty}^{\infty} f(s)U(t + s)xds - \int_{-\infty}^{\infty} U(t_0 + s)xds\right)\\ &= \frac{1}{t - t_0}\left(\int_{-\infty}^{\infty} f(r - t)U(r)xdr - \int_{-\infty}^{\infty} f(r - t_0)U(r)xdr\right)\\ &= \int_{-\infty}^{\infty} \frac{f(r - t) - f(r - t_0)}{t - t_0} U(r)xdr\end{aligned}$$
for all $t_0 \ne t$. Since $f' \in C_0^\infty(\mathbb{R})$, the norm of the integrand on the right-hand side is dominated by $\|f'\|_\infty \chi_{\operatorname{supp} f'}\|x\|$ (using Lagrange's Mean-value Theorem), so we can apply Dominated Convergence Theorem, which yields that
$$\lim_{t\to t_0} \frac{U(t) - U(t_0)}{t - t_0} x_f = -\int_{-\infty}^{\infty} f'(r - t_0)U(r)xdr,$$
which is contained in D because $f'(\cdot - t_0) \in C_0^\infty(\mathbb{R})$. In particular, $U'(0)x_f = x_{-f'}$.

(3) First, it is easy to check that $AU(t)x = U(t)Ax = -i\frac{dU}{dt}(t)x$. For $x, y \in \mathscr{H}$ and $f, g \in C_0^\infty(\mathbb{R})$,
$$\begin{aligned}(Ax_f, y_g) &= \lim_{t\to0}\left(\left(\frac{U(t) - I}{it}\right) x_f, y_g\right)\\ &= \lim_{t\to0}\left(x_f, \left(\frac{I - U(-t)}{t}\right) y_g\right)\\ &= \left(x_f, \frac{1}{i} y_{-g'}\right)\\ &= (x, Ay_g)\end{aligned}$$
Hence A is symmetric. Now suppose $x \in D(A^*)$ such that $A^*x = ix$, then
$$\frac{d}{dt}(x, U(t)y) = \left(x, \frac{dU}{dt}(t)y\right) = (x, iAU(t)y) = -i(A^*x, U(t)y) = (x, U(t)y), \quad \forall y \in D.$$

This differential equation with initial condition $U(0) = I$ is solved by $(x, U(t)y) = (x, y)e^t$. Notice that $|(x, U(t)y)| \le \|x\| \, \|U(t)y\| = \|x\| \, \|y\|$ for all t, we must have $x \in D^\perp$, thus $x = 0$ since D is dense. We have proved that $\ker(A^* - iI) = \{0\}$. Similarly it can be shown that $\ker(A^* + iI) = \{0\}$. Therefore A is essentially self-adjoint.

(4) It suffices to show that $U(t)x - V(t)x = 0$ for $x \in D$. Let $y(t) = U(t)x - V(t)x$. This is well-defined, because $V(t)x \in D(\bar{A})$. Differentiate with respect to t, $y'(t) = U'(t)x - V'(t)x = iAU(t)x - i\bar{A}V(t)x = i\bar{A}y(t)$, hence $\frac{d}{dt}\|y(t)\|^2 = -i(\bar{A}y(t), y(t)) + i(y(t), \bar{A}y(t)) = 0$. Finally $y(0) = 0$ and thus $y(t) = 0$ for all t. □

3.3 Let A_n be a sequence of self-adjoint operators on $\mathscr{H}$. Suppose that for EACH $x \in \mathscr{H}$ and each $t \in \mathbb{R}$, e^{itA_n} converges strongly in $\mathscr{H}$. Show that there exists a self-adjoint operator A such that $A_n \to A$ s.r.s.

Proof. Define $T(t) = s\text{-}\lim_{n\to\infty} e^{itA_n}$. We want to show that $\{T(t) : t \in \mathbb{R}\}$ is a unitary group.

