1 Basics of Algebras

1.1 Let ϕ be a non-zero linear functional on an algebra $\mathscr{A}$ over complex numbers, satisfying $\langle \phi, ab \rangle = \langle \phi, a \rangle \langle \phi, b \rangle$. Such a linear functional ϕ is also called a complex homomorphism. Show that

(1) if $\mathscr{A}$ has identity e then $\phi(e) = 1$;

(2) for any invertible $a \in \mathscr{A}$, it holds that $\phi(a) \neq 0$.

Proof. (1) Since ϕ is nonzero, there exists a such that $\phi(a) \neq 0$. Then $\phi(a) = \phi(ae) = \phi(a)\phi(e)$, and thus $\phi(e) = 1$.

(2) $1 = \phi(e) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1})$. □

1.2 Let J be an ideal of algebra $\mathscr{A}$. Show that J is maximal iff $\mathscr{A}/J$ does not contain a non-zero ideal.

Proof. `Only if': Let J be a maximal ideal of $\mathscr{A}$. Suppose that $\mathscr{B} = \mathscr{A}/J$ contains a non-zero ideal J_B. Consider the natural maps $\phi : \mathscr{A} \to \mathscr{B}$ and $\psi : \mathscr{B} \to \mathscr{B}/J_B$. It is clear that both ϕ and ψ are non-trivial homomorphisms, and thus $\ker(\psi \circ \phi)$ is an ideal of $\mathscr{A}$ containing J. Since J_B is non-zero, there exists $a \in \mathscr{A}$ such that $[a] \in J_B$ and $[a] \neq [0]$. Therefore $a \in \ker(\psi \cdot \phi)$ but $a \notin J$, which contradicts with the maximality of J.

`If': Suppose $\mathscr{B} = \mathscr{A}/J$ does not contain a non-zero ideal but $J' \supset J$ is a bigger ideal of $\mathscr{A}$. Consider $J_B = \{[x] \in \mathscr{B} : x \in J' \setminus J\}$. Since $J' \setminus J \neq \emptyset$, we know that $J_B \neq \emptyset$. From $J' \neq \mathscr{A}$ we also know that $J_B \neq \mathscr{B}$. Lastly, for all $[a] \in \mathscr{B}$ and $[j] \in J_B$, it holds that $[a][j] = [aj] = [j]$ and $[j][a] = [ja] = [j]$ because $ja \in J'$ and $aj \in J'$ as J' is an ideal. We have found that J_B is a non-zero ideal of $\mathscr{B}$. Contradiction. □

2 Banach Algebra

2.1 Let $\mathscr{A}$ be a Banach algebra with identity and $G(\mathscr{A})$ be the set of all invertible elements in $\mathscr{A}$. Show that $G(\mathscr{A})$ is open and $a \mapsto a^{-1}$ is continuous.

Proof. We shall use the next problem in the proof. Let $a \in G(\mathscr{A})$. Then for all $b \in B(a, \frac{1}{\|a^{-1}\|})$, we have that $\|a^{-1}(b-a)\| < 1$. Hence $e + a^{-1}(b-a)$ is invertible and $b = a(e + a^{-1}(b-a))$ is invertible.

To show the continuity of inverse map, we first observe that for $a, b \in G(\mathscr{A})$ it holds that

$$
\begin{aligned}
\|b^{-1} - a^{-1}\| &= \|(a+\eta)^{-1} - a^{-1}\| \\
&= \|(a(e + a^{-1}\eta))^{-1} - a^{-1}\| \\
&= \|((e + a^{-1}\eta)^{-1} - e)a^{-1}\| \\
&\leq \|(e + a^{-1}\eta)^{-1} - e\| \, \|a^{-1}\| \\
&\leq \frac{\|a^{-1}\eta\|}{1 - \|a^{-1}\eta\|} \|a^{-1}\| \\
&\leq \frac{\|a^{-1}\| \, \|\eta\|}{1 - \|a^{-1}\| \, \|\eta\|} \|a^{-1}\|
\end{aligned}
$$

For a given $\epsilon > 0$ we can choose δ such that

$$\frac{\delta \|a^{-1}\|^2}{1 - \delta \|a^{-1}\|} < \epsilon,$$

then $\|b^{-1} - a^{-1}\| < \epsilon$ whenever $\|b - a\| < \delta$. □

2.2 Let $\mathscr{A}$ be a Banach algebra with identity and $a \in \mathscr{A}$ with $\|a\| < 1$. Show that $e - a \in G(\mathscr{A})$ and

$$(e-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof. Let $y_N = \sum_{n=0}^{N} a^n$. Since $\|a\| < 1$, it is easy to show that $\{y_N\}$ is Cauchy and hence y_N converges to some $y \in \mathscr{A}$. Observe that $(e-a)y_N = y_N(e-a) = e - a^{N+1} \to e$ as $N \to \infty$. Since multiplication is continuous, we have that $(e-a)y = y(e-a) = e$, which shows that $e - a \in G(\mathscr{A})$ and $(e-a)^{-1} = y$. □

2.3 Let $\mathscr{A}$ be a Banach algebra with identity and $a \in \partial(G(\mathscr{A}))$. Prove that

(1) If $a_n \in G(\mathscr{A})$, $a_n \to a$, then $\lim_{n\to\infty} \|a_n^{-1}\| = \infty$.

(2) There exists $b_n \in \mathscr{A}$, $\|b_n\| = 1$, such that $\lim_{n\to\infty} ab_n = \lim_{n\to\infty} b_n a = 0$.

Proof. (1) If $\|a_n^{-1}\| \leq L$ for all n, then $\|a_n^{-1}(a - a_n)\| < 1$ for n sufficiently large, which means that $e + a_n^{-1}(a - a_n) \in G(\mathscr{A})$ and $a = a_n(e + a_n^{-1}(a - a_n)) \in G(\mathscr{A})$. Contradiction.

