1 Closed Operators

1.1 Show that every bounded operator on a Hilbert space is closable and every finite-rank closable operator is bounded.

Proof. For the first part, see Theorem 2.3.12. Now we prove the second part. Suppose that A is a finite-rank closable operator, i.e., if $\{x_n\} \subseteq D(A)$, $x_n \to 0$ and $Ax_n \to y$ then $y = 0$. If A is not bounded, then there exist $\{y_n\}$ such that $\|Ay_n\| \geq n\|y_n\|$. Let $x_n = y_n/\|Ay_n\|$, then $\|Ax_n\| = 1$ and $\|x_n\| \leq \frac{1}{n}$. Hence $x_n \to 0$. Note that A is finite-rank and recall that the unit sphere is sequentially compact in a finite dimensional space, thus we can choose a subsequence of $\{x_n\}$, still denoted by x_n, such that $Ax_n \to z$ for some z. Since A is closable, we must have $z = 0$, which contradicts with $\|x_n\| = 1$. □

1.2 Show that a linear operator T is closed if and only if $D(T)$ is complete under graph norm.

Proof. It is clear that $\{x_n\}$ is Cauchy in $D(T)$ under graph norm if and only if $\langle x_n, Tx_n\rangle$ is Cauchy in $\mathscr{X} \times \mathscr{Y}$. The conclusion follows immediately. □

1.3 Let T be a closable operator. Show that $\overline{T}^* = T^*$.

Proof. It is easy to see that ${}^\perp S = {}^\perp \overline{S}$ for any $S \subseteq \mathscr{X}$. Hence, $\Gamma(T^*) = {}^\perp (V\Gamma(T)) = {}^\perp (\overline{V\Gamma(T)}) = {}^\perp (V\overline{\Gamma(T)}) = {}^\perp (V\Gamma(\overline{T})) = \Gamma(\overline{T}^*)$, which implies that $\overline{T}^* = T^*$. □

1.4 Let T be a densely-defined linear symmetric operator on a Hilbert space, show that

(1) T is closed $\iff T = T^{**} \subset T^*$;

(2) T is essentially self-adjoint $\iff T \subset T^{**} = T^*$;

(3) T is self-adjoint $\iff T = T^{**} = T^*$.

Proof. (1) In the proof of 6.1.4, we have seen that $\Gamma(T^{**}) = \overline{\Gamma(T)}$. Hence $T = T^{**} \iff \Gamma(T^{**}) = \Gamma(T) \iff \overline{\Gamma(T)} = \Gamma(T) \iff T$ is closed. From the definition of symmetric operators, $T \subset T^*$ is automatic.

(2) `$\Rightarrow$': T is closable implies that $\Gamma(\overline{T}) = \overline{\Gamma(T)} = \Gamma(T^{**})$, and thus $T \subset T^{**}$, and from the previous problem, $\overline{T}^* = T^*$. Also, $\overline{T}$ is self-adjoint, $\overline{T} = \overline{T}^* = T^*$. Taking conjugate on both sides, $\overline{T}^* = T^{**}$, i.e., $T^* = T^{**}$.
`$\Leftarrow$': T is symmetric, thus T is closable and $\overline{T} = T^{**}$ (Theorem 6.1.4). Also $T^{**} = T^* = \overline{T}^*$ (Problem 6.1.3), it follows that $\overline{T} = \overline{T}^*$ and $\overline{T}$ is self-adjoint.

(3) T is self-adjoint $\iff$ (by definition) $T = T^* \Longrightarrow T^* = T^{**}$. □

1.5 Let T be a densely-defined operator on Hilbert space $\mathscr{H}$. Show that $D(T^*) = \{0\}$ if and only if $\Gamma(T)$ is dense in $\mathscr{H} \times \mathscr{H}$.

Proof. It suffices to show that

$$\Gamma(T^*) = {}^\perp (V\Gamma(T)) = \{0\} \iff \Gamma(T) \text{ is dense in } \mathscr{H} \times \mathscr{H},$$

which is obvious, since ${}^\perp(V\Gamma(T)) = \{0\}$ iff $V\Gamma(T)$ is dense iff $\Gamma(T)$ is dense. □

1.6 Determine whether the following statement is true: Let T be a densely-defined operator on $\mathscr{H}$ such that $(Tx, x) = 0$ for all $x \in D(T)$, then $Tx = 0$ for all $x \in D(T)$.

Proof. This is false. Consider the differential operator $T : x \mapsto \frac{d}{dt}$ defined on $C_0^\infty(\mathbb{R})$, which is a dense subset of $L^2(\mathbb{R})$. Suppose $x \in C_0^\infty(\mathbb{R})$, then

$$\int_{\mathbb{R}} \left(\frac{dx}{dt} \cdot x\right) dt = x^2\Big|_{-\infty}^{+\infty} - \left(\int_{\mathbb{R}} x \cdot \frac{dx}{dt}\right) dt = -\left(\int_{\mathbb{R}} x \cdot \frac{dx}{dt}\right) dt,$$

hence $\langle Tx, x\rangle = 0$ for all $x \in C_0^\infty(\mathbb{R})$. Obviously $Tx \neq 0$ for some $x \in C_0^\infty(\mathbb{R})$. □

1.7 Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces, and $\mathscr{Y}$ is reflexive. $T : \mathscr{X} \to \mathscr{Y}$ is a densely-defined operator. Show that T is closable if and only if T^* is densely-defined. Also let $J_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}^{**}$ and $J_{\mathscr{Y}} : \mathscr{Y} \to \mathscr{Y}^{**}$ be natural embeddings, show that when T is closable, $T = J_{\mathscr{Y}}^{-1} T^{**} J_{\mathscr{X}}$.

Proof. \`If': Since T^* is densely-defined, T^{**} is a closed operator, and

$$\Gamma(T^{**}) = ^{\perp} V\Gamma(T^*) = ^{\perp} V ^{\perp} V\Gamma(T) = ^{\perp} (^{\perp}V^2\Gamma(T)) = ^{\perp} (^{\perp}\Gamma(T)) = \overline{\Gamma(\tilde{T})},$$

where $\tilde{T} : \mathscr{X}^{**} \to \mathscr{Y}^{**}$ is the natural lift of $T : \mathscr{X} \to \mathscr{Y}$. It is clear to see that $\overline{\Gamma(\tilde{T})}$ restricted on im $J_{\mathscr{X}} \times \mathscr{Y}^{**}$ can be brought down to $\mathscr{X} \times \mathscr{Y}$ and become $\overline{\Gamma(T)}$. To summarize, $\overline{T} = J_{\mathscr{Y}}^{-1} T^{**} J_{\mathscr{X}}$.

\`Only if': Suppose that T is closable. If $D(T^*)$ is not dense, then there exists $y_0 \in \mathscr{Y}^{**}$, $y_0 \neq 0$, such that $y_0 \in^{\perp} D(T^*)$, thus $\langle y_0, 0\rangle \in^{\perp} \Gamma(T^*)$. Obviously $\langle 0, y_0\rangle \in^{\perp} V\Gamma(T^*)$, which implies that $^{\perp}V\Gamma(T^*)$ can not be a graph of some linear operator. But on the other hand, $^{\perp}V\Gamma(T^*) =^{\perp} V^{\perp}V\Gamma(T)$, which is, as shown above, the graph of the lift of $\overline{T}$, contradiction. Therefore T^* is densely-defined. □

1.8 Let f be a bounded and measurable function on $\mathbb{R}^1$, but $f \notin L^2(\mathbb{R}^1)$. Let

$$D = \left\{\psi \in L^2(\mathbb{R}^1) : \int |f(x)\psi(x)| dx < \infty\right\}.$$

Suppose that $\psi_0 \in L^2(\mathbb{R}^1)$ and define

$$T\psi = (f, \psi)\psi_0, \quad \forall \psi \in D.$$

Prove that T is densely-defined and find T^*.

Proof. Obviously $C_0^\infty(\mathbb{R}) \subset D$ and we know that $C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, therefore D is dense in $L^2(\mathbb{R})$ and T is densely-defined. Let $f_n = f\chi_{[-n,n]}$, then $\langle f, f_n\rangle = \|f_n\|_2^2$. Note that $\|f_n\| \to \infty$ as $n \to \infty$, this implies that (f, x) is not a bounded functional on D. Suppose $y \in D(T^*)$, which requires that there exists M_y such that

$$|(y, Tx)| = |(y, (f, x)\phi_0)| = |(f, x)|\,|(y, \phi_0)| \le M_y\|x\|, \quad \forall x \in D.$$

Since (f, x) is not a bounded functional, we must have $(y, \phi_0) = 0$. It is also easy to see that all y such that $(y, \phi_0) = 0$ is contained in $D(T^*)$, and therefore $D(T^*) = \{y \in L^2 : (y, \phi_0) = 0\}$. Since $(T^*y, x) = (y, Tx) = (f, x)(y, \phi_0) = 0$ for all $x \in D$. Since D is dense, it must hold that $T^*y = 0$. Hence $T^* = 0$. □

1.9 Let T be a linear operator in Hilbert space $\mathscr{H}$. Define its kernel as $N(T) = \{x \in D(T) : Tx = 0\}$. Show that

(1) If $D(T)$ is dense in $\mathscr{X}$ then $N(T^*) = R(T)^\perp \cap D(T^*)$;

(2) If T is closed, then $N(T) = R(T^*)^\perp \cap D(T)$.

Proof. (1) \`$\subseteq$': Let $y^* \in N(T^*)$, then $(y^*, Tx) = (T^*y^*, x)$ for all $x \in D(T)$. Since $T^*y^* = 0$, it follows that $(y^*, Tx) = 0$, which implies that $y^* \perp R(T)$.

\`$\supseteq$': Let $y^* \in R(T)^\perp \cap D(T^*)$, then $0 = (y^*, Tx) = (T^*y^*, x)$ for all $x \in D(T)$, which means that $T^*y^* \perp D(T)$. Since $D(T)$ is dense, it must hold that $T^*y^* = 0$, i.e., $y^* \in \ker T^*$.

(2) Since T is closed, T^* is densely-defined.
`$\subseteq$': Suppose that $x \in R(T^*)^\perp \cap D(T)$, then $(T^*y^*, x) = 0$ for all $y^* \in D(T)$. Then $(y^*, Tx) = (T^*y^*, x) = 0$ for all $y^* \in D(T^*)$. Since $D(T^*)$ is dense, we must have $Tx = 0$, or, $x \in \ker T$.
`$\supseteq$': Suppose that $x \in \ker T$. Then $0 = (y^*, Tx) = (T^*y^*, x)$ for all $y^* \in D(T)$, which implies that $x \perp R(T^*)$. □

1.10 Let T be an injective linear operator on $\mathscr{H}$. Consider some assumptions about T:

(1) T is closed;
(2) $\operatorname{im} T$ is dense;
(3) $\operatorname{im} T$ is closed;
(4) $\exists c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in D(T)$.

Show that

(1) Conditions (1), (2) and (3) imply (4);
(2) Conditions (2), (3) and (4) imply (1);
(3) Conditions (1) and (4) imply (3);

Proof. (1) The conditions (2) and (3) imply that $\operatorname{im} T = \mathscr{H}$, since $\mathscr{H}$ is injective, we must have $D(T) = \mathscr{H}$, which is closed. It follows condition (1) and Closed Operator Theorem that T is continuous. Also T is bijective, Open Mapping Theorem asserts that T^{-1} is bounded, which is exactly condition (4).

(2) From the same argument as in subproblem (1), we know that $D(T)$ is bijective. Condition (4) implies that T^{-1} is continuous. Suppose that $x_n \to x$ and $y_n \to y$, $y_n = Tx_n$, then $x_n = T^{-1}y_n$. Taking limits on both slides yields $x = T^{-1}y$, i.e., $y = Tx$. Therefore T is closed.

(3) Suppose that $\{Tx_n\}$ is a Cauchy sequence. Condition (4) implies that $\{x_n\}$ is a Cauchy sequence. Suppose that $Tx_n \to y$ and $x_n \to x$. Condition (1) says that $x \in D(A)$ and $y = Tx \in \operatorname{im} T$, hence $\operatorname{im} T$ is closed. □

1.11 Let $\mathscr{H} = L^2[0,1]$, $T_1 = i\frac{d}{dt}$, $T_2 = i\frac{d}{dt}$.

$$D(T_1) = \{u \in \mathscr{H} : u \textit{ is absolutely continuous}\},$$
$$D(T_2) = \{u \in \mathscr{H} : u(0) = 0,\ u \textit{ is absolutely continuous}\},$$

Show that both T_1 and T_2 are closed operators.