First, it is easy to see that $T(t) \le 1$ and $T(0) = I$. Then it is easy to verify that $T(s+t) = T(s)T(t)$. Using functional calculus, it is easy to see that $T(t)$ is unitary. Hence $\{T(t) : t \in \mathbb{R}\}$ is a one-parameter group of unitary operators. For each x and $y \in \mathscr{H}$, $(T(t)x, y)$ is the limit of a sequence of continuous functions and thus measurable. Hence $\{U(t)\}$ is strongly continuous (see Remark 2 after Philips Theorem). Now, by Stone's Theorem, there exists a self-adjoint operator A such that $T(t) = \exp(itA)$. The conclusion follows immediately from Example 6.6.7 (Trotter's Theorem). □

3.4 Let U be an unitary operator on $\mathscr{H}$ then the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n x = \bar{x}$$

exists and $U\bar{x} = \bar{x}$.

Proof. Consider the spectrum decomposition of U,

$$U = \int_0^{2\pi} e^{i\theta} dF_\theta,$$

then

$$\frac{1}{N} \sum_{n=0}^{N-1} U^n = \int_0^{2\pi} \left(\frac{1}{N} \sum_{n=0}^{N-1} e^{in\theta} \right) dF_\theta.$$

Note that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{in\theta} = \begin{cases} 1, & \theta = 0; \\ 0, & 0 < \theta < 2\pi. \end{cases}$$

Denote the right-hand side by $f(\theta)$. It is easy to see that

$$E(\{1\})x = \int_0^{2\pi} f(\theta) dF_\theta x.$$

In fact, $E(\{1\})$ is the orthogonal projection of U onto the eigenvalue associated with eigenvalue 1. By Dominated Convergence Theorem,

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} U^n x - E(\{1\})x \right\|^2 = \int_0^{2\pi} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{in\theta} - f(\theta) \right|^2 d\|F_\theta x\|^2 \to 0$$

as $N \to \infty$. Therefore

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n x = E(\{1\})x.$$

It follows immediately from functional calculus that $UE(\{1\})x = E(\{1\})x$. □

3.5 Let $(\Omega, \mathscr{B}, \sigma)$ be a measure space of finite measure. Suppose that $\{\Gamma_t : t \in \mathbb{R}\}$ is an ergodic group of measure-preserving transformations, show that

(1) $\forall f, g \in L^2(\Omega, \mathscr{B}, \sigma)$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (f(\Gamma_t x), g) dt = \frac{1}{\sigma(\Omega)} \int_\Omega f d\sigma \int_\Omega g d\sigma$$

(2) Let $A, B \in \mathscr{B}$, then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sigma(\Gamma_t A \cap B) dt = \frac{1}{\sigma(\Omega)} \sigma(A)\sigma(B)$$

Proof. (1) Since $\{\Gamma_t\}$ is ergodic and $\sigma(\Omega) < +\infty$,

$$\lim_{T\to\infty} \int_0^T f(\Gamma_t x) dx = \frac{1}{\sigma(\Omega)} \int_\Omega f d\sigma.$$

It follows immediately from continuity of inner product that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (f(\Gamma_t x), g) dt = \lim_{T\to\infty} \left(\int_0^T f(\Gamma_t x) dx, g \right) = \frac{1}{\sigma(\Omega)} \int_\Omega f d\sigma \int_\Omega g d\sigma$$

(2) In the previous part, let $f = \chi_A$ and $g = \chi_B$ and note that

$$(\chi_A(\Gamma_t x), \chi_B) = \int_\Omega \chi_A(\Gamma_t x) \chi_B(x) dx = \sigma(\Gamma_t^{-1} A \cap \Gamma_B) = \sigma(\Gamma_{-t} A \cap \Gamma_B)$$

and (with substitution $t' = -t$

$$\frac{1}{T} \int_0^T \sigma(\Gamma_t A \cap B) dt = \frac{1}{-T} \int_0^{-T} \sigma(\Gamma_{-t'} A \cap B) dt'.$$