(2) Suppose that $a_n \to a$ with $a_n \in G(\mathscr{A})$. Let $b_n = a_n^{-1}/\|a_n^{-1}\|$, then

$$\begin{aligned}
\|ab_n\| &= \frac{\|aa_n^{-1}\|}{\|a_n^{-1}\|} \\
&= \frac{\|(a - a_n + a_n)a_n^{-1}\|}{\|a_n^{-1}\|} \\
&= \frac{\|(a - a_n)a_n^{-1} + e\|}{\|a_n^{-1}\|} \\
&\leq \frac{\|(a - a_n)a_n^{-1}\| + \|e\|}{\|a_n^{-1}\|} \\
&\leq \|a - a_n\| + \frac{\|e\|}{\|a_n^{-1}\|} \to 0
\end{aligned}$$

as $n \to \infty$. Hence $ab_n \to 0$. Similarly we can show that $b_n a \to 0$.

□

2.4 Let

$$\mathscr{A} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

be an algebra under the usual addition and multiplication of matrices. Show that $\mathscr{A}$ is a Banach algebra under the norm

$$\left\| \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \right\| = |\alpha| + |\beta|.$$

Proof. The only less trivial part is to show the completeness. Suppose that $\{A_n\} \subseteq \mathscr{A}$ is a Cauchy sequence. Since $\|A_n - A_m\| = |\alpha_n - \alpha_m| + |\beta_n - \beta_m|$, we know that $\{\alpha_n\}$ and $\{\alpha_m\}$ are Cauchy sequences, hence $\alpha_n \to \alpha$ and $\beta_n \to \beta$ for some α and β. It is then straightforward to see that $A_n \to \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$. □

2.5 Let $\mathscr{A}$ be a Banach algebra with identity and $\phi : \mathscr{A} \to \mathbb{C}$ a homomorphism. Then $|\phi(a)| \leq \|a\|$ for all $a \in \mathscr{A}$.

Proof. Suppose that $|\phi(a)| > \|a\|$ for some a. Let $b = a/|\phi(a)|$, then $|\phi(b)| = 1 > \|b\|$, and $e - b$ is invertible. Thus $1 = \phi(e) = \phi(e-b)\phi((e-b)^{-1})$. Note that $\phi(e-b) = \phi(e) - \phi(b) = 0$. Contradiction. □

2.6 Let $\mathscr{A}$ be a commutative Banach algebra with identity. Show that $a \in \mathscr{A}$ is invertible if and only if $\phi(a) \neq 0$ for all non-trivial continuous homomorphism $\phi : \mathscr{A} \to \mathbb{C}$.

Proof. `Only if': Trivial, as $1 = \phi(e) = \phi(a)\phi(a^{-1})$.

`If': Suppose that a is not invertible. Let J be a maximal ideal containing a (J exists because $a\mathscr{A}$ is an ideal). Then by Gelfand-Mazur Theorem, $\mathscr{A}/J$ is isomorphic to $\mathbb{C}$, and there exists a natural continuous homomorphism from $\phi_J : \mathscr{A} \to \mathbb{C}$ such that $\phi_J(a) = 0$. Contradiction. □

2.7 Let $\mathscr{A}$ be a Banach algebra with identity and $a \in \mathscr{A}$. Show that

(1) $\sigma(a)$ is compact;

(2) $\sigma(a)$ is not empty.

Proof. (1) Let $\lambda \in \sigma(s)$. It is clear that $|\lambda| \geq \|a\|$, otherwise $\lambda e - a = \lambda(e - a/\lambda)$ would be invertible as $\|a/\lambda\| < 1$. Hence $\sigma(a)$ is bounded.

Now we show that $\rho(a)$ is open. Suppose that $\lambda \in \rho(a)$, then $\lambda e - a \in G(\mathscr{A})$. Recall that $G(\mathscr{A})$ is open, hence there exists ϵ such that $\lambda e - a + \eta \in G(A)$ whenever $\|\eta\| < \epsilon$. In particular, choose $\eta = \delta e$, where $|\delta| < \epsilon/\|e\|$, it follows that $(\lambda + \delta)e - a$ is invertible.

(2) Suppose that $\sigma(a) = \emptyset$, then $\lambda e - a$ is invertible for all λ. Define $r(\lambda) = (\lambda e - a)^{-1}$. The rest of the proof goes as in the proof of Gelfand-Mazur Theorem. Finally we arrive at $r(\lambda) = 0$ for all λ. Contradiction. □

2.8 Let $\mathscr{A}$ be a Banach algebra with identity and $a, b \in \mathscr{A}$. Show that

(1) If $e - ab$ is invertible then $e - ba$ is invertible, too;

(2) If $\lambda \in \sigma(ab)$, $\lambda \neq 0$, then $\lambda \in \sigma(ba)$;

(3) If a is invertible then $\sigma(ab) = \sigma(ba)$.

Proof. (1) $(e - ba)^{-1} = e + b(e - ab)^{-1}a$.

(2) $\lambda e - ba$ is invertible $\Longrightarrow e - \lambda^{-1}ba$ is invertible $\Longrightarrow$ (by the first part) $e - \lambda^{-1}ab$ is invertible $\Longrightarrow \lambda e - ab$ is invertible. Contradiction.

(3) $\lambda e - ab$ is invertible $\Longleftrightarrow \lambda a^{-1} - b$ is invertible $\Longleftrightarrow \lambda e - ba$ is invertible. □

2.9 Let $\mathscr{A}$ and $\mathscr{B}$ be commutative Banach algebras with identity and $\mathscr{B}$ semi-simple. Let $\phi : \mathscr{A} \to \mathscr{B}$ be a homomorphism, show that ϕ is continuous.