Proof. Suppose that $\{x_n\} \subseteq D(T_2)$, $x_n \to x$ and $i\frac{dx_n}{dt} \to iy$. Since x_n is absolutely continuous,

$$x_n(t) = \int_0^t x_n'(s)ds.$$

Note that

$$\int_0^t |x_n'(s) - y(s)|ds \leq \sqrt{t} \cdot \|x_n' - y\|_2 \leq \|x_n' - y\|_2 \to 0, \quad ,n \to \infty,$$

it follows that

$$x_n(t) \to \int_0^t y(s)ds$$

uniformly on $[0,1]$. Hence $\|x_n - \int y\|_2 \leq \|x_n - \int y\|_\infty^2 \to 0$. From the uniqueness of limit, we see that

$$x(t) = \int_0^t y(s)ds,$$

which is contained in $D(T_2)$ and $T_2 x = iy$. Therefore T_2 is closed.

Now suppose that $\{x_n\} \subseteq D(T_2)$, $x_n \to x$ and $i\frac{dx_n}{dt} \to iy$. Since L^2 convergence implies convergence in measure, and Riesz theorem ensures an a.e. pointwise convergent subsequence in a subsequence of functions converging in measure, we may assume that $x_n \to x$ pointwise a.e. Define $f(t) = \int_0^t y(s)ds$, from the preceding argument, we conclude that $x_n(t) - x_n(0) \to f(t)$ everywhere. Recall that $x_n(t) \to x(t)$ a.e., we must have that $x_n(0) \to a$ for some a and $x(t) = f(t) + a$ a.e.. Note that $f(t)$ is absolutely continuous, hence $x(t)$ is absolutely continuous, too. This implies that T_1 is closed. □

1.12 Let $\mathscr{X}$ be a separable Hilbert space and $\{e_n\}_{n=1}^{\infty}$ an orthonormal basis. Suppose that $a \in \mathscr{X}$, a is not a finite linear combination of $\{e_n\}$. Let D be the set of finite combinations of $\{e_n\}$ and a, and define on D

$$T(\beta a + \sum a_i e_i) = \beta a,$$

where in the summand there are only finitely many non-zero a_i's. Show that $\langle a, a\rangle \in \overline{\Gamma(T)}$, $\langle a, 0\rangle \in \overline{\Gamma(T)}$ and thus $\Gamma(T)$ is not the graph of any linear operator.

Proof. It is trivial that $\langle a, a\rangle \in \Gamma(T)$. Let $a_n = \sum_{i=1}^{n}(a, e_i)e_i$, then $a_n \to a$ and $Ta_n = 0$. Hence $\langle a, 0\rangle \in \overline{\Gamma(T)}$. □

1.13 Let $\mathscr{H} = l^2$ and

$$D(T) = \left\{ a \in l^2 : \exists N \text{ such that whenever } n > N, a_n = 0 \text{ and } \sum_{j=0}^{N} a_j = 0 \right\}.$$

Define $Ta \in l^2$ for $a \in l^2$ as

$$(Ta)_n = i\left(\sum_{j=1}^{n-1} a_j + \sum_{j=1}^{n} a_j\right).$$

Show that

(1) T is densely-defined and symmetric;
(2) $R(T + i)$ is dense in l^2;
(3) $(1, 0, 0, \dots) \in D(T^*)$ and $(T^* + i)(1, 0, 0, \dots) = 0$.

Proof. (1) To show that $D(T)$ is dense, it suffices to show that $D(T)$ is dense in span$\{e_n\}$, where $\{e_n\}$ is the natural orthonormal basis in l^2. Furthermore, it suffices to show that each e_n can be approximated by elements in $D(T)$. Take e_1 for example. Let

$$a_n = \Big(1 - \frac{1}{n}, \underbrace{-\frac{1}{n}(1 - \frac{1}{n}), \dots, -\frac{1}{n}(1 - \frac{1}{n})}_{n \text{ times}}, 0, 0, \dots\Big).$$

Then

$$\|a_n - e_1\|^2 = \frac{1}{n^2} + n\Big(\frac{1}{n}\Big(1 - \frac{1}{n}\Big)\Big)^2 \to 0$$

as $n \to \infty$. We have seen that $a_n \to e_1$. The approximation to general e_m is similar, just right shift $\{a_n\}$ by m positions.

Now we show that $(Tx, y) = (x, Ty)$ for all $x, y \in D(T)$, to prove that T is symmetric. Suppose that N is the maximum of the two N's corresponding to x and y.

$$(Tx, y) = i\sum_{n=1}^{N} \overline{y_n}\left(\sum_{j=1}^{n-1} x_j + \sum_{j=1}^{n} x_j\right)$$

$$
\begin{aligned}
&= i\left(\sum_{n=1}^{N}\sum_{j=1}^{n-1} x_j\overline{y_n} + \sum_{n=1}^{N}\sum_{j=1}^{n} x_j\overline{y_n}\right)\\
&= i\left(\sum_{j=1}^{N-1}\sum_{n=j+1}^{N} x_j\overline{y_n} + \sum_{j=1}^{N}\sum_{n=j}^{N} x_j\overline{y_n}\right)\\
&= i\sum_{j=1}^{N} x_j\left(\sum_{n=j+1}^{N}\overline{y_n} + \sum_{n=j}^{N}\overline{y_n}\right)\\
&= i\sum_{j=1}^{N} x_j\left(-\sum_{n=1}^{j}\overline{y_n} - \sum_{n=1}^{j-1}\overline{y_n}\right)\\
&= (x, Ty).
\end{aligned}
$$

(2) Note that $(T+i)a = 2i(a_1, a_1+a_2, \ldots, a_1+a_2+\cdots+a_n, \ldots)$. Hence $(T+i)(\frac{1}{2i}a) = e_1$, where $a = (1, -1, 0, \ldots)$. Similarly we can show that $\{e_n\} \subseteq R(T+i)$, which implies that $R(T+i)$ is dense.

(3) Let $y^* = (1, 0, 0, \ldots)$, then $(y^*, Tx) = \overline{(Tx)_1} = -i\overline{x_1}$. Let $x^* = (-i, 0, 0, \ldots) = -y^*$, then $(x^*, x) = -i\overline{x_1}$. Hence $T^*y^* = -y^*$, $y^* \in D(T^*)$ and $(T+i)y^* = 0$. □

1.14 Let T be a symmetric operator on $\mathscr{X}$ with domain D. Suppose that $D_1 \subseteq D$ is a dense linear set and $T|_{D_1}$ is T restricted to D_1. If $T|_{D_1}$ is essential self-adjoint, so is T and $\overline{T} = \overline{T|_{D_1}}$.

Proof. Since D_1 is dense in D, we can use diagonal technique to show that $\overline{\Gamma(T)} = \overline{\Gamma(T|_{D_1})} = \Gamma(\overline{T|_{D_1}})$. Hence T is closable and $\overline{T} = \overline{T|_{D_1}}$. Now we show that $\overline{T}$ is self-adjoint. Since $\overline{T|_{D_1}}$ is self-adjoint, we have that $\overline{T|_{D_1}}^* = \overline{T|_{D_1}}$ and therefore $\overline{T}^* = \overline{T|_{D_1}}^* = \overline{T|_{D_1}} = \overline{T}$. □

1.15 Let $\mathscr{H} = L^2(\mathbb{R}^1)$ and

$$D(T) = \left\{u \in \mathscr{H} : \int_{-\infty}^{\infty} x^2|u(x)|^2dx < \infty\right\}.$$

Define T as $(Tu)(x) = xu(x)$ for $u \in D(T)$. Show that T is unbounded and closed.

Proof. It is clear that $\|T\chi_{[0,n]}\| = \frac{1}{\sqrt{3}}n^{\frac{3}{2}}$ and $\|\chi_{[0,n]}\| = \sqrt{n}$, $\frac{\|T\chi_{[0,n]}\|}{\|\chi_{[0,n]}\|} \to \infty$ as $n \to \infty$, hence T is unbounded.

Suppose that $u_n \to u$ and $xu_n \to v$ in L_2. We know that $u_n \to u$ in measure and Riesz's Theorem enables us to pick a subsequence, still denoted by u_n, which is convergent to u almost everywhere. So $u_n \to u$ in L^2 and pointwise a.e., thus $xu_n \to xu$ a.e. A similar argument shows that there is a subsequence of $\{xu_n\}$, again denoted by $\{xu_n\}$, converges to v pointwise a.e. Therefore it must hold that $xu = v$ a.e., which implies that T is closed. □

1.16 Suppose that T is a densely-defined closed operator on $\mathscr{H}$. Show that for all $a, b \in \mathscr{X}$, the system of equations

$$
\begin{aligned}
-Tx + y &= a\\
x + T^*y &= b
\end{aligned}
$$

has a unique solution $x \in D(T)$ and $y \in D(T^*)$.

Proof. `Existence': Consider the set $S \subseteq \mathscr{H} \times \mathscr{H}$ of all pairs (a, b) which make the system of equations have at least one solution. It is clear that S is a linear set, $V\Gamma(T) \in S$ and $\Gamma(T^*) \in S$. Note that $\Gamma(T^*) = (V\Gamma(T))^{\perp}$. Since $\Gamma(T)$ is closed, we know that $V\Gamma(T)$ is closed and $\Gamma(T^*) + V\Gamma(T) = \mathscr{H}$. Therefore $S = \mathscr{H}$.

`Uniqueness': It suffices to show that

$$-Tx + y = 0$$

$$x + T^*y = 0$$

has solution $x = 0$, $y = 0$ only. A solution satisfies $(y, Tx') = (T^*y, x')$ for all $x' \in D(T)$. In particular $(x' = x)$ we have that $(y, y) = -(x, x)$, it must hold that $(y, y) = (x, x) = 0$ from non-negativity of inner product, and therefore $x = 0$ and $y = 0$. □

2 Cayley Transform and Spectral Decomposition of Self-Adjoint Operators

2.1 Consider the operator $Au = iu'$ on $L^2(\mathbb{R}^1)$. Define $D(A) = \{u \in l^2(\mathbb{R}) : u$ is absolutely continuous and $u' \in L^2(\mathbb{R}^1)\}$. Show that A is self-adjoint.

Proof. It is clear that $C_0^\infty(\mathbb{R})$ is contained in $D(A)$ and thus $D(A)$ is dense.

Suppose that $u \in D(A)$ and $\epsilon > 0$. Since $u' \in L^2$ there exists δ_0 such that $\int_x^{x+\delta} |u'|^2 < \epsilon$ for all x and $\delta < \delta_0$. Let $\delta_1 = \min\{\delta_0, \epsilon\}$. Then for all $\delta < \delta_1$,

$$|u(x+\delta) - u(x)| = \left| \int_x^{x+\delta} u'(t)dt \right| \le \sqrt{\delta_1} \sqrt{\int_x^{x+\delta_1} |u'(t)|^2 dt} \le \sqrt{\epsilon} \cdot \sqrt{\epsilon} = \epsilon.$$

Now we are ready to show that $u(\pm\infty) = 0$. If not, without loss of generality, suppose that there exists $\epsilon_0 > 0$ and $x_n \to +\infty$ such that $|u(x_n)| \ge \epsilon_0$ for all n. We have seen that u is uniformly continuous, so we can find δ such that $|u(x) - u(y)| < \frac{\epsilon_0}{2}$ whenever $|x - y| < \delta$. Therefore, we have that $|u(x)| \ge \frac{\epsilon_0}{2}$ on $(x_n - \delta, x_n + \delta)$ for all n. Without loss of generality, assume that $x_{n+1} - x_n \ge 2\delta$. Then

$$\int_{\mathbb{R}} |u|^2 \ge \sum_{n=1}^\infty \int_{x_n-\delta}^{x_n+\delta} |u|^2 \ge \sum_{n=1}^\infty 2\delta \cdot \frac{\epsilon_0^2}{4} = \infty,$$

which contradicts with $u \in L^2(\mathbb{R})$. Hence $u(\pm\infty) = 0$, then

$$(Au, v) = i \int_{\mathbb{R}} u'\overline{v} = iu\overline{v}\Big|_{-\infty}^{\infty} - i \int u\overline{v'} = -i \int u\overline{v'} = (u, Av).$$

Using the same technique in Problem 6.1.11, we can show that A is closed. It is easy to see that $\ker(A^* + iI) = \{0\}$ as $A \subseteq A^*$ and $\ker(A + iI) = \{0\}$. It follows from Theorem 6.2.4 that A is self-adjoint. □

2.2 Prove Corollary 6.2.5: Let A be a symmetric operator on a Hilbert space, then the following statements are equivalent:

(1) A is essentially self-adjoint;

(2) $\ker(A^* \pm iI) = \{0\}$;

(3) $\overline{R(A \mp iI)} = \mathscr{H}$.