□

3.6 Let A and B be positive self-adjoint operators. Suppose that $A+B$ is self-adjoint on $D(A) \cap D(B)$, $-A$, $-B$ and $-(A+B)$ can generate strongly continuous contraction semigroup, denoted by $\{T^A(t) : t \geq 0\}$, $\{T^B(t) : t \geq 0\}$ and $\{T^{A+B}(t) : t \geq 0\}$, respectively. Show that

$$T^{A+B}(t) = s\text{-}\lim_{n\to\infty} \left(T^A\left(\frac{t}{n}\right) T^B\left(\frac{t}{n}\right) \right)^n$$

Proof. The proof follows the same outline as that of Trotter's Product Formula (Theorem 7.3.14). Let $x \in D := D(A) \cap D(B)$, then

$$s\text{-}\lim_{s\to 0^+} \frac{1}{s}(T^A(s)T^B(s)x - x) = s\text{-}\lim_{s\to 0^+} \frac{1}{s}(T^A(s) - x) + s\text{-}\lim_{s\to 0^+} \frac{1}{s} T^A(s)(T^B(s)x - x) = -Ax - Bx$$

and

$$s\text{-}\lim_{s\to 0^+} \frac{1}{s}(T^{A+B}(s)x - x) = -(A+B)x.$$

Let $T_s = \frac{1}{s}(T^A(s)T^B(s) - T^{A+B}(s))$, then $T_s x \to 0$ as $s \to 0^+$ and $T_s \to 0$ as $s \to +\infty$. Hence for any $x \in D$, $\|T_s x\|$ is bounded and thus $\|T_s\|$ is uniformly bounded by uniform boundedness principle. It follows that $T_s x \to 0$ uniformly on any compact set $K \subset D$.

Choose $K = \{T^{A+B}(r)x : r \in [-1, 1]\}$ (a continuous map maps a compact set to a compact set). We have

$$s\text{-}\lim_{s\to 0^+} \frac{1}{s}(T^A(s)T^B(s) - T^{A+B}(s))T^{A+B}(r)x = 0$$

uniformly on $s \in [-1, 1]$.

From the interpolation

$$(T^A(s)T^B(s))^n x - (T^{A+B}(s))^n x = \sum_{k=0}^{n-1}(T^A(s)T^B(s))^k(T^A(s)T^B(s) - T^{A+B}(s))(T^{A+B}(s))^{n-1-k}x$$

it follows that

$$\begin{aligned}
&\left\|\left(T^A\left(\frac{t}{n}\right)T^B\left(\frac{t}{n}\right)\right)^n x - \left(T^{A+B}\left(\frac{t}{n}\right)\right)^n x\right\| \\
&\leq n \max_{|s|<t}\left\|\left(T^A\left(\frac{t}{n}\right)T^B\left(\frac{t}{n}\right) - T^{A+B}\left(\frac{t}{n}\right)\right)T^{A+B}(s)x\right\| \\
&\leq |t| \max_{|s|<t}\left\|\frac{n}{t}\left(T^A\left(\frac{t}{n}\right)T^B\left(\frac{t}{n}\right) - T^{A+B}\left(\frac{t}{n}\right)\right)T^{A+B}(s)x\right\| \\
&= |t| \max_{|s|<t}\left\|T_{\frac{t}{n}}T^{A+B}(s)x\right\| \to 0
\end{aligned}$$

as $n \to \infty$. Therefore we have established

$$s\text{-}\lim_{n\to\infty}\left(T^A\left(\frac{t}{n}\right)T^B\left(\frac{t}{n}\right)\right)^n x = T^{A+B}(t)x$$

for all $x \in D$. Since D is dense and the semi-groups are contractions, the limits holds for all $x \in \mathscr{H}$. □

4 Markov Processes

No exercises.

5 Scattering Theory

No exercises.

6 Evolution Equations

No exercises.