Proof. We shall show that ϕ is closed, whence the continuity follows from Closed Graph Theorem. Suppose that $a_n \to 0$ in $\mathscr{A}$ and $\phi(a_n) \to b$ in $\mathscr{B}$. We want to show that $b = 0$. If not, since $\mathscr{B}$ is semi-simple, there exists a maximal ideal J such that $b \notin J$. By Gelfand-Mazur Theorem, there exists an isomorphism $i : \mathscr{B}/J \to \mathbb{C}$, and $i([b]) \neq 0$. Note that $i \circ \psi \circ \phi : \mathscr{A} \to \mathbb{C}$ is a homomorphism, where ψ is the natural homomorphism from $\mathscr{B}$ to $\mathscr{B}/J$. By Problem 2.4, $|i \circ \psi \circ \phi(a_n)| \leq \|a_n\|$ for all n. Let $n \to \infty$, we find that $|i \circ \psi(b)| \leq 0$, thus $i \circ \psi(b) = 0$ and $\psi(b) = 0$. Contradiction. □

2.10 Let $\mathscr{A}$ be a Banach algebra with identity. Let

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\},$$

which is called the spectral radius of a. Show that for all $a, b \in \mathscr{A}$,

(1) $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$;

(2) $r(ab) = r(ba)$;

(3) if $ab = ba$ then $r(a + b) \leq r(a) + r(b)$ and $r(ab) \leq r(a)r(b)$.

Proof. (1) According to Cauchy-Hadamard test, when $\lambda > \limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}}$, it holds that

$$(\lambda e - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}.$$

Therefore $r(a) \le \limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. Next we show that $\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \le r(a)$. For all $\phi \in \mathscr{A}^*$, the function $\lambda \mapsto \langle \phi, (\lambda e - a)^{-1} \rangle$ is analytic in the region $|\lambda| > r(a)$. Hence

$$\langle \phi, (\lambda e - a)^{-1} \rangle = \sum_{n=0}^{\infty} \frac{\langle \phi, a^n \rangle}{\lambda^{n+1}}.$$

and

$$\langle \phi, a^n \rangle = \frac{1}{2\pi i} \oint_{|\lambda| = r(a)+\epsilon} \langle \phi, (\lambda e - a)^{-1} \rangle \lambda^n d\lambda$$

Let $M = \max_{|\lambda|=r(a)+\epsilon} \|(\lambda e - a)^{-1}\|$, then $M < \infty$ because $(\lambda e - a)^{-1}$ is continuous w.r.t. λ. Thus

$$|\langle \phi, a^n \rangle| \le \|\phi\| M (r(a) + \epsilon)^{n+1}$$

for all $\phi \in \mathscr{A}^*$. Hence by Hahn-Banach Theorem,

$$\|a_n\| \le M(r(a) + \epsilon)^{n+1}.$$

Therefore

$$\lim_{n\to\infty} \|a_n\|^{\frac{1}{n}} \le r(a) + \epsilon.$$

Let $\epsilon \to 0$ and we conclude that $\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} = r(a)$.
Now we show that $r(a) \le \liminf_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. Note that

$$\lambda^n e - a^n = (\lambda e - a)P(a) = P(a)(\lambda e - a),$$

where

$$P(a) = \lambda^{n-1} e + \lambda^{n-2} a + \cdots + a^{n-1},$$

whence we see that $\lambda \in \sigma(a)$ implies that $\lambda^n \in \sigma(a^n)$. It follows that $|\lambda^n| \le \|a^n\|$, and $|\lambda| \le \liminf_{n\to\infty} \|a^n\|^{\frac{1}{n}}$.

(2) By Exercise 2.8(2), $\sigma(ab)$ and $\sigma(ba)$ differ by 0 only at most. If both $\sigma(ab)$ and $\sigma(ba)$ contain a non-zero number, we would have $r(ab) = r(ba)$. Now suppose that $\sigma(ab) = \{0\}$, then $\lambda e - ab$ is invertible for any $\lambda \ne 0$. By Exercise 2.8(2), $\lambda e - ba$ is invertible, too, and thus $\sigma(ba) = \{0\}$ since $\sigma(ba) \ne \emptyset$.

(3) Since $ab = ba$, it holds that $\|(ab)^n\|^{\frac{1}{n}} = \|a^n b^n\|^{\frac{1}{n}} \le \|a^n\|^{\frac{1}{n}} \|b^n\|^{\frac{1}{n}}$. It follows immediately from part (1) that $r(ab) \le r(a)r(b)$.
Pick $\alpha > r(a)$ and $\beta > r(b)$, let $x = a/\alpha$ and $y = b/\beta$. Then

$$\|(a+b)^n\|^{\frac{1}{n}} = \left\| \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \right\|^{\frac{1}{n}} \le \left(\sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \|x^k\| \, \|y^{n-k}\| \right)^{\frac{1}{n}}.$$

Let

$$k_n = \arg\max_k \|x^k\| \, \|y^{n-k}\|,$$

so

$$\|(a+b)^n\|^{\frac{1}{n}} \le \|x^{k_n}\|^{\frac{1}{n}} \, \|y^{n-k_n}\|^{\frac{1}{n}} (\alpha + \beta)$$

for all n. Since $0 \le k_n/n \le 1$ we can choose a subsequence such that $k_{n_i}/n_i \to \delta$ for some δ as i tends to infinity. Denote this subsequence by k_n. If $\delta = 0$ then

$$\limsup_{n\to\infty} \|x^{k_n}\|^{\frac{1}{n}} \le \limsup_{n\to\infty} \|x\|^{\frac{k_n}{n}} \le 1,$$

otherwise $k_{n_i} \neq 0$ for i big enough and thus

$$\limsup_{n\to\infty} \|x^{k_n}\|^{\frac{1}{n}} = \limsup_{n\to\infty} \left(\|x^{k_n}\|^{\frac{1}{k_n}}\right)^{\frac{k_n}{n}} = r(x)^\delta \leq 1$$

because $r(x) \leq \|x\| < 1$. Therefore $r(a+b) \leq \alpha + \beta$ and the conclusion follows by letting $\alpha \to r(a)^+$ and $\beta \to r(b)^+$. ☐

2.11 Let $\mathscr{A} = \{f \in C^1[0,1]\}$ with norm

$$\|f\|_{C^1} = \|f\| + \|f'\|.$$

Show that $\mathscr{A}$ is a semi-simple commutative Banach algebra.