Proof. Theorem 6.2.3 implies that (2) and (3) are equivalent, and a symmetric operator is closable. Now suppose that A is essentially self-adjoint, so $\overline{A}$ is self-adjoint and $\overline{A}^* = A^*$. It follows from Proposition 6.2.1 that $\ker(A^* \pm iI) = \ker(\overline{A}^* \pm iI) = \{0\}$. Conversely, if (2) holds then it holds that $\ker(\overline{A}^* \pm iI) = \{0\}$ and by Theorem 6.2.4 we know that $\overline{A}$ is self-adjoint, which implies that A is essentially self-adjoint. □

2.3 Consider $Au = iu'$ as an operator on $L^2[0, \infty)$ with domain $C_0^\infty[0, +\infty)$. Is A essentially self-adjoint?

Proof. From Problem 1 we know that A is symmetric. It is easy to see that $e^{-x} \in D(A^*)$ and $D^*e^{-x} = -ie^{-x}$ since $(e^{-x}, u') = (ie^{-x}, u)$ for all $u \in C_0^\infty[0, +\infty)$. Therefore $e^{-x} \in \ker(A^* - iI)$ and $\ker(A^* - iI) \ne \{0\}$. Corollary 6.2.5 tells us that A is not essentially self-adjoint. □

2.4 Let A be a densely-defined symmetric operator, A is positive ($(Ax,x) \geq 0\ \forall x \in D(A)$), show that

(1) $\|(A+I)x\|^2 \geq \|x\|^2 + \|Ax\|^2$;

(2) A is a closed operator if and only if $R(A+I)$ is a closed set;

(3) A is essentially self-adjoint if and only if $A^*y = -y$ has solution $y=0$ only.

Proof. (1) Since A is symmetric, we have that $(Ax,x) = (x,Ax)$. Hence $((A+I)x,(A+I)x) = (Ax,Ax) + 2(Ax,x) + (x,x) \geq (Ax,Ax) + (x,x)$.

(2) `Only if': Suppose that A is closed. Let $\{y_n\} \subseteq R(A+I)$ be a Cauchy sequence. Suppose that $y_n = Ax_n + x_n$. From part (1) we know that $\{x_n\}$ and $\{Ax_n\}$ are Cauchy, thus $x_n \to x$ and $Ax_n \to y$ for some x and y. Since A is closed, $x \in D(A)$ and $y = Ax$, thus $y_n \to (A+I)x \in R(A+I)$. Therefore $R(A+I)$ is closed.
`If': Suppose that $x_n \to x$ and $Ax_n \to y$. Then $(A+I)x_n \to x+y \in R(A+I)$, there exists a $z \in D(A)$ such that $Az + z = x + y$. Hence $(A+I)(x_n - z) \to 0$. From part (1) we see that $x_n \to z$, hence $x = z \in D(A)$ and $Az = y$, showing that A is closed.

(3) `Only if': Suppose that A is essentially self-adjoint, then A is closable and $A^* = \overline{A}^* = \overline{A}$. Let $y \in D(\overline{A})$ be a solution of $A^*y = -y$. Then $(\overline{A}+I)x, y) = (x, (A^*+I)y) = 0$ for all $x \in D(\overline{A})$. In particular, let $x = y$, we have $((A+I)y,y) = 0$, i.e., $0 = \|y\|^2 + (Ay,y) \geq \|y\|^2$, it must hold that $y = 0$.
`If': Since T is symmetric and densely-defined, T is closable, thus $\bar{T}^* = T^*$, and $\bar{T} = T^{**} \subseteq (\bar{T})^*$ (because $\overline{T} \subseteq T^*$. Hence $\bar{T}$ is symmetric. It suffices to show that $D(T^*) \subseteq D(\bar{T})$. Let $y \in D(T^*)$ and $x = (T^*+I)y$. For this, we shall first prove that $R(\bar{T}+I)$ is closed. Clearly $\bar{T}$ is positive. Then let $\{y_n\}$ be a Cauchy sequence in $R(\bar{T}+I)$ and suppose that $y_n = (\bar{T}+I)x_n$. Then

$$(y_n, x_n) = ((\bar{T}+I)x_n, x_n) \geq \|x_n\|^2,$$

and note the Cauchy-Schwarz Inequality $(y_n,x_n) \leq \|y_n\|\,\|x_n\|$ it follows that $\|x_n\| \leq \|y_n\|$. Hence $\{x_n\}$ is bounded as $\{y_n\}$ is bounded. Then

$$\|x_n - x_m\|^2 \leq (y_n - y_m, x_n - x_m) \leq (\|x_n\| + \|x_m\|)\|y_n - y_m\|,$$

whence we see that $\{x_n\}$ is Cauchy. Since $\bar{T}$ is closed, we have $x_n \to x$ and $y_n \to (\bar{T}+I)x \in R(\bar{T}+I)$. Note that $\ker(T^*+I) \oplus R(\bar{T}+I) = \mathscr{H}$, it follows from $\ker(T^*+I) = \{0\}$ that $R(\bar{T}+I) = \mathscr{H}$. Thus there exists $y' \in D(\bar{T})$ such that

$$(\bar{T}+I)y' = (\bar{T}^*+I)y' = x = (T^*+I)y.$$

Since T^*+I is injective, it must hold that $y = y' \in D(\bar{T})$, and $D(T^*) \subseteq D(\bar{T})$. □

2.5 Let

$$\mathscr{H} = \left\{ f(z) = \sum_{n=0}^{\infty} c_n z^n, |z| < 1 : \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\},$$

then $\mathscr{H}$ is a Hilbert space under the norm $\|f\| = (\sum |c_n|^2)^{\frac{1}{2}}$. Define operators U and A on $\mathscr{H}$ as

$$(Uf)(z) = zf(z),$$
$$(Af)(z) = i\frac{1+z}{1-z}f(z).$$

Show that A is a symmetric operator on $\mathscr{H}$, U is the Cayley transform of A and find $R(A+iI)$ and $R(A-iI)$.

Proof. Suppose that $f(z) = \sum c_n z^n$, then

$$(Af)(z) = i\sum_{n=0}^{\infty}\left(2\sum_{k=0}^{n-1} c_k + c_n\right) z^n.$$

Since $\mathscr{H}$ is isomorphic to l^2 via $f \leftrightarrow \{c_n\}$, the operator A in this problem corresponds to T in Exercise 6.1.13. We can therefore define $D(A)$ as $D(T)$ in Exercise 6.1.13, and it follows that A is densely-defined and symmetric.

Direct computation shows that

$$(U(A+iI)f)(z) = \left(U\left(\frac{2i}{1-z}f(z)\right)\right)(z) = \frac{2iz}{1-z}f(z)$$
$$((A-iI)f)(z) = \frac{2iz}{1-z}f(z),$$

hence $A - iI = U(A+iI)$. Hence $U = (A-iI)(A+iI)^{-1}$, which is exactly the Cayley transform of A.

It is clear that $R(A+iI)$ consists of polynomials, and $R(A-iI)$ polynomials with a zero constant term. □

2.6 Let C be a symmetric operator on $\mathscr{H}$ and A a linear operator on $\mathscr{H}$. Suppose that $A \subset C$ and $R(A+iI) = R(C+iI)$, show that $A = C$.

Proof. For any $y \in R(C+iI)$ we have $x \in D(C)$ and $z \in D(A)$ such that $(C+iI)z = (A+iI)x = y$. Since $A \subset C$, we have also $(C+iI)x = y$. Note that $C + iI$ is injective (Proposition 6.2.1), it must hold that $z = x \in R(A)$. This implies that $R(C) \subseteq R(A)$ and therefore $A = C$. □

2.7 Let A be a symmetric operator on $\mathscr{H}$, $R(A+iI) = \mathscr{H}$ and $R(A-iI) \neq \mathscr{H}$. Show that A has no self-adjoint extensions.

Proof. Suppose that B is a self-adjoint extension of A, then $B^* \subset A^*$, and $R(B \pm iI) = \mathscr{H}$. It follows from the previous problem that $A = B$, and thus $R(A-iI) = R(B-iI) = \mathscr{H}$. Contradiction. Therefore A cannot have a self-adjoint extension. □

2.8 Let V be an isometry on $\mathscr{H}$: $\|Vx\| = \|x\|$ for all $x \in D(V)$. Show that

(1) $(Vx, Vy) = (x, y)$ for all $x, y \in D(V)$;
(2) If $R(I-V)$ is dense in $\mathscr{H}$ then $I - V$ is injective;
(3) If one of $D(V)$, $R(V)$, $\Gamma(V)$ is closed, so are the other two.

Proof. (1) This is a direct corollary of polarisation identity.

(2) Suppose that $(I-V)y = 0$, i.e., $y = Vy$. From part (1), $(Vx, Vy) = (x, y)$ for all $x \in D(V)$. Replacing Vy by y yields $(Vx - x, y) = 0$ for all $y \in D(V)$. Since $R(I-V)$ is dense, it must hold that $y = 0$, i.e., $\ker(I-V) = \{0\}$.

(3) It follows easily from $\|x\| = \|Vx\|$ that $D(V)$ is closed if and only if $R(V)$ is closed. The graph norm $\|x\|_G = \|x\| + \|Vx\| = 2\|x\|$. Hence $\Gamma(V)$ is closed if and only if $D(V)$ is closed. □

2.9 Let T be a closed operator on Hilbert space $\mathscr{H}$. Show that $\rho(T)$ is open. For $z \in \rho(T)$ define $R_z(T) = (zI - T)^{-1}$, show that $R_z(T)$ is an analytic function with respect to t on each connected component of $\rho(T)$ and satisfies the first resolvent formula:

$$R_{z_1}(T) - R_{z_2}(T) = (z_2 - z_1)R_{z_1}(T)R_{z_2}(T).$$

Proof. See the proof of Corollary 2.6.7, Lemma 2.6.8 and Theorem 2.6.9. □

2.10 Prove Proposition 6.2.16, 6.2.17 and 6.2.18.

Proposition 6.2.16: Let A be a self-adjoint operator and $\{E_\lambda\}$ its spectral family. Then $\lambda_0 \in \sigma_p(A)$ if and only if $E_{\lambda_0} - E_{\lambda_0^-} \neq 0$.

Proof. Note that $\lambda_0 I - A = \int_{\mathbb{R}} (\lambda_0 - \lambda) dE_\lambda$ and

$$\|(\lambda_0 I - A)x\|^2 = \int_{\mathbb{R}} (\lambda_0 - \lambda)^2 d\|E_\lambda x\|^2, \quad x \in D(A).$$

Thus by $E_{-\infty} = 0$ and the right continuity of $\|E_\lambda x\|^2$ in λ, we see that $\lambda_0 x = Ax$ iff

$$E_\lambda x = E_{\lambda_0^+} x = E_\lambda x \quad \forall \lambda \geq \lambda_0$$
$$E_\lambda x = E_{\lambda_0^-} x = 0 \quad \forall \lambda < \lambda_0,$$

that is, $\lambda_0 x = Ax$ iff $(E_{\lambda_0} - E_{\lambda_0^-})x = x$. □

Proposition 6.2.17: Let A be a self-adjoint operator then $\sigma_r(A) = \emptyset$.

Proof. Suppose $\lambda \in \sigma_r(A)$ then λ is real. Since $\overline{R(\lambda I - A)} \neq \mathscr{H}$, there exists $y \neq 0$ such that $y \perp \overline{(\lambda I - A)}$, i.e., $((\lambda I - A)x, y) = 0$ for all $x \in D(A)$. Hence $(Ax, y) = (\lambda x, y) = (x, \lambda y)$ and $y \in D(A^*) = D(A)$ as A is self-adjoint, and $D^* y = \lambda y$. Since $D^* = D$, we find that $y \in \sigma_p(A)$ and thus meet a contradiction. □

Proposition 6.2.18: Let A be a self-adjoint operator with spectral family $\{E_\lambda\}$, then $\lambda_0 \in \sigma(A)$ if and only if for all $\epsilon > 0$ it holds that $E(\lambda_0 - \epsilon, \lambda_0 + \epsilon) \neq 0$.

Proof. From the previous problem we see that $\rho(A)$ is open, and thus $\sigma(A)$ is closed. The rest of the proof is exactly the same as the proof of Theorem 5.5.19. □

2.11 Prove Proposition 6.2.20: Let A be a self-adjoint operator with spectral family $\{E_\lambda\}$, then $\lambda_0 \in \sigma_{\text{ess}}(A)$ if and only if, $\forall \epsilon > 0$, $\dim R(E(\lambda - \epsilon, \lambda + \epsilon)) = \infty$.

Proof. `Only if': Let $\lambda_0 \in \sigma_{\text{ess}}(A)$ but $\dim R(E(\lambda - \epsilon, \lambda + \epsilon)) < \infty$ for some ϵ. Since $\lambda_0 \in \sigma(A)$, the argument in the proof of Theorem 5.5.21 gives that λ_0 is an isolated point of $\sigma(A)$ and thus belongs to $\sigma_p(A)$ (use Proposition 6.2.16 and 6.2.18), however, $\ker(\lambda_0 I - A) = \dim R(E(\{\lambda_0\})) \leq \dim R(E(\lambda - \epsilon, \lambda + \epsilon)) < \infty$, contradiction with the assumption that $\lambda_0 \in \sigma_{\text{ess}}(A)$.