Proof. It is clear that $\mathscr{A}$ is a commutative Banach algebra with identity. We shall show the semi-simpleness by showing

$$\lim_{n\to\infty} \|f^n\|^{\frac{1}{n}} = 0 \Longrightarrow f = 0.$$

In fact, $r(f) = 0$ means that $\lambda e - f$ is invertible for all $\lambda \neq 0$. If $f(x_0) = \lambda \neq 0$ for some x_0 then $\lambda e - f$ would not be invertible. Hence $f(x) = 0$ for all $x \in [0,1]$. ☐

2.12 Let $\mathscr{A}$ be a commutative Banach algebra and $r = \inf\limits_{a\neq 0} \dfrac{\|a^2\|}{\|a\|^2}$ and $s = \inf\limits_{a\neq 0} \dfrac{\|\hat{a}\|_\infty}{\|a\|}$. Show that $s^2 \leq r \leq s$.

Proof. Note that $\|\Gamma a\|^2 = \|\Gamma a^2\| \leq \|a^2\|$, it follows that $s^2 \leq r$. Now we show the second half. Starting from $\|a^2\| \geq r\|a\|^2$, then $\|a^4\| \geq r\|a^2\|^2 \geq r^3\|a\|^4$. By induction one can show that $\|a^{2^k}\| \geq r^{2^k-1}\|a\|^{2^k}$, hence

$$\|a^{2^k}\|^{\frac{1}{2^k}} \geq r^{1-\frac{1}{2^k}}\|a\|$$

Letting $k \to \infty$, we obtain that $\|\hat{a}\|_\infty \geq r\|a\|$. It follows immediately that $s \geq r$. ☐

3 Examples and Applications

3.1 Let

$$\mathscr{A} = \left\{ f : \mathbb{Z} \to \mathbb{C} : \|f\| = \sum_{n=-\infty}^{\infty} |f(n)| 2^{|n|} < \infty \right\}$$

under the usual addition of scalar multiplication and the following multiplication

$$f * g(n) = \sum_{k=-\infty}^{\infty} f(n-k)g(k).$$

Show that

(1) $\mathscr{A}$ is a commutative Banach algebra;

(2) Let $K = \{z \in \mathbb{C} : \frac{1}{2} \leq |z| \leq 2\}$ then K is one-to-one correspondent with $\mathfrak{M}$ and the Gelfand representation of $\mathscr{A}$ is the Laurent series that are absolutely convergent on K.

Proof. (1) It is easy to verify that $\mathscr{A}$ is a commutative algebra, and

$$\|f * g\| = \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} f(n-k)g(k) \right| 2^{|n|}$$

$$
\begin{aligned}
&\leq \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |f(n-k)|\,|g(k)|2^{|n|} \\
&= \sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} |f(n-k)|2^{|n|}\right)|g(k)| \\
&\leq \sum_{k=-\infty}^{\infty} (\|f\|2^{|k|})|g(k)| \\
&= \|f\|\,\|g\|.
\end{aligned}
$$

The only thing left is to show that $\mathscr{A}$ is complete. Suppose that $\{f_n\}$ is a Cauchy sequence, i.e., for given ϵ there exists N such that for all $m > n \geq N$ it holds that $\sum_{k\in\mathbb{Z}} |f_m(k) - f_n(k)|2^{|k|} < \epsilon$. This implies that $\{f_n(k)\}$ is Cauchy and thus $f_n(k) \to f(k)$ for some f. Note that $\sum_{k\in\mathbb{Z}} |f_m(k) - f(k)|2^{|k|} \leq \epsilon$, i.e., $f_m - f \in \mathscr{A}$, and therefore $f \in \mathscr{A}$.

(2) For $f \in \mathscr{A}$ define $g_f(z) = \sum_{n\in Z} f(n)z^n$, which is well-defined on K. Given $z_0 \in K$, consider homomorphism $\phi_{z_0} : f \mapsto g_f(z_0)$, then $J_{z_0} = \ker\phi_{z_0}$ is a maximal ideal, since $g_f(z)$ is continuous. Obviously $z_0 \mapsto J_{z_0}$ is injective, and we shall show that this mapping is surjective also. Let $J \in \mathfrak{M}$, we want to find $z_0 \in K$ such that $\phi_J = \phi_{z_0}$, i.e., $\langle\phi_J, f\rangle = g_f(z_0)$ for all $f \in \mathscr{A}$. Let $h \in A$ such that $g_h(z) = z$, then $g_{h^n}(z) = z^n$. Since $|\langle\phi_J, h\rangle^n| = |\langle\phi_J, h^n\rangle| \leq \|h^n\| = 2^{|n|}$ for all n, it follows that $\langle\phi_J, h\rangle \in K$, say z_0, then by continuity of ϕ_J, $\langle\phi_J, f\rangle = g_f(z_0)$ for all $f \in \mathscr{A}$. Actually $f \mapsto g_f$ is the Gelfand representation of $\mathscr{A}$. □

3.2 Let $\mathscr{A}$ be the semi-simple commutative Banach algebra in Problem 2.11. Find its maximal ideal space $\mathfrak{M}$. For $x \in [0,1]$ define

$$J = \{f \in \mathscr{A} : f(x) = f'(x) = 0\},$$

show that J is a closed ideal of $\mathscr{A}$ and $\mathscr{A}/J$ is a two-dimensional algebra with one-dimensional radical.

Proof. Similarly to Theorem 5.3.1, we have that $\mathfrak{M}$ is homeomorphic and isomorphic to $[0,1]$. For a given $x \in [0,1]$, it is obvious that J is a closed ideal. It is easy to verify that $\phi : \mathscr{A}/J \to \mathbb{C}^2$ as $\phi(f) = (f(x), f'(x))$ is an isomorphism, where the multiplication of $\mathbb{C}^2$ is defined as

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, x_1y_2 + x_2y_1).$$

It is then easy to see that the identity element of $\mathbb{C}^2$ is $(1,0)$ and all the non-invertible elements have the form $(0, z)$. In fact, those non-invertible elements constitute the only maximal ideal in $\mathscr{A}$. Hence the radical is one-dimensional. □

3.3 Let $\mathfrak{M}$ be a compact T_2 space. Show that there exists a one-to-one correspondence between the set of all closed subsets of M and the set of closed ideals of $C(M)$.

Proof. Suppose X is a closed subset of M. Define $J_X = \{f \in C(M) : f(x) = 0 \text{ for all } x \in X\}$, which is clearly a closed ideal of $C(M)$. The map $X \mapsto J_X$ is clearly injective by Urysohn's Lemma. Now we prove the converse. Suppose that J is a closed ideal of $C(M)$, let $X = \bigcap_{f\in J}\{x : f(x) = 0\}$, which is the intersection of closed sets, and thus closed. From Theorem 5.3.1 we know that X is non-empty and thus $J \subseteq J_X$. Let $f \in J_X$. Given any positive ϵ let $F_\epsilon = \{x \in M : |f(x)| \geq \epsilon\}$ then F_ϵ is compact and disjoint from X. If there exists $g_\epsilon \in J$ such that $g_\epsilon = 1$ on F_ϵ and $|g_\epsilon| \leq 1$ on M, then $fg_\epsilon \in J$ and $\|f - fg_\epsilon\| \leq \epsilon$, which implies that $f \in I$ since J is closed. Therefore $J = J_X$.

Now we shall construct such g_ϵ. For any $x \notin F_\epsilon$ there exists $f_x \in J$ such that $f_x(x) \neq 0$ and thus $f_x(x)$ is non-zero on a neighbhourhood of x. Since F_ϵ is compact, we can choose a finite cover of the neighbourhoods corresponding to $x_1, \ldots, x_n$. Then

$$h_\epsilon(x) = \sum_{i=1}^{n} f_{x_i}(x)\overline{f_{x_i}(x)} = \sum_{i=1}^{n} |f_{x_i}(x)|^2$$

is contained in J and positive on F_ϵ. Since F_ϵ is continuous, h_ϵ attains minimum c at some x_0. Now let $k_\epsilon(x) = \max\{h_\epsilon(x), c\}$, then $k_\epsilon \in C(M)$, $k_\epsilon > 0$ everywhere and $k_\epsilon = h_\epsilon$ on F_ϵ. Finally let $g_\epsilon = k_\epsilon^{-1} h_\epsilon$. ☐

3.4 Let $\mathscr{A} = C^n[0,1]$ with norm

$$\|f\| = \sup_{0 \le t \le 1} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!}.$$

Show that, under the usual addition, multiplication and scalar multiplication of functions, $\mathscr{A}$ is a Banach algebra. How to characterise its maximal ideals?

Proof. It is easy to verify that $\mathscr{A}$ is an algebra and $\|fg\| \le \|f\| \, \|g\|$. To show the completeness of $\mathscr{A}$ recall that if $u_n \to u$ and $u_n' \to v$ uniformly, where u_n, u, u_n', v are continuous, then $u' = v$. It is then straightforward to see that $\mathscr{A}$ is complete, and thus a Banach algebra. Similar to the case of $C^1[0,1]$, the maximal ideals are homomorphic and isomorphic to $[0,1]$. ☐

3.5 Define positionwise multiplication on ℓ^1. Show that ℓ^1 is a commutative Banach algebra without identity. Furthermore, show that

(1) There exists a one-to-one correspondence between $\mathfrak{M}$ and $\mathbb{Z}$;

(2) Gelfand topology is discrete topology;

(3) There exists a one-to-one correspondence between the set of closed ideals of ℓ^1 and the set of subsets of $\mathbb{Z}$.

Proof. It is clear that ℓ^1 is commutative. If it has identity e, then $e = (1, 1, 1, \dots)$, which is not in ℓ^1. Hence ℓ^1 has no identity. It is also clear that $\|xy\| \le \|x\| \, \|y\|$. Therefore ℓ^1 is a Banach algebra. For the next problems, we can replace $\mathbb{Z}$ by $\mathbb{Z}^+$, the set of non-negative integers.

(1) Let $n \in \mathbb{Z}^+$. Consider $J_n = \{x \in \ell^1 : x_n = 0\}$, which is clearly an ideal. It is also easy to see that J_n is maximal, and the map $n \mapsto J_n$ is injective.
For $n = 0$, consider $J_0 = \{x \in \ell^1 : \exists N_x \forall n \ge N_x,\ x_n = 0\}$, which is also clearly a maximal ideal.
Now we prove the converse. Let J be a maximal ideal. If for any k there exists $x_k \in J$ such that $x_{k,k} \neq 0$, then $x_{k,k} e_k \in J$ and thus $e_k \in J$. For any $x \in \ell^1$, all finite truncations of x are in J. We conclude that $J_0 \subseteq J$ and by maximality of J, we have that $J = J_0$. Otherwise, there exists n such that $x_n = 0$ for all $x \in J$, so that $J \subseteq J_n$ and thus $J = J_n$ by maximality of J.

(2) From part (1) it is clear that $\hat{x}(n) = x_n$ for all $x \in \ell^1$. There exists $x \in \ell^1$ such that $x_i \neq x_j$ for all pair $i \neq j$. Since $\hat{x}$ is continuous under Gelfand topology, it must be discrete.