`If': See the proof of Theorem 5.5.21. □

3 Spectral Transform of Unbounded Normal Operators

3.1 Suppose that N be a normal operator, show that N^* is a normal operator also.

Proof. Theorem 6.1.4 tells us that $N = \overline{N} = N^{**}$, then $N^{**} N^* = NN^* = N^* N = N^* N^{**}$. From the same theorem we know that N^* is densely defined, and $\Gamma(N^*) = (V\Gamma(N))^\perp$ is closed, which implies that N^* is closed. Therefore N^* is normal. □

3.2 Suppose that T is a densely-defined closed operator, $D(T) = D(T^*)$, $\|Tx\| = \|T^* x\|$ for all $x \in D(T)$. Show that T is normal.

Proof. From $D(T) = D(T^*)$ it is easy to see that $D(T^*T) = D(TT^*)$. Since $\|Tx\| = \|T^*x\|$ for all $x \in D(T)$, it follows from polarisation identity that $\langle Tx, Ty \rangle = \langle T^*x, T^*y \rangle$ for all $x, y \in D(T)$. Then for $x \in D(T^*T)$ and $y \in D(T)$, it is immediate that $(T^*Tx, y) = (TT^*x, y)$. Since $D(T)$ is dense in $\mathscr{H}$, we must have that $T^*Tx = TT^*x$ for all $x \in D(T^*T)$, which, together with $D(T^*T) = D(TT^*)$, implies that $TT^* = T^*T$ and T is normal. □

3.3 Let $L \in L(\mathscr{H})$ and M, N unbounded normal operator on $\mathscr{H}$. Suppose that $LM \subset NL$, show that $LM^* \subset N^*L$.

Proof. First consider the case where $M = N$. Let E be the spectral decomposition of M. Then $E(\Delta)L = LE(\Delta)$ for every Borel set Δ (Theorem 6.3.11). It follows that

$$(LM^*x, y) = (M^*x, L^*y) = \int \bar{z} d(E(z)x, L^*y) = \int \bar{z} d(LE(z)x, y) = \int \bar{z} d(E(z)Lx, y) = (M^*Lx, y)$$

for all $x \in D(M^*) = D(M)$ and $y \in \mathscr{H}$. This implies that $LM^* \subseteq M^*L$.

Now we consider the general case. Define $\hat{M}$ on $D(M) \times D(N) \subseteq \mathscr{H} \times \mathscr{H}$ as $\hat{M}(x, y) = (Mx, Ny)$. It is clear that $\hat{M}$ is normal. Also define $\hat{L}$ on $\mathscr{H} \times \mathscr{H}$ as $\hat{L}(x, y) = (Ly, 0)$, which is bounded. Then it is easy to verify that $\hat{L}\hat{M} \subset \hat{M}\hat{L}$. Applying the previous case where $M = N$, we obtain that $\hat{L}\hat{M}^* \subset \hat{M}^*\hat{L}$, that is, $LM^* \subset N^*L$. □

3.4 Show that a densely-defined closed operator N on $\mathscr{H}$ is an unbounded normal operator if and only if the following conditions hold simultaneously:

(1) $D(N) = D(N^*)$;

(2) $\overline{N + N^*}$, $\overline{i(N - N^*)}$ are self-adjoint, and their spectral families are commutative.

3.5 Let N be a densely-defined closed operator on $\mathscr{H}$. Show that N is normal if and only if there exist decomposition of the form $N = A + iB$, A, B are self-adjoint, and their spectral families are commutative.

Proof. `Only if': Suppose that N is normal. Let $A = \frac{N+N^*}{2}$ and $B = i\frac{N^*-N}{2}$. Note that $D(N) = D(N^*)$, it follows easily that A, B are self-adjoint and $AB = BA$. □

3.6 Prove that every normal operator N in $\mathscr{H}$ has a polar decomposition

$$N = UP = PU,$$

where U is unitary, P self-adjoint, $P \geq 0$, and $D(P) = D(N)$.

Proof. Put $p(z) = |z|$ and $u(z) = z/|z|$ if $z \neq 0$, $u(0) = 1$. Then p and u are Borel functions on $\sigma(N)$, $D_{p(z)} = D_z = D(N)$ and $D_{u(z)} = \mathscr{H}$. Put $P = \Phi p$ and $U = \Phi u$. Since $p \geq 0$, we know that $P \geq 0$. Since $u\bar{u} = 1$, $QQ^* = Q^*Q = I$. Since $z = p(z)u(z)$, the relation $N = PU = UP$ would follow immediately from the symbolic calculus. □

3.7 Suppose that N is an unbounded normal operator and $(\mathbb{C}, \mathscr{B}, E)$ is its spectral family. Show that

(1) $z \in \sigma_p(N) \Leftrightarrow E(\{z\}) \neq 0$;

(2) $\sigma_r(N) = \emptyset$;

(3) $z \in \sigma(N) \Leftrightarrow \forall$ Borel set $\Delta, z \in \Delta,$ it holds that $E(\Delta) \neq 0$.

Proof. With the spectral theorem, the proof is almost identical to the case of bounded normal operator. See Problem 2.10, Theorem 5.5.18 and 5.5.19. □

3.8 Suppose that N is an unbounded normal operator and E is its spectral family. Let

$$\sigma_{\text{ess}}(N) = \{z \in \sigma(N) : z \text{ has a Borel neighbourhood } \Delta \text{ such that } \dim R(E(\Delta)) = +\infty.\},$$
$$\sigma_d(N) = \sigma(N) \setminus \sigma_{\text{ess}}(N),$$

show that $z \in \sigma_d(N)$ if and only if z is a finite isolated eigenvalue, $z \in \sigma_{\text{ess}}(N)$ if and only if z is a limit point of $\sigma(N)$ or an infinite eigenvalue.

Proof. See Theorem 5.5.21. □

3.9 Suppose that $\mathscr{H}$ is a Hilbert space, $(\mathbb{C}, \mathscr{B}, E)$ a spectral family and f, g Borel-measurable functions. Show that $\Phi(f)\Phi(g) = \Phi(fg)$ if and only if $D_{fg} \subset D_g$, where $\Phi(f)$ and D_f are defined in (6.3.11) and (6.3.8) respectively.

Proof. Theorem 6.3.4 says that $\Phi(f)\Phi(g) \subset \Phi(fg)$ and $D(\Phi(f)\Phi(g)) = D_g \cap D_{fg}$.

`Only if': Suppose that $\Phi(f)\Phi(g) = \Phi(fg)$, then $D(\Phi(f)\Phi(g)) = D(\Phi(fg))$, that is, $D_g \cap D_{fg} = D_{fg}$, hence $D_{fg} \subseteq D_g$.

`If': Suppose that $D_{fg} \subset D_g$, then $D(\Phi(f)\Phi(g)) = D_{fg} = D(\Phi(fg))$, and thus $\Phi(f)\Phi(g) = \Phi(fg)$. □

3.10 Let $\mathscr{H}$ be a Hilbert space, $(\mathbb{C}, \mathscr{B}, E)$ an arbitrary spectral family and f a bounded Borel-measurable function. Show that under the operator norm, the integral

$$\int_{\mathbb{C}} f(z) dE(z)$$

is convergent in the sense of Lebesgue integral, and

$$\Phi(f) = \int_{\mathbb{C}} f(z) dE(z),$$

where $\Phi(f)$ is defined as in (6.3.1).

Proof. See the remark following Theorem 5.5.14. □

3.11 Let $\mathscr{H}$ be a Hilbert space, $(\mathbb{C}, \mathscr{B}, E)$ an arbitrary spectral family and f a Borel-measurable function. Define $\Delta_n = \{z : |f(z)| \leq n\}$, $f_n(z) = \chi_{\Delta_n}(z) f(z)$, show that

$$\Phi(f) = s - \lim \Phi(f_n),$$

where $\Phi(f)$ is defined as in (6.3.11).

Proof. Since f_n is bounded, it holds that $D_f = D_{f-f_n}$. For each $x \in D_f$, it follows from Dominated Convergence Theorem that

$$\|\Phi(f)x - \Phi(f_n)x\| \leq \int_{\mathbb{C}} |f - f_n|^2 d\|E(z)x\|^2 \to 0$$

as $n \to \infty$. □

4 Extension of Self-Adjoint Operators

4.1 Let A_n be a symmetric operator on a Hilbert space $\mathscr{H}_n$ for $n = 1, 2, \ldots$. Define

$$D = \left\{ u = (u_1, u_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathscr{H}_n : u_n \in D(A_n), \text{ only finitely many } u_n\text{'s are non-zeroes} \right\}.$$

Show that

(1) $A = \sum_{n=1}^{\infty} A_n$ is symmetric on D;
(2) $n_{\pm}(A) = \sum_{n=1}^{\infty} n_{\pm}(A_n)$.

Proof. (1) It is not difficult to see that D is dense and $A = \sum_{n=1}^{\infty} A_n$ is linear. It is straightforward to verify that $(Ax, y) = (x, Ay)$ for $x, y \in D$, thus A is symmetric.

(2) We only show that $n_+(A) = \sum_{n=1}^\infty n_+(A_n)$ ($n_-(A)$ can be proved similarly), for which it suffices to show that

$$\ker(A^* - iI) = \bigoplus_{n=1}^\infty \ker(A_n^* - iI).$$

The left-hand side is $R(A+iI)^\perp$. Suppose that $v = (v_1, v_2, \dots) \in R(A+iI)$, then $\sum((A_n+iI)u_n, v_n) = 0$ for all $(u_1, u_2, \dots) \in D$, which reduces to $((A_n + iI)u_n, v_n) = 0$ for all n and $u_n \in D(A_n)$. This implies that $v_n \in R(A_n + iI)^\perp = \ker(A_n^* - iI)$, giving $\ker(A^* - iI) \subseteq \sum_{n=1}^\infty \ker(A_n^* - iI)$.

Conversely, suppose that $v_n \in \ker(A_n^* - iI) = R(A_n + iI)^\perp$, i.e., $((A_n+iI)u_n, v_n) = 0$ for all $u_n \in D(A_n)$, then $\sum((A_n + iI)u_n, v_n) = 0$ for all $(u_1, u_2, \dots) \in D$, indicating that $(v_1, v_2, \dots) \in R(A + iI)^\perp = \ker(A^* - iI)$. Hence $\sum_{n=1}^\infty \ker(A_n^* - iI) \subseteq \ker(A^* - iI)$.

Finally consider the decomposition of 0. Suppose that $(A + iI)(u_1, u_2, \dots) = 0$, i.e., $(A_1u_1 + iu_1, A_2u_2 + iu_2, \dots) = 0$, which implies that $(A_n + iI)u_n = 0$ for all n. Since A_n is symmetric, it must hold that $u_n = 0$. Hence the sum is a direct sum. ☐

4.2 Define $T_1 = i\frac{d}{dx}$ with domain $C_0^\infty[0,\infty)$ in $L^2[0,\infty)$ and $T_2 = i\frac{d}{dx}$ with domain $C_0^\infty(-\infty, 0]$ in $L^2(-\infty, 0]$. Show that $\operatorname{def}(T_1) = (0, 1)$ and $\operatorname{def}(T_2) = (1, 0)$. Show how to construct a symmetric operator with any given pair of deficiency indices.

Proof. Integration by parts shows that T_1 is symmetric. The range of $T_1 - iI$ contains all functions f of form

$$i\frac{d}{dx}u - iu = f, \quad u \in C_0^\infty[0,\infty).$$

Hence $f \in C_0^\infty[0,\infty)$. Multiply by e^{-x},

$$i\frac{d}{dx}(e^{-x}u) = e^{-x}f.$$

Since u has compact support, we obtain that

$$\int_0^\infty e^{-x} f = 0 \tag{1}$$

Conversely, every C_0^∞ function f satisfying the condition above belongs to the range of $T_1 - iI$ as we can define u by

$$u(x) = -i\int_0^x e^{-(y-x)} f(y)dy.$$

It is clear that $u \in C_0^\infty[0,\infty)$. Therefore $f \in C_0^\infty[0,\infty)$ is contained in $R(T_1 - iI)$ if and only if f satisfies (1). Note that $e^{-x} \in L^2[0,\infty)$, it follows that $R(T_1 - iI)^\perp$ is a one-dimensional subspace spanned by e^{-x}, and $n_-(T_1) = 1$.

Now consider the range of $T_1 + iI$. Similarly we conclude that $f \in C_0^\infty[0,\infty)$ is contained in $R(T_1 + iI)$ if and only if

$$\int_0^\infty e^x f = 0$$

Since $e^x \notin L^2[0,\infty)$, f satisfies the equation above is dense in $C_0^\infty[0,\infty)$. Therefore $R(T_1 + iI)$ is dense and thus $n_+(T_1) = 0$.

A similar argument shows that $\operatorname{def}(T_2) = (1, 0)$. Now combining with Problem 1, we see that on

$$D = \left\{ u \in \bigoplus^p L^2[0,\infty) \oplus \bigoplus^q L^2(-\infty, 0] : u_i \in C_0^\infty[0,\infty) \text{ for } 1 \le i \le p \text{ and } u_i \in C_0^\infty(-\infty, 0] \text{ for } p+1 \le i \le p+q \right\}$$

the operator $\sum^{p+q} i\dfrac{d}{dx}$ has deficiency indices (p, q). ☐

4.3 Suppose that $p(x)$ is a polynomial with real coefficients. Let $A = p(i\frac{d}{dx})$ with domain $C_0^\infty[0,\infty)$ in $L^2[0,\infty)$. Show that

(1) A is symmetric;

(2) if p has no odd powers, then the deficiency indices of A are equal;

(3) if the degree of p is odd, then the deficiency indices of A are unequal.

Proof. (1) Straightforward integration by parts.

(2) If p has no odd-degree terms, then $\overline{(A+iI)u} = (A-iI)\overline{u}$, which implies that $R(A+iI)$ is isomorphic to $R(A-iI)$. The conclusion follows easily.

(3) The approach is similar to that in Problem 4.2.
The range of $A-iI$ contains all functions f of form $Au-iu=f$, $u \in C_0^\infty[0,\infty)$. From ODE Theory, we conclude that f is contained in the range of $A-iI$ if and only if $\int_0^\infty fg = 0$ for all g that are solutions to $(A+iI)g = 0$, where we formally extend the domain of A to $C^\infty[0,\infty) \cap L^2[0,\infty)$. The deficiency index concerns only those g that are contained in L^2, hence we are only concerned with $\int_0^\infty x^k e^{zx} f(x)dx = 0$, where z is the root of $p(iz)+i=0$ with $\Re z < 0$. In fact, $n_+(A)$ is the number of the roots of $p(iz)+i=0$ lying in $\Re z < 0$. Similarly, $n_-(A)$ is the number of the roots of $p(iz)-i=0$ lying in $\Re z > 0$. Note that $p(ix) \pm i = 0$ has no pure imaginary roots, and $z \leftrightarrow -\bar{z}$ is a bijection between the roots of the two equations. We conclude that $n_+ + n_- = \deg g$, which is odd, therefore n_+ and n_- can never be equal. □

4.4 Let M and N be two subspaces of $\mathscr{H}$ and $\dim M > \dim N$. Show that there exists $u \in M$, $\|u\| = 1$, such that $u \in N^\perp$.

Proof. By considering a subspace of M, if necessary, we can assume that both M and N are finite-dimensional. Take orthonormal basis $\{x_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^n$, $m > n$, for M and N, respectively. Consider $x = \sum a_i x_i \in M$. We want $(x, y_j) = \sum_j a_i(x_i, y_j) = 0$ for all $1 \le j \le n$. This is a system of linear equations that can be rewritten as $Ax = 0$, where $A_{ij} = (x_i, y_j)$. Note that A has more rows (m rows) than columns (n columns), the linear system has a non-zero solution. □

4.5 Let A be a closed symmetric operator. Show that $\sigma(A)$ must be one of the four cases:

(1) the closed upper half plane;

(2) the closed lower half plane;

(3) the entire plane;

(4) a subset of the real axis.

Proof. Suppose that $z_0 \in \rho(A)$. First suppose that $\operatorname{im} z_0 < 0$, then $\dim\ker(A^* + zI) = n_- = \dim\ker(A^* + z_0 I)$ for all $\operatorname{im} z < 0$. Since $A - z_0 I$ is invertible, $R(A - z_0 I) = \mathscr{H}$ and $n_- = 0$. Hence $\ker(A^* + zI) = \{0\}$ for all $\operatorname{im} z < 0$, that is, $R(A - zI) = \mathscr{H}$ for all $\operatorname{im} z < 0$ (because $R(A - zI)$ is closed when A is closed and symmetric). Note also symmetry of A implies that $A - zI$ is injective. Hence $A - zI$ is bijective for $\operatorname{im} z < 0$, and $z \in \rho(A)$. Similarly, if $\operatorname{im} z_0 > 0$ then the entire open half-plane is contained in $\rho(A)$. □

4.6 Let A be a closed symmetric operator. If $\rho(A)$ contains a real number then A is self-adjoint.

Proof. Since $\rho(A)$ contains a real number, the spectrum $\sigma(A)$ must be in case (4), that is, $\sigma(A) \subset \mathbb{R}$. Then $\operatorname{def}(A) = (0,0)$ and it follows from von Neumann Theorem that A is self-adjoint. (See also Theorem 6.4.5) □

4.7 Let A be a symmetric operator. If A_1 is a symmetric extension of A, then $A_1 \subset A^*$. Define a sesquilinear form on $D(A^*)$ as

$$\{x, y\} = (A^*x, y) - (x, A^*y).$$

Show that $\{x, y\} = 0$ for all $x, y \in D(A_1)$.

Proof. $A \subset A_1 \Rightarrow A_1^* \subset A^*$. Also A_1 is symmetric, $A_1 \subset A_1^*$ and $\{x, y\} = 0$. ☐

4.8 Suppose that A is a symmetric operator and D a linear subspace such that $D(A) \subset D \subset D(A^*)$ and $\{x, y\} = 0$ on $D \times D$. Show that there exists a symmetric extension, denoted A_1, of A such that $D(A_1) = D$.

Proof. Let $A_1 = A^*|_D$, then it is symmetric because $\{x, y\} = 0$ on $D \times D$. Also, $A \subset A^*$ and $D(A) \subset A$, we see that $A \subset A_1$. ☐

4.9 Let A be a symmetric operator. Define an inner product on $D(A^*)$ as

$$(x, y)_A = (x, y) + (A^*x, A^*y),$$

then $D(A^*)$ with $(\cdot, \cdot)_A$ forms a Hilbert space. Show that

(1) The sesquilinear form defined in Problem 6.4.7 is continuous under the topology induced by $(\cdot, \cdot)_A$;

(2) Suppose that A_1 is a restriction of A. Show that A_1 is a closed operator if and only if $D(A_1)$ is closed under the topology induced by $(\cdot, \cdot)_A$.

Proof. (1) Suppose that $x_n \to x$ and $y_n \to y$ under $\|\cdot\|_A$, then $x_n \to x$, $y_n \to y$, $A^*x_n \to A^*x$, $A^*y_n \to A^*y$ (because A^* is closed -- the dual of any densely-defined operator is closed) under the usual norm. It follows that

$$\{x_n, y_n\} = (A^*x_n, y_n) - (x_n, A^*y_n) \to (A^*x, y) - (x, A^*y) = \{x, y\},$$

where we use the fact that the usual inner product is continuous w.r.t. the usual norm.

(2) Note that the graph norm of A_1 coincides with $(\cdot, \cdot)_A$. ☐

4.10 Let A be a symmetric operator and view $D(A^*)$ as a Hilbert space with inner product $(\cdot, \cdot)_A$. Let S be a subset of $D(A^*)$. We say S is symmetric if $\{x, y\} = 0$ on $S \times S$. Show that there is a one-to-one correspondence between the closed symmetric subspaces of $D(A^*)$ that contain $D(A)$ and all the closed symmetric subspaces of $D_+ \oplus D_-$, where $D_+ = \ker(A^* - iI)$ and $D_- = \ker(A^* + iI)$. Moreover, if $D \supset D(A)$ is closed and symmetric and corresponds to $\tilde{D}$, a closed and symmetric subspace of $D_+ \oplus D_-$, then $D = D(\bar{A}) \oplus \tilde{D}$.

Proof. First it is clear that A is closable, and $\bar{A}^* = A^*$. Observe that any closed subspace of $D(A^*)$ that contains $D(A)$ also contains $D(\bar{A})$, we may assume that A is closed.

Suppose $D \supset D(A)$ is a closed subspace of $D(A^*)$. Note that $D(A^*) = D(A) \oplus D_+ \oplus D_-$, for any $x \in D$ we can write $x = x_A + x_+ + x_-$ in a unique way. Let $\tilde{D}$ be spanned by those x_+'s and x_-'s. We claim that $\tilde{D}$ is a closed symmetric subspace of $D_+ \oplus D_-$. The closedness of $\tilde{D}$ follows from the closedness of D and $D(A)$. We show that $\tilde{D}$ is symmetric, i.e. (after some algebra), $(x_+, y_+) = (x_-, y_-)$ for all $x, y \in \tilde{D}$. This is not hard to obtain from the symmetry of D, $A^*x = Ax + ix_+ - ix_-$ together with the assumption that A is symmetric. It is clear that $D = D(\bar{A}) \oplus \tilde{D}$ from the construction of $\tilde{D}$, which implies that $D \leftrightarrow \tilde{D}$ is a one-to-one correspondence. ☐

4.11 Suppose that A is a symmetric operator, A^2 is densely-defined, show that $A^*\bar{A}$ is a Friedrichs self-adjoint extension of A^2.

Proof. Without loss of generality, assume that A is closed. It is clear that A^2 is symmetric. Define $a(u, v) = (A^2u, v) + (u, v)$, then $a(u, v)$ is a positive-definite sesquilinear form on $D(A^2) \subseteq D(A)$. Consider the completion of $D(A^2)$ with respect to a, denoted by D. Note that $a(u, u) = \|Au\|^2 + \|u\|^2$ and $D(A)$ is closed under this norm (equivalent to the graph norm), the completion of $D(A^2)$, denoted by D, is the intersection of all subspaces of $D(A)$ that are closed under the graph norm. We shall show that $D = D(Q)$, where $D(Q)$ is defined in Corollary 6.4.21. Then it follows from the uniqueness of the extension (Theorem 6.4.20) that A^*A is the self-adjoint extension of A^2 (Theorem 6.4.21).

Obviously $D \subseteq D(Q)$, thus it suffices to show that $D(Q) \subseteq D$. This is because $D(Q)$ is closed and is dense in $D(A)$. ☐

4.12 Suppose that A is a lower semi-bounded closed symmetric operator, $A \geq -M$. Then $\dim\ker(A^* - zI)$ is a constant on $\mathbb{C} \setminus [-M, \infty)$.

Proof. The proof is the same as that of Theorem 6.4.4. To connect the upper and lower half-planes, notice that the proof is valid for real $z \in (-\infty, -M)$. In fact, suppose that $u \in D(A)$, $(A - zI)u = x$,

$$(x, u) = ((A - zI)u, u) \geq (-M - z)\|u\|^2,$$

implying that

$$\|x\| \geq \sqrt{(-M - z)}\|u\|.$$

□

4.13 Let A be a closed symmetric operator that is semi-bounded from below. Suppose that $n_+(A) = n_-(A) < \infty$, show that any self-adjoint extension of A is semi-bounded from below.

Proof. Suppose that A_1 is a self-adjoint extension of A. From Problem 4.10, we know that $D(A_1) = D(A) \oplus S$, where S is a finite-dimensional linear space. Suppose that M is the lower bound of A and pick $K < M$. Then $\dim P_{(-\infty,K]} \leq \dim S$, where P_Ω is the projection-valued measure of A_1. Otherwise, we can find $x \in D(A) \cap R(P_{(-\infty,K]})$, so that

$$(Ax, x) = \int_{\mathbb{R}} z d\|E(z)x\|^2 \leq K\|E(K)x\|^2 \leq M\|x\|^2,$$

contradicting with $A \geq M$. We have established that $\dim P_{(-\infty,K]} < \infty$, this implies that $\sigma(A_1)$ has only finitely many elements in $(-\infty, K]$, and they are eigenvalues. Therefore, A_1 is bounded below. □

4.14 Suppose that T is a densely-defined closed operator in a Hilbert space. Show that there exist a positive self-adjoint operator A with $D(A) = D(T)$ and an isometry $V : (\ker T)^\perp \to \overline{R(T)}$ such that

$$T = VA.$$

This is called polar decomposition of closed operator.

Proof. Since T is densely-defined and closed, we have that T^*T is positive self-adjoint. Let $A = (T^*T)^{\frac{1}{2}}$. For $x \in D(T^*T)$ we clearly have $\|Tx\|^2 = (T^*Tx, x) = (A^2x, x) = \|Ax\|^2$. Since $D(T^*T)$ is dense in $D(T)$, we can extend A to $D(T)$ by continuity such that $\|Tx\| = \|Ax\|$ for all $x \in D(T)$.

Define $V : R(A) \to R(T)$ such that $VAx = Tx$, it is clear that V is well-defined and norm preserving. Thus V extends to an isometry from $\overline{R(A)}$ to $\overline{R(T)}$ by continuity. Since A is self-adjoint, $\overline{R(A)} = (\ker A)^\perp = (\ker T)^\perp$.