(3) For any non-empty subset $I \subseteq \mathbb{Z}^+$, consider $J_I = \{x \in \ell^1 : x_n = 0 \text{ for all } n \in I\}$. It is clear that J_I is a closed ideal, and the map $I \mapsto J_I$ is injective. We shall show the converse. Let J be a closed ideal. Consider $I = \bigcap_{x \in J} Z(x)$ where $Z(x) = \{n \in \mathbb{Z}^+ : x_n = 0\}$. We claim that $I \neq \emptyset$.
If for any k there exists $x_k \in J$ such that $x_{k,k} \neq 0$, then $x_{k,k} e_k \in J$ and thus $e_k \in J$. As a consequence, for any $x \in \ell^1$, all finite truncations of x are in J. Since J is closed, we conclude that $J = \ell^1$. Contradiction. Hence $I \neq \emptyset$.
We have seen that $J \subseteq J_I$. Now we shall show that $J = J_I$. From a similar argument to the above, we have $e_k \in J$ for all $k \notin I$, hence any $x \in J_I$ can be approximated by its finite truncations, and $\overline{J} = J_I$. The conclusion follows from the closedness of J. ☐

3.6 Let $\mathscr{A}$ be a semi-simple commutative Banach algebra. Prove that $\Gamma(\mathscr{A})$ is closed in $C(\mathfrak{M})$ if and only if there exists a constant K such that $\|a\|^2 \le K \|a^2\|$ for all $a \in \mathscr{A}$.

Proof. We use the notations of Problem 5.2.12.

`If': The assumptions implies that $r \geq \frac{1}{K} > 0$, thus $s > r > 0$, i.e., $s\|a\| \leq \|\hat{a}\|$ for all $a \in \mathscr{A}$. Suppose that $\{\widehat{a_n}\}$ is a Cauchy sequence in $C(\mathfrak{M})$, then $\|a_n\|$ is a Cauchy sequence in $\mathscr{A}$. Hence there exists $a \in \mathscr{A}$ such that $a_n \to a$. By continuity of Γ, it holds that $\widehat{a_n} \to \widehat{a}$, whence it follows that $\Gamma(\mathscr{A})$ is closed.

`Only if': It suffices to show that $s \neq 0$ then $r \geq s^2 > 0$. The Gelfand transform $a \mapsto \hat{a}$ is a continuous isomorphism between two Banach spaces, $\mathscr{A}$ and $C(\mathfrak{M})$, as $\Gamma(\mathscr{A})$ is closed and $\mathscr{A}$ is semisimple. By Open Mapping Theorem, Γ^{-1} exists and is continuous, that is, there exists c such that $\|\Gamma^{-1}\hat{a}\| \leq c\|\hat{a}\|$ for all $\hat{a} \in C(\mathscr{A})$, i.e., $\|a\| \leq c\|\hat{a}\|$ for all $a \in \mathscr{A}$. □

4 C^*-algebra

4.1 Let $\mathscr{A}$ be a commutative Banach algebra. Suppose that $\mathscr{A}$ is semi-simple, then every involution on $\mathscr{A}$ is continuous.

Proof. By Closed Graph Theorem it suffices to show that involution is closed, that is, suppose that $a_m \to 0$ and $a_m^* \to a$, we need to show that $a = 0$. Since $\mathscr{A}$ is semi-simple, we need only to show that $r(a) = 0$. In factm $r(a) \leq r(a - a_m^*) + r(a_m^*) \leq \|a - a_m^*\| + \|a_m^*\| \to 0$. □

4.2 Verify that $L^1(\mathbb{R})$ is a Banach algebra under the multiplication of convolution

$$(x * y)(t) = \int_{\mathbb{R}} x(s)y(t-s)ds$$

and involution

$$x^*(t) = \overline{x(-t)}.$$

Is it a C^*-algebra?

Proof. We have that

$$\begin{aligned}(x * y)^*(t) = \overline{(x * y)(-t)} = \overline{\int_{\mathbb{R}} x(s)y(-t-s)ds} = \int_{\mathbb{R}} \overline{x(s)y(t-s)}ds \\ = \int_{\mathbb{R}} \overline{x(-s)y(-t+s)}ds = \int_{\mathbb{R}} x^*(s)y^*(t-s)ds = x^*y^* = y^*x^*\end{aligned}$$

since convolution is commutative. The rest assumptions are easy to verify. Hence $L^1(\mathbb{R})$ is a Banach algebra with involution as defined. It is not a C^*-algebra. Take $f(x) = (\operatorname{sgn} x)\chi_{[-1,1]}(x)$, then

$$(f^*f)(x) = \begin{cases} 2 - 3|s|, & |s| \leq 1; \\ -2 + |s|, & 1 < |s| \leq 2; \\ 0, & \text{otherwise.} \end{cases}$$

and $\|f^*f\| = \frac{8}{3}$ while $\|f\| = 2$, so $\|f^*f\| \neq \|f\|^2$. □

4.3 Consider the algebra of analytic functions $A_0(\mathbb{D})$. ~~Show that conjugation $f \mapsto \overline{f}$ is an involution on $A_0(\mathbb{D})$ and under which $A_0(\mathbb{D})$ is a C^*-algebra.~~ Show the map $* : f \mapsto f^*(z) = \overline{f(\bar{z})}$ is an involution, too. Is $A_0(\mathbb{D})$ a C^*-algebra with this involution?

Proof. The statement crossed out does not hold, as $\overline{f}$ is not necessarily analytic when f is analytic.

It is obvious that $*$ is an involution. Let $f(z) = iz + 1$, then $f^*(z) = -iz + 1$, so $\|f^*f\| = \sup_{|z|\leq 1} |1 + z^2| = 3$ while $\|f\|^2 = 4$. Hence $A_0(\mathbb{D})$ with $*$ as involution is not a C^*-algebra. □

4.4 Let $\mathscr{A}$ be a Banach algebra with involution $*$ and S be a subset of A. We say that S is *regular* if

(1) S is commutative, i.e., $ab = ba$ for any $a, b \in S$;

(2) S is closed under involution, i.e., whenever $a \in S$ it holds that $a^* \in S$.

Obviously for any $a \in S$ we have $aa^* = a^*a$. A regular subset is said to be *maximal* if it is not a proper subset of any normal subset. Let $\mathscr{B}$ be a maximal regular subset of $\mathscr{A}$, show that

(1) $\mathscr{B}$ is a closed commutative subalgebra of $\mathscr{A}$;

(2) $\forall a \in \mathscr{B}$, it holds that $\sigma_{\mathscr{B}}(a) = \sigma_{\mathscr{A}}(a)$.

Proof. (1) Let $a, b \in \mathscr{B}$. It is clear that $a+b$ commutes with $\mathscr{B}$ because each of a and b commutes with $\mathscr{B}$. Hence $\mathscr{B}$ is closed under addition since $\mathscr{B}$ is maximal. Similarly it can be shown that S is closed under multiplication and scalar multiplication. The laws of associativity and distributivity are inherited from $\mathscr{A}$. Therefore $\mathscr{B}$ is a commutative subalgebra.
Now we shall show that $\mathscr{B}$ is closed. Suppose $a_n \to a$ and $\{a_n\} \subseteq \mathscr{B}$. From the continuity of multiplication, we have that a commutes with $\mathscr{B}$. Since $\mathscr{B}$ is maximal, it must hold that $a \in \mathscr{B}$. Therefore $\mathscr{B}$ is closed.

(2) Clearly $\sigma_{\mathscr{A}}(a) \subseteq \sigma_{\mathscr{B}}(a)$. If $\lambda \notin \sigma_{\mathscr{A}} a$ then $(\lambda e - a)^{-1}$ exists. Since $(\lambda e - a)^{-1}$ commutes with $\mathscr{B}$ and $\mathscr{B}$ is maximal, we have $(\lambda e - a)^{-1} \in \mathscr{B}$ and therefore $\lambda \notin \sigma_{\mathscr{B}}(a)$. Hence $\sigma_{\mathscr{B}}(a) = \sigma_{\mathscr{A}}(a)$. □

4.5 Let $\mathscr{A}$ be a C^*-algebra, show that

(1) Let a be an hermitian element, then $\sigma(a) \subset \mathbb{R}^1$;

(2) If a is normal ($aa^* = a^*a$) then $\|a\| = r(a)$;

(3) $\|a\|^2 = r(aa^*)$.

Proof. Let $\mathscr{B}$ be the closure of the subalgebra generated by e, a, elements of form $(\lambda e - a)^{-1}$ for $\lambda \in \rho(a)$. Then $\mathscr{B}$ is commutative and $\sigma_{\mathscr{B}}(a) = \sigma_{\mathscr{A}}(a)$. We consider Gelfand transform on $\mathscr{B}$ for part (1) and (2).

(1) It follows immediately from Arens' Lemma.

(2) In fact, normality of a means that a and a^* is commutative. Hence the proof of Theorem 5.4.8 (3) holds, and $\|a\| = \|\Gamma a\| = r(a)$.

(3) It is clear that aa^* is normal, hence by part (2), $r(aa^*) = \|aa^*\| = \|a^*\|^2 = \|a\|^2$. □

4.6 Let $\mathscr{A}$ be a C^*-algebra and $a \in \mathscr{A}$. We say a is positive, denoted by $a \geq 0$, if a is hermitian and $\sigma(a) \subseteq [0, +\infty]$. Show that

(1) $\forall a \in \mathscr{A}$, $aa^* \geq 0$;

(2) If $a, b \in \mathscr{A}$, $a \geq 0$, $b \geq 0$ then $a + b \geq 0$;

(3) For all $a \in \mathscr{A}$, $e + aa^*$ is invertible in $\mathscr{A}$.

Proof. (1) It is trivial that aa^* is hermitian. As in the previous problem, we consider Gelfand transform in $C^*(a)$, which is commutative since $a = a^*$. Thus $\Gamma aa^* = \Gamma a \Gamma a^* = |\Gamma a|^2 \geq 0$, therefore $aa^* \geq 0$.

(2) First consider $C^*(a)$, which is commutative. We claim that $\|\lambda e - a\| \leq \lambda$ for all $\lambda \geq \|a\|$. In fact, $\Gamma(\lambda e - a) = \lambda - \Gamma(a)$. Since $a \geq 0$, $\Gamma(a)(J) \in [0, \lambda]$, we have that $\Gamma(\lambda e - a)(J) \in [0, \lambda]$ and thus $\|\lambda e - a\| = \|\Gamma(\lambda e - a)\| \leq \lambda$. Similarly for $b \geq 0$, we can find μ such that $\|\mu e - b\| \leq \mu$. Hence $\|(\lambda + \mu)e - (a + b)\| \leq \|\lambda e - a\| + \|\mu e - b\| \leq \lambda + \mu$. Now in $C^*(a + b)$,

$$\lambda + \mu \geq \|\lambda + \mu)e - (a + b)\| = \|\Gamma((\lambda + \mu)e - (a + b))\| \geq |(\lambda + \mu) - \Gamma(a + b)(J)| \geq (\lambda + \mu) - \Gamma(a + b)(J)$$

for all J, whence it follows that $\Gamma(a + b)(J) \geq 0$ for all J, that is, $a + b \geq 0$.
Similarly, by choosing an appropriate subalgebra, we may, without loss of generality, assume that $\mathscr{A}$ is commutative. Then $\Gamma(a + b) = \Gamma a + \Gamma b \geq 0$. Hence $a + b \geq 0$.

(3) This is a direct corollary of part (1). $\square$

4.7 Let $\mathscr{X}$ be a Hilbert space and $\mathscr{A}$ a C^*-algebra of $L(\mathscr{X})$. Define

$$\mathscr{A}^c = \{T \in L(\mathscr{X}) : TA = AT, \forall A \in \mathscr{A}\},$$

which is called the *centre* of $\mathscr{A}$. Show that $\mathscr{A}^c$ is a C^*-algebra and closed under weak topology.