Suppose that $T = V'A'$ is another decomposition. Then $T^*T = A'^*V'^*VA' = A'^*A' = A'^2$, thus $A = A'$ on $D(T^*T)$ because $\sqrt{T^*T}$ is unique. It follows immediately that $A = A'$ on $D(T)$ and $V' = V$. □

4.15 Let A be a symmetric operator in a Hilbert space. Show that A is essentially self-adjoint if and only if $\dim\ker(A^* \mp iI) \triangleq n_\pm = 0$.

Proof. This is Corollary 6.2.5 (Exercise 6.2.2). □

4.16 Denote the Schwartz space by $\mathscr{S}(\mathbb{R}^3)$. Let $K_1(\mathbb{R}^3)$ be the closure of $\mathscr{S}(\mathbb{R}^3)$ under the norm of $\int_{\mathbb{R}^3} |\nabla u|^2 dx$. Let $\mathscr{H} = K_1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and define an inner product in $\mathscr{H}$ as

$$(\langle f_1, f_2\rangle, \langle g_1, g_2\rangle) = \int_{\mathbb{R}^3} (\nabla f_1 \cdot \overline{\nabla g_1} + f_2\overline{g_2}) dx.$$

Consider the following operator in $\mathscr{H}$:

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad D(A) = \mathscr{S}(\mathbb{R}^3) \times \mathscr{S}(\mathbb{R}^3).$$

Show that

(1) iA is symmetric;

(2) iA is essentially self-adjoint.

Proof. (1) For $\langle f_1, f_2\rangle, \langle g_1, g_2\rangle \in D(A)$, it holds that

$$\begin{aligned}(iA\langle f_1, f_2\rangle, \langle g_1, g_2\rangle) &= (i\langle f_2, \Delta f_1\rangle, \langle g_1, g_2\rangle)\\ &= i\int_{\mathbb{R}^3}(\nabla f_2\cdot\overline{\nabla g_1} + \Delta f_1\cdot\overline{g_2})dx\\ &= -i\int_{\mathbb{R}^3}(f_2\cdot\overline{\Delta g_1} + \nabla f_1\cdot\overline{\nabla g_2})dx\\ &= (\langle f_1, f_2\rangle, i\langle g_2, \Delta g_1\rangle)\\ &= (\langle f_1, f_2\rangle, iA\langle g_1, g_2\rangle).\end{aligned}$$

(2) We shall show that $R(A\pm iI)$ is dense in $\mathscr{H}$. We first show that $R(A+iI)$ is dense. Note that

$$(A+iI)\langle f_1, f_2\rangle = i\langle f_2 + f_1, \Delta f_1 + f_2\rangle,$$

it suffices to show that the system of equations

$$\begin{aligned}v + u &= f\\ \Delta u + v &= g\end{aligned}$$

has solution $u, v\in\mathscr{S}(\mathbb{R}^3)$ if $f, g\in\mathscr{S}(\mathbb{R}^3)$, which can be easily reduced to show that

$$\Delta u - u = h$$

has solution $u\in\mathscr{S}(\mathbb{R}^3)$ if $h\in\mathscr{S}(\mathbb{R}^3)$. Take Fourier transform on both sides,

$$-4\pi^2|\xi|^2\hat{u} - \hat{u} = \hat{h}.$$

Solve for $\hat{u}$,

$$\hat{u} = -\frac{\hat{h}}{1+4\pi^2|\xi|^2},$$

which is clearly in $\mathscr{S}(\mathbb{R}^3)$. Hence by taking inverse Fourier transform we obtain a solution $u\in\mathscr{S}(\mathbb{R}^3)$. Similarly, to show that $R(A-iI)$ is dense, it suffices to show that

$$\begin{aligned}v - u &= f\\ \Delta u - v &= g\end{aligned}$$

has solution $u, v\in\mathscr{S}(\mathbb{R}^3)$ if $f, g\in\mathscr{S}(\mathbb{R}^3)$, which reduced to the same problem as above. □

5 Perturbation of Self-Adjoint Operators

5.1 Let A be self-adjoint and B be symmetric. Suppose that B is A-bounded with relative bound equal to a. Prove that

$$a = \lim_{n\to\infty}\|B(A+in)^{-1}\|.$$

Proof. Note that $\|(A+in)u\|^2 = \|Au\|^2 + n^2\|u\|^2$ for all $u\in D(A)$. Since A is self-adjoint, $A+in$ is invertible and $R(A+in) = \mathscr{H}$. Replace u by $(A+in)^{-1}x$,

$$\|x\|^2 = \|A(A+in)^{-1}x\|^2 + n^2\|(A+in)^{-1}x\|^2. \tag{2}$$

Suppose that $\|Bu\|^2 \leq a'^2\|Au\| + b'^2\|u\|^2$ for all $u \in D(A)$. Replace u by $(A+in)^{-1}x$ and use (2),

$$\begin{aligned}\|B(A+in)^{-1}x\|^2 &\leq a'^2\|A(A+in)^{-1}x\| + b'^2\|(A+in)^{-1}x\|^2 \\ &\leq a'^2(\|x\|^2 - n^2\|(A+in)^{-1}x\|^2) + b'^2\|(A+in)^{-1}x\|^2 \\ &\leq a'^2\|x\|^2\end{aligned}$$

when n is large enough. This implies that $a' \geq \overline{\lim}\,\|B(A+in)^{-1}\|$ and thus $a \geq \overline{\lim}\,\|B(A+in)^{-1}\|$. The conclusion follows easily if $a = 0$, so we assume $a > 0$ henceforth.

On the other hand, By the definition of relative bound, we know that for any $\epsilon > 0$ small enough, $b > 0$, there exists $u \in D(A)$ such that

$$\|Bu\|^2 > (a-\epsilon)^2\|Au\|^2 + b^2\|u\|^2.$$

Use the same technique as before,

$$\|B(A+in)^{-1}x\|^2 > (a-\epsilon)^2\|x\|^2 + (b^2 - (a-\epsilon)^2n^2)\|(A+in)^{-1}x\|^2$$

Choose $b = (a-\epsilon)n$, we know that for any $\epsilon > 0$ there exists x such that

$$\|B(A+in)^{-1}x\|^2 > (a-\epsilon)^2\|x\|^2$$

which implies that $\|B(A+in)^{-1}\| \geq a - \epsilon$. This result holds for all n, thus $\underline{\lim}\,\|B(A+in)^{-1}\| \geq a-\epsilon$, and let $\epsilon \to 0$, $a \leq \underline{\lim}\,\|B(A+in)^{-1}\|$, whence the conclusion follows. □

5.2 Let A be a densely defined closed operator and B a closable operator. If $D(A) \subset D(B)$, show that B is A-bounded.

Proof. Since A is closed, $X = (D(A), \|\cdot\|_{\Gamma(A)})$ is a Banach space. Without loss of generality, we may assume that B is closed. To show that B is A-bounded, i.e., B is continuous on X, it suffices to show that $B|_X$ is a closed operator then the Closed Graph Theorem applies. In fact, suppose that $x_n \to x$ in X and $Bx_n \to y$. Then $x_n \to x$ in $\mathscr{H}$. Since B is closed, we must have $Bx = y$, which shows that $B|_X$ is closed. □

5.3 Suppose that A and B are densely-defined operators in $\mathscr{H}$, B is A-bounded, then there exist $a, b \geq 0$ such that

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \quad \forall x \in D(A).$$

Show that

(1) B is $(A+B)$-bounded and the relative bound is at most $\frac{a}{1-a}$;
(2) if C is A-bounded with relative bound c, then C is $(A+B)$-bounded with relative bound at most $\frac{c}{1-a}$.

Proof. (1) Note that

$$\|(A+B)x\| \geq \|Ax\| - \|Bx\| \geq \|Ax\| - (a\|Ax\| + b\|x\|) = (1-a)\|Ax\| - b\|x\|$$

Then

$$\|Ax\| \leq \frac{\|(A+B)x\| + b\|x\|}{1-a} \tag{3}$$

and

$$\|Bx\| \leq a\|Ax\| + b\|x\| \leq a\frac{\|(A+B)x\| + b\|x\|}{1-a} + b\|x\| = \frac{a}{1-a}\|(A+B)x\| + \frac{b(1+a)}{1-a}\|x\|.$$

(2) For any $\epsilon > 0$ there exists $d \geq 0$ such that

$$\|Cx\| \leq (c+\epsilon)\|Ax\| + d\|x\| \leq \frac{c+\epsilon}{1-a}\|(A+B)x\| + \left(\frac{c+\epsilon}{1-a} + d\right)\|x\|,$$

thus C is $(A+B)$-bounded with relative bound at most $\frac{c+\epsilon}{1-a}$. Let $\epsilon \to 0$, completing the proof. □

5.4 Let $\mathscr{H}$ be a Hilbert space. Suppose that A is a densely defined closed operator and B is A-bounded such that

$$\|Bx\| \le a\|Ax\| + b\|x\|.$$

Let $\lambda \in \rho(A)$ such that

$$a\|AR_\lambda(A)\| + b\|R_\lambda(A)\| < 1,$$

where $R_\lambda(A) = (\lambda I - A)^{-1}$ is the resolvent operator of A. Show that $A+B$ is closed, $\lambda \in \rho(A+B)$ and

$$\|R_\lambda(A+B)\| \le \|R_\lambda(A)\|(1 - a\|AR_\lambda(A)\| - b\|R_\lambda(A)\|)^{-1}.$$

Proof. First we show that $A+B$ is closed. Suppose that $x_n \to x$ and $(A+B)x_n \to y$. From (3) we see that $\{Ax_n\}$ is Cauchy and thus $Ax_n \to z$ for some z. Since A is closed, we have that $x \in D(A)$ and $z = Ax$. Thus $Bx_n \to y - z$. Also, since B is A-bounded, it holds that $Bx_n \to Bx$. Therefore $y - z = Bx$ and $(A+B)x_n \to (A+B)x$.

Denote $c = a\|AR_\lambda(A)\| + b\|R_\lambda(A)\|$. Replacing x by $R_\lambda(A)y$ in $\|Bx\| \le a\|Ax\| + b\|x\|$, we obtain that

$$\|BR_\lambda(A)y\| \le a\|AR_\lambda(A)y\| + b\|R_\lambda(A)y\| \le c\|y\|.$$

Then

$$\|(A+B-\lambda I)x\| \ge \|(A-\lambda I)x\| - \|Bx\| \ge \|y\| - c\|y\| = (1-c)\|y\| \ge \frac{1-c}{\|R_\lambda(A)\|}\|x\|,$$

which implies that $\lambda \in \rho(A+B)$ and $\|R_\lambda(A+B)\| \le \frac{\|R_\lambda(A)\|}{1-c}$. □

5.5 Let A and B be densely defined operators in $\mathscr{H}$. Suppose that $A^{-1} \in L(\mathscr{H})$ and B is A-bounded such that

$$\|Bx\| \le a\|Ax\| + b\|x\|, \quad x \in D(A).$$

Suppose that $a + b\|A^{-1}\| < 1$, prove that

(1) $A+B$ is closed and invertible;

(2) $\|(A+B)^{-1}\| \le \|A^{-1}\|(1-a-b\|A^{-1}\|)^{-1}$, $\|(A+B)^{-1} - A^{-1}\| \le \|A^{-1}\|(a+b\|A^{-1}\|)(1-a-b\|A^{-1}\|)^{-1}$;

(3) if A^{-1} is compact, $(A+B)^{-1}$ is also compact.

Proof. It has been proved in the previous exercise that $A+B$ is closed. Similarly, Replacing x by $A^{-1}y$ in $\|Bx\| \le a\|Ax\| + b\|x\|$, we obtain that

$$\|BA^{-1}y\| \le a\|y\| + b\|A^{-1}y\| \le c\|y\|,$$

where $c = a + b\|A^{-1}\| < 1$. Then

$$\|(A+B)x\| = \|y + BA^{-1}y\| \ge \|y\| - c\|y\| = \frac{1-c}{\|A^{-1}\|}\|x\|,$$

which shows that $A+B$ is invertible and $\|(A+B)^{-1}\| \le \frac{\|A^{-1}\|}{1-c}$. Denote $T = (A+B)^{-1} - A^{-1}$. Now,

$$\|T\| \le \|(A+B)^{-1}\|\|(A+B)T\| = \|(A+B)^{-1}\|\|BA^{-1}\| \le \|(A+B)^{-1}\|c.$$

Since $\|BA^{-1}\| < 1$, we see that $I + BA^{-1}$ is invertible, then $(A+B)^{-1} = A^{-1}(I + BA^{-1})^{-1}$ is compact by Theorem 4.1.2(6). □

5.6 Suppose that A and B are densely defined operators, B is A-bounded and $\dim R(B) < \infty$. Show that B is A-compact.

Proof. Suppose $\{x_n\}$ and $\{Ax_n\}$ are bounded sequences. Since B is A-bounded, $\{Bx_n\}$ is a bounded sequence, too. Then $\{Bx_n\}$ has a convergent subsequence because $R(B)$ is finite-dimensional. □

5.7 Suppose that A and B are symmetric operators, $D(A) = D(B) = D$, and

$$\|(A-B)x\| \leq a'\|Ax\| + a''\|Bx\| + b\|x\|, \forall x \in D,$$

where $0 < a', a'' < 1$, $b > 0$. Show that A is essentially self-adjoint if and only if B is essentially self-adjoint, and when they are self-adjoint it holds that $D(\bar{A}) = D(\bar{B})$.

Proof. Use Corollary 6.5.12 instead of Theorem 6.5.2 in the proof of Corollary 6.5.4. □

5.8 Suppose that A is self-adjoint and B is symmetric. Show that B is A-compact if and only if

(1) $D(B) \supset D(A)$;

(2) $\forall\lambda \in \rho(A)$, $B(\lambda I - A)^{-1}$ is compact.

Furthermore, the condition (2) can be replaced by

(2') $\exists\lambda \in \rho(A)$ such that $B(\lambda I - A)^{-1}$ is compact.

Proof. `If': Suppose that $\{x_n\}$ and $\{Ax_n\}$ are bounded sequences, then $\{(\lambda I - A)x_n\}$ is bounded. Hence $\{Bx_n\} = \{B(\lambda I - A)^{-1}((\lambda I - A)x_n)\}$ has a convergent subsequence.

`Only if': Suppose that $\{x_n\}$ is a bounded sequence, then $\{(\lambda I - A)^{-1}x_n\}$ is bounded, $\{A(\lambda I - A)^{-1}x_n\}$ is also bounded since $A(\lambda I - A)^{-1} = \lambda(\lambda I - A)^{-1} - I$. Since B is A-compact, $\{B(\lambda I - A)^{-1}x_n\}$ has a convergent subsequence.

It is clear that we need only $\exists\lambda$ instead of $\forall\lambda$ in the `only if' part. □

5.9 Let $V \in \mathscr{H} = L^2(\mathbb{R}^3)$ and $\lambda > 0$. Show that

$$\lim_{\lambda\to\infty} \|V(-\Delta + \lambda)^{-1}\| = 0,$$

and that V is $(-\Delta)$-compact.

Proof. It is easy to see that $-\Delta + \lambda$ is invertible on C_0^∞ using Fourier Transform and $(-\Delta + \lambda)^{-1}u$ is in Schwartz space for $u \in C_0^\infty(\mathbb{R}^3)$. More precisely, using Green's function,

$$((-\Delta + \lambda)^{-1}u)(x) = \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|} u(y)dy,$$

Then

$$((V(-\Delta + \lambda)^{-1})u)(x) = \int_{\mathbb{R}^3} |V(x)| \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|} u(y)dy,$$

where the integral kernel

$$K_\lambda(x,y) = |V(x)| \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \in L^2(\mathbb{R}^6).$$

Now,

$$\|V((-\Delta + \lambda)^{-1}\| \leq \|K_\lambda\| \cdot \frac{1}{\lambda} \to 0$$

as $\lambda \to \infty$. Also, since $K_\lambda(x,y) \in L^2(\mathbb{R}^6)$, it is a Hilbert-Schmidt kernel and $V(-\Delta+\lambda)^{-1}$ is compact. It follows from the previous problem that V is $(-\Delta)$-compact. □

5.10 Let A be essentially self-adjoint and B bounded symmetric. Show that $A + B$ is essentially self-adjoint.

Proof. Obviously B is A-bounded with relative bound 0. The conclusion follows immediately from Corollary 6.5.12. □

5.11 Let A be self-adjoint and B symmetric with $D(A) \subset D(B)$ and $B^2 \leq A^2 + b^2 I$, where b is a constant. Show that $A + B$ is essentially self-adjoint.

Proof. Since

$$\|Bx\|^2 = (Bx, Bx) = (B^2 x, x) \leq (A^2 x, x) + b^2(x, x) = (Ax, Ax) + b^2\|x\|^2 = \|Ax\|^2 + b^2\|x\|^2,$$

the conclusion follows immediately from Theorem 6.5.14. □

5.12 Let $\mathscr{H}$ be a Hilbert space, A self-adjoint, $A \geq 0$, B symmetric with $D(B) \supset D(A)$. Suppose that

$$\|Bx\| \leq \|Ax\|, \quad \forall x \in D(A).$$

Show that $|(Bx, x)| \leq (Ax, x)$.

Proof. For any $t \in (-1, 1)$, tB is symmetric and A-bounded with relative bound $|t| < 1$. Hence $A + tB \geq 0$ from Theorem 6.5.16. It means that $t(Bx, x) \geq -(Ax, x)$ for all $t \in (-1, 1)$. The conclusion follows from letting $t \to \pm 1$. □

5.13 Suppose that $V_1, V_2 \in L^2(\mathbb{R}^3)$ are real-valued functions and view $V_i(x_i)$ $(i = 1, 2)$ as multiplication operator. Show that $-\Delta + V_1(x_1) + V_2(x_2)$ is essentially self-adjoint with domain $C_0^\infty(\mathbb{R}^6)$.

Proof. In the proof of Example 6.5.11, we see that given any $a > 0$ there exists $b > 0$ such that

$$\|u\|_\infty \leq a\|\Delta u\|_2 + b\|u\|_2$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, which is `equivalent' to

$$\|u\|_\infty^2 \leq a^2\|\Delta u\|_2^2 + b^2\|u\|_2^2.$$

Now let $u \in C_0^\infty(\mathbb{R}^6)$,

$$\begin{aligned}
\|V_1 u\|_2^2 &\leq a^2 \int |-\Delta_1 u(x_1, x_2)|^2 dx_1 dx_2 + b^2 \int |u(x_1, x_2)|^2 dx_1 dx_2 \\
&= a^2 \int \left| \sum_{i=1}^{3} p_i^2 \hat{u}(p_1, \ldots, p_6) \right|^2 dp_1 \cdots dp_6 + b^2\|u\|_2^2 \\
&\leq a^2 \int \left| \sum_{i=1}^{6} p_i^2 \hat{u}(p_1, \ldots, p_6) \right|^2 dp_1 \cdots dp_6 + b^2\|u\|_2^2 \\
&= a^2\|-\Delta u\|_2^2 + b^2\|u\|_2^2,
\end{aligned}$$

A result with the same right-hand side holds for $V_2 u$. It follows that

$$\|V_1(x_1)u + V_2(x_2)u\|^2 \leq 2a^2\|-\Delta u\|_2^2 + 2b^2\|u\|_2^2.$$

Since we can choose a as small as we want, $V_1(x_1) + V_2(x_2)$ is infinitesimally small with respect to $-\Delta$. Thus, by Kato-Rellich Theorem, $-\Delta + V_1(x_1) + V_2(x_2)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^6)$. □

5.14 Let A be a self-adjoint operator and B a bounded symmetric operator. Show that $A+B$ is self-adjoint, and

$$d(\sigma(A), \sigma(A+B)) \leq \|B\|,$$

i.e.,

$$\sup_{\lambda \in \sigma(A)} d(\lambda, \sigma(A+B)) \leq \|B\|, \tag{4}$$

$$\sup_{\lambda \in \sigma(A+B)} d(\sigma(A), \lambda) \leq \|B\|. \tag{5}$$

Proof. It is clear that $A+B$ is symmetric. Also $D(A^*+B^*) = D(A^*) = D(A) = D(A+B)$ because B is defined or can be extended to the entire $\mathscr{H}$. Therefore $A+B$ is self-adjoint.

To show (4), it suffices to show that for any $\lambda \in \sigma(A)$ and $\epsilon > 0$, it holds that

$$(\lambda - \|B\| - \epsilon, \lambda + \|B\| + \epsilon) \cap \sigma(A+B) \neq \emptyset.$$

Suppose it holds that

$$(\lambda - \|B\| - \epsilon, \lambda + \|B\| + \epsilon) \subset \rho(A+B),$$

then

$$\begin{aligned} \|(\lambda I - A - B)x\|^2 &= \int_{\mathbb{R}} (\lambda - \zeta)^2 d\|E_\zeta^{A+B} x\|^2 \\ &= \int_{\mathbb{R} \setminus (\lambda - \|B\| - \epsilon, \lambda + \|B\| + \epsilon)} (\lambda - \zeta)^2 d\|E_\zeta^{A+B} x\|^2 \\ &\geq (\|B\| + \epsilon)^2 \|x\|^2. \end{aligned}$$

So

$$\|(\lambda I - A - B)^{-1}\| \leq \frac{1}{\|B\| + \epsilon},$$

and $\|B(\lambda I - A - B)^{-1}\| < 1$, hence $I + B(\lambda I - A - B)^{-1}$ is invertible and so is

$$\lambda I - A = (I + B(\lambda I - A - B)^{-1})(\lambda I - A - B).$$

Contradiction.

For the second half, just notice that (5) is (4) applied to $(A+B) + (-B) = A$. □

5.15 Let A be a self-adjoint operator, $D \subset \mathbb{C}$ be a Borel-measurable set with smooth boundary $\Gamma = \partial D$. Suppose that $\Gamma \subset \rho(A)$, show that

$$E(D) = \frac{1}{2\pi i} \oint_\Gamma (zI - A)^{-1} dz,$$

where E is the spectral family of A.

Proof. Note that $\rho(A)$ is an open set and $\sigma(A) \subset \mathbb{R}$, hence such a boundary Γ separates $\sigma(A)$. Then the proof follows the same line as in Exercise 5.5.15. □

5.16 Let A be a self-adjoint operator and C a compact operator, then

$$\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A+C).$$

5.17 Suppose that $V \in L^2(\mathbb{R}^3)$ is real-valued, show that $\sigma_{\text{ess}}(-\Delta + V) = [0, \infty)$.

Proof. Using Fourier transform we can easily obtain that $\sigma_{\text{ess}}(-\Delta) = [0, \infty)$. Since V is symmetric (because it is real-valued) and $(-\Delta)$-compact (Exercise 5.9), it immediately follows from Weyl's Theorem that $\sigma_{\text{ess}}(-\Delta + V) = [0, \infty)$. □

6 Convergence of Unbounded Operators

6.1 Let A_n and A be self-adjoint operators and suppose that for all $x, y \in \mathscr{H}$ and all λ with $\operatorname{im}\lambda \neq 0$, $(R_\lambda(A_n)x, y) \to (R_\lambda(A)x, y)$. Prove that $A_n \to A$ s.r.s.

Proof.

$$\begin{aligned}\|(R_\lambda(A_n) - R_\lambda(A))x\|^2 &= ((R_\lambda(A_n) - R_\lambda(A))x, (R_\lambda(A_n) - R_\lambda(A))x) \\ &= (R_\lambda(A_n)x, R_\lambda(A_n)x) - 2\Re(R_\lambda(A_n)x, R_\lambda(A)x) + (R_\lambda(A)x, R_\lambda(A)x).\end{aligned}$$

Since $A_n \to A$ w.r.s, it is clear that $(R_\lambda(A_n)x, R_\lambda(A)x) \to (R_\lambda(A)x, R_\lambda(A)x)$. Also,

$$\begin{aligned}(R_\lambda(A_n)x, R_\lambda(A_n)x) &= (R_{\bar\lambda}(A_n)R_\lambda(A_n)x, x) \\ &= \left(-\frac{R_{\bar\lambda}(A_n) - R_\lambda(A_n)}{\bar\lambda - \lambda}x, x\right) \\ &\to \left(-\frac{R_{\bar\lambda}(A) - R_\lambda(A)}{\bar\lambda - \lambda}x, x\right) \\ &= (R_{\bar\lambda}(A)R_\lambda(A)x, x) \\ &= (R_\lambda(A)x, R_\lambda(A)x)\end{aligned}$$

Therefore,

$$\|(R_\lambda(A_n) - R_\lambda(A))x\|^2 \to 0.$$

□

Remark. This problem is exactly weak resolvent convergence implies strong resolvent convergence.

6.2 Let A_n and A be positive self-adjoint operators, show that $A_n \to A$ s.r.s if and only if $(A_n + I)^{-1} \to (A + I)^{-1}$ strongly.

Proof. `If': Let $\lambda_0 = -1$. Examine the proof of Theorem 6.6.3, we see that the power series

$$R_\lambda(A) = \sum_{k=0}^{\infty}(\lambda_0 - \lambda)^k (R_{-1}(A))^{k+1} \tag{6}$$

converges in norm in $|\lambda - \lambda_0| < 1$, because $\sigma(A) \subset [0, \infty)$. So does the power series of $R_\lambda(A_n)$. Hence there exists λ, $\operatorname{im}\lambda \neq 0$ such that $R_\lambda(A_n) \to R_\lambda(A)$ strongly. Theorem 6.6.3 then applies.

`Only if': Note that $\lambda_0 = -1 + i$ is contained in $\rho(A_n)$ and $\rho(A)$. The power series (6) converges in norm in $|\lambda - \lambda_0| < \sqrt{2}$ because $\sigma(A) \subset [0, \infty)$. So does the power series of $R_\lambda(A_n)$. Hence $R_\lambda(A_n) \to R_\lambda(A)$ s.r.s in $|\lambda - \lambda_0| < \sqrt{2}$. Let $\lambda = -1$. □

6.3 Let A be a self-adjoint operator. Show that

(1) n.r.s-$\lim_{t \to t_0} tA = t_0 A$, where $t_0 \neq 0$;

(2) $\lim_{t \to t_0} \|e^{itA} - e^{it_0 A}\| = 0$ if and only if A is bounded.

Proof. (1) Let $\lambda \in \mathbb{C}$ with $\operatorname{im}\lambda \neq 0$. Then

$$\begin{aligned}\|R_\lambda(t_0A) - R_\lambda(tA)\| &= \|(\lambda I - tA)^{-1})(t_0A - tA)(\lambda I - t_0A)^{-1}\| \\ &\le \|(\lambda I - tA)^{-1})\| \, \|t_0A - tA\| \, \|(\lambda I - t_0A)^{-1}\| \\ &= \left\| t^{-1}\left(\frac{\lambda}{t}I - A\right)^{-1} \right\| |t_0 - t| \, \|A\| \, \|(\lambda I - t_0A)^{-1}\| \\ &\le |t^{-1}| \, |\operatorname{im}\lambda/t|^{-1} \, |t_0 - t| \, \|A\| \, \|(\lambda I - t_0A)^{-1}\| \to 0.\end{aligned}$$

(2) `If': Suppose that E is the spectral family of A. Since A is bounded, $\sigma(A)$ is compact. Suppose that $\sigma(A) \subset [-N, N]$ and $|t - t_0| < 1/N$. It follows from

$$\begin{aligned}\|e^{itA}x - e^{it_0A}x\|^2 &= \int_{\mathbb{R}} |e^{it\lambda} - e^{it_0\lambda}|^2 d\|E_\lambda x\|^2 \\ &= \int_{-N}^{N} |e^{i(t-t_0)\lambda} - 1|^2 d\|E_\lambda x\|^2 \\ &= 2\int_{-N}^{N} (1 - \cos((t-t_0)\lambda)) d\|E_\lambda x\|^2 \\ &\le 2(1 - \cos((t-t_0)N)) \int_{\mathbb{R}} d\|E_\lambda x\|^2 \\ &= 2(1 - \cos((t-t_0)N))\|x\|^2\end{aligned}$$

that

$$\|e^{itA} - e^{it_0A}\| \le \sqrt{2(1 - \cos((t-t_0)N))} \to 0$$

as $t \to t_0$.

`Only if': Assume that $t_0 = 0$ for simplicity. For any operator Z that differs from I by an operator of norm < 1 we can define

$$\ln Z = \ln(I + (Z - I)) = Z - I - \frac{(Z-I)^2}{2} + \cdots .$$

Since $\|e^{itA} - I\| \to 0$, there exists t such that $\|e^{itA} - I\| < \frac{1}{3}$. We can define $\ln e^{itA}$ according to the expansion of $\ln Z$ above. Then $\ln e^{itA}$ is bounded. On the other hand, from functional calculus we see that $\ln e^{itA} = itA$. Therefore A is bounded. □

6.4 Let A_n and A be uniformly bounded self-adjoint operators. Show that

$$A_n \to A \text{ s.r.s} \iff A_n \to A \text{ strongly}.$$

Proof. `⇒': Suppose that $A_n \to A$ s.r.s, then for all λ, $\operatorname{im}\lambda \ne 0$ and all x, $(R_\lambda(A_n) - R_\lambda(A))x \to 0$. Note that

$$A - A_n = (\lambda I - A_n) - (\lambda I - A) = (\lambda I - A_n)(R_\lambda(A) - R_\lambda(A_n))(\lambda I - A),$$

hence

$$\begin{aligned}\|Ax - A_nx\| &\le \|(\lambda I - A_n)\|\, \|(R_\lambda(A) - R_\lambda(A_n))(\lambda I - A)x\| \\ &\le (M + |\lambda|)\|(R_\lambda(A) - R_\lambda(A_n))(\lambda I - A)x\| \to 0,\end{aligned}$$

where M is the uniform bound of A_n.

`⇐': Suppose that $A_n \to A$ strongly. For any x, there exists $y \in D(A)$ such that $x = (\lambda I - A)y$. Then

$$(\lambda I - A)^{-1}x - (\lambda I - A_n)^{-1}x = (\lambda I - A_n)^{-1}(A - A_n)y \to 0$$

because

$$\|(\lambda I - A_n)^{-1}\| \le |\operatorname{im}\lambda|^{-1}.$$

Therefore $(\lambda I - A_n)^{-1} \to (\lambda I - A)^{-1}$ strongly. □

6.5 Show that if $A_n \to A$ s.r.s then $e^{itA_n} \to e^{itA}$ uniformly strongly for t in any finite interval.

Proof. Let $f_s(t) = e^{its}$. A careful examination of the proof of Theorem 6.6.6(2) reveals that we need to prove

$$\|f_s(A_n)g_{m_0}(t)x - f_s(A)g_{m_0}(t)x\| < \epsilon/3$$

for all s in a finite interval when n is big enough. Since $|f_s(t)| = 1$ regardless of s and t, the other lines in the proof of Theorem 6.6.6(2) still carries through for s in a finite interval.

Fix m. Note that $f_s(t)g_m(t) = e^{-\frac{t^2}{m}+its}$ and

$$\begin{aligned}\sup_{t\in\mathbb{R}} |f_{s_1}(t)g_m(t) - f_{s_2}(t)g_m(t)| &= \sup_{t\in\mathbb{R}} e^{-\frac{t^2}{m}}|e^{i(s_1-s_2)t} - 1| \\ &= \sup_{t\in\mathbb{R}} \sqrt{2}e^{-\frac{t^2}{m}}\sqrt{1 - 2\cos(s_1 - s_2)t}\end{aligned}$$

By splitting $\mathbb{R}$ into $|t| < T$ and $|t| \geq T$, it is easy to see that

$$\sup_{t\in\mathbb{R}} |f_{s_1}(t)g_m(t) - f_{s_2}(t)g_m(t)| < \epsilon$$

for $|s_1 - s_2|$ small enough (depending on ϵ and independent of s_1 or s_2). This fact shows that the following line in the proof of Theorem 6.6.6

$$\sup_{x\in\mathbb{R}^1} \left| f_s(x)g_{m_0}(x) - P\left(\frac{1}{x+i}, \frac{1}{x-i}\right)\right| \leq \frac{\epsilon}{3}.$$

holds for all s inside any interval with a small length $L(\epsilon)$. Consequently, for any of such interval, there exists N such that whenever $n > N$ it holds that

$$\|f_s(A_n)x - f(A)x\| \leq \epsilon$$

holds for all s inside the small interval. The final step is to divide a finite interval into pieces, each has length $L(\epsilon)$. □

6.6 Let A_n and A be uniformly bounded self-adjoint operators. Suppose that $A_n \to A$ weakly but not strongly. Does $A_n \to A$ W.R.S?

Proof. No. If $A_n \to A$ W.R.S, then $A_n \to A$ S.R.S and thus $A_n \to A$ strongly by Exercise 6.6.4. □

6.7 Let A_n and A be positive self-adjoint operators. Suppose that $e^{-tA_n} \to e^{-tA}$ strongly for all $t > 0$. Show that S.R.S-$\lim_{n\to\infty} A_n = A$.

Proof. One can show that for positive self-adjoint operator A,

$$\phi(A) = \int_0^\infty \phi(\lambda)dE_\lambda$$

for Borel measurable ϕ that is bounded on $[0, \infty)$. Then following the same outline of Example 6.6.7, we obtain that

$$R_{-1}(A)u = -\int_0^\infty e^{-t}e^{-tA}udt,$$

and thus

$$\|R_{-1}(A_n)u - R_{-1}(A)u\| \leq \int_0^\infty e^{-t}\|e^{-tA_n}u - e^{-tA}u\|dt.$$

It follows from Dominated Convergence Theorem that $R_{-1}(A_n) \to R_{-1}(A)$ strongly, and thence $A_n \to A$ S.R.S by Problem 6.2. □

6.8 Let $\{A_n\}$ be a sequence of symmetric operators. Define $D^S_\infty = \{x : \exists y \in \mathscr{H}, \langle x, y\rangle \in \Gamma^S_\infty\}$. If D^S_∞ is dense in $\mathscr{H}$, show that $\{A_n\}$ has a strong graph limit and the limit operator is also symmetric. Moreover, the limit operator is closed.

Proof. First we show that Γ^S_∞ is the graph of an operator, for which we need only to show that the operator is well-defined, i.e., suppose $x_n, x'_n \in D(A_n)$ and $x_n \to x$, $x'_n \to x'$, $A_n x_n \to y$ and $A_n x'_n \to y'$, we must have $y = y'$. Indeed, let u be an arbitrary element in D^S_∞, then there exists $u_n \in D(A_n)$ such that $u_n \to u$ and $Au_n \to v$. Thus,

$$(y - y', u) = \lim_{n\to\infty} (A_n(x_n - x'_n), u_n) = \lim_{n\to\infty} (x_n - x'_n, A_n u_n) = 0. \tag{7}$$

Since D^S_∞ is dense, it follows immediately that $y = y'$. So $\{A_n\}$ has a strong graph limit, say A.

Now we show that A is symmetric. Let $x, y \in D^S_\infty$. There exist $u_n \to x$ and $v_n \to y$ such that $u_n, v_n \in D(A_n)$, $A_n u_n \to Ax$ for some Ax and $A_n v_n \to Ay$ for some Ay. Then

$$(x, Ay) = \lim_{n\to\infty} (u_n, A_n v_n) = \lim_{n\to\infty} (A_n u_n, v_n) = (Ax, y). \tag{8}$$

Moreover, A is closed: suppose that $x_n \to x$ and $Ax_n \to y$. There exist $x_{nm} \to x_n$ and $A_m x_{nm} \to Ax_n$ for each n. We can pick $x_{n_m m} \to x$ and $A_m x_{n_m m} \to y$, hence $x \in D(A)$ and $y = Ax$. □

6.9 Let $\{A_n\}$ be a sequence of operators on $\mathscr{H}$. Define $\Gamma^w_\infty = \{\langle u, v\rangle \in \mathscr{H} \times \mathscr{H} : \exists u_n \in D(A_n), u_n \to u, A_n u_n \rightharpoonup v\}$. If Γ^w_∞ is the graph of some linear operator A, we say A is the weak graph limit of $\{A_n\}$, denoted by $A = \text{wg-}\lim_{n\to\infty} A_n$. Suppose that A_n and A are uniformly bounded self-adjoint operators, show that $A = \text{wg-}\lim_{n\to\infty} A_n$ if and only if $A_n \to A$ weakly.

Proof. Suppose that the uniform bound of A_n and A is M.

`Only if': We want to prove that $A_n u \rightharpoonup Au$ for all u. There exist u_n such that $u_n \to u$ and $A_n u_n \rightharpoonup Au$. Since A_n and A are bounded, they can be extended to the entire $\mathscr{H}$. Notice that

$$|(A_n u - A_n u_n, y)| \leq M\|u - u_n\|\,\|y\| \to 0, \quad \forall y \in \mathscr{H}$$

it follows immediately that

$$\lim_{n\to\infty} (A_n u, y) = \lim_{n\to\infty} (A_n u_n, y) = (Au, y), \quad \forall y \in \mathscr{H}$$

or, $A_n u \rightharpoonup Au$.

`If': Suppose that $A_n u \rightharpoonup Au$ for all u. We want to find $\{u_n\}$ such that $u_n \to u$ and $A_n u_n \rightharpoonup Au$. Note that $D(A_n)$ is dense, we can easily find $u_n \in D(A_n)$ such that $u_n \to u$. Now, as above, it automatically holds that

$$\lim_{n\to\infty} (A_n u_n, y) = \lim_{n\to\infty} (A_n u, y) = (Au, y), \quad \forall y \in \mathscr{H}.$$ □

6.10 Let $\{A_n\}$ be a sequence of symmetric operators. Define $D^w_\infty = \{x : \exists y \in \mathscr{H}, \langle x, y\rangle \in \Gamma^w_\infty\}$. If D^S_∞ is dense in $\mathscr{H}$, show that Γ^w_∞ is the graph of some symmetric operator.

Proof. The proof follows the same line as that of Exercise 6.8. Recall Exercise 2.5.18: If $a_n \rightharpoonup a$ and $b_n \to b$ then $(a_n, b_n) \to (a, b)$. Hence (7) and (8) still hold. □