Proof. It is easy to see that $\mathscr{A}^c$ is closed under addition, multiplication, scalar multiplication and involution, whence it follows that $\mathscr{A}^c$ is a C^*-algebra. Now we shall show it is closed under weak topology. Suppose that $T_n \rightharpoonup T$, $\{T_n\} \subseteq \mathscr{A}^c$. Note that $\mathscr{X}$ is a Hilbert space and thus reflexive, we have $T_n x \to Tx$ for all $x \in H$. Hence $AT_n x \to ATx$, that is, $T_n Ax \to ATx$ for all $A \in \mathscr{A}$ and $x \in H$. Recall that multiplication is continuous, $T_n A \to TA$, hence $TAx = ATx$ for all $x \in H$, that is exactly $TA = AT$ for all $A \in \mathscr{A}$. Therefore $T \in \mathscr{A}^c$ and $\mathscr{A}^c$ is closed under weak topology. $\square$

5 Normal Operators in Hilbert spaces

5.1 Let N be a normal operator in a Hilbert space. Show that

(1) If $\phi \in C(\sigma(N))$ then $\sigma(\phi(N)) = \phi(\sigma(N))$;

(2) If $\phi \in C(\sigma(N))$, $\psi \in C(\sigma(\phi(N)))$, then $(\psi \circ \phi)(N) = \psi(\phi(N))$.

Proof. (1) Since $\lambda I - \phi(N) = (\lambda - \phi)(N)$, we have that $\lambda I - \phi(N)$ is invertible $\Leftrightarrow \phi(z) \neq \lambda$ for all $z \in \sigma(N)$ $\Leftrightarrow \lambda \notin \phi(\sigma(N))$.

(2) By part (1), $\psi(\phi(N))$ is well-defined. It is clear that the conclusion holds when ψ is a polynomial of z and $\bar{z}$. For a general ψ, pick a sequence of polynomials $\psi_n \to \psi$, then $\psi_n \circ \phi \to \psi \circ \phi$, the conclusion follows from the continuity of the isomorphism between $\mathscr{A}_N$ and $C(\sigma(N))$. $\square$

5.2 Show that N is normal iff $\|Nx\| = \|N^*x\|$ for all x.

Proof. $N^*N = NN^* \Leftrightarrow \langle N^*Nx, x\rangle = \langle NN^*x, x\rangle \Leftrightarrow \langle Nx, Nx\rangle = \langle N^*x, N^*x\rangle \Leftrightarrow \|Nx\|^2 = \|N^*x\|^2$. $\square$

5.3 Let N be a normal operator. Show that

(1) $\|N\| = \sup\{|\lambda| : \lambda \in \sigma(N)\}$, and if P is a polynomial then

$$\|P(N)\| = \sup\{|P(\lambda)| : \lambda \in \sigma(N)\}$$

(2) for $A \in L(\mathscr{X})$ it holds that $\|A\|^2 = r(AA^*)$.

Proof. (1) When N is normal, $P(N)$ is normal, too. The conclusion follows from Problem 5.4.5(2).

(2) Problem 5.4.5(3). $\square$

5.4 Show that the product of two positive operators is positive.

Proof. Notice that $\Gamma(ab) = \Gamma a \Gamma b$. $\square$

5.5 Let $A, B \in L(\mathscr{X})$, $0 \leq A \leq B$. Suppose that A and B are commutative, then $A^2 \leq B^2$. However, this is not necessarily true when A and B are not commutative.

Proof. Note that $B^2 - A^2 = (B + A)(B - A)$ because A and B are commutative. The conclusion then follows from the previous problem and Problem 5.4.6.

Take $\mathscr{X} = \mathbb{R}^2$. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}$, then $B \geq A \geq 0$ but $AB \neq BA$. Then $B^2 - A^2 = \begin{pmatrix} 4 & 32 \\ 10 & 79 \end{pmatrix}$, which is not positive. □

5.6 Let N be a normal operator, then there exists $P, Q \in L(\mathscr{X})$, P is positive and unique, Q is unitary, such that

$$N = PQ = QP.$$

This is called polar decomposition of N.

Proof. Put $p(z) = |z|$ and $q(z) = z/|z|$ if $z \neq 0$, $q(0) = 1$. Then p and q are continuous functions on $\sigma(N)$. Put $P = \tilde{\Gamma}^{-1}p$ and $Q = \tilde{\Gamma}^{-1}q$. Since $p \geq 0$, we know that $P \geq 0$. Since $q\bar{q} = 1$, $QQ^* = Q^*Q = I$. Since $z = p(z)q(z)$, the relation $N = PQ = QP$ follows from the symbolic calculus.

Now we prove that uniqueness of P. If $N = QP$, P positive and Q unitary, then $N^*N = P^*Q^*QP = P^*P = P^2$. The uniquesness of P follows from the uniqueness of $(N^*N)^{\frac{1}{2}}$. □

5.7 Let $\mathscr{X}$ be a locally compact topological space and $\mathscr{H}$ a Hilbert space. Definition 5.5.13 gives a spectral family $(\mathscr{X}, \mathscr{B}, E)$, show that if $\Delta_1, \Delta_2 \in \mathscr{B}$ then

$$E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2).$$

Proof. First we show that if $\Delta_1 \cap \Delta_2 =$ then $E(\Delta_1)E(\Delta_2) = 0$. Note that $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$ is a projector, hence $(E(\Delta_1) + E(\Delta_2))^2 = E(\Delta_1) + E(\Delta_2)$, whence it follows that $E(\Delta_1)E(\Delta_2) = 0$.

In the general case, $E(\Delta_1) = E(\Delta_1 \setminus \Delta_2) + E(\Delta_1 \cap \Delta_2)$ and $E(\Delta_2) = E(\Delta_1 \setminus \Delta_1) + E(\Delta_1 \cap \Delta_2)$. Hence

$$E(\Delta_1)E(\Delta_2) = (E(\Delta_1 \setminus \Delta_2) + E(\Delta_1 \cap \Delta_2))(E(\Delta_1 \setminus \Delta_1) + E(\Delta_1 \cap \Delta_2)).$$

Notice that $\Delta_1 \setminus \Delta_2$, $\Delta_2 \setminus \Delta_1$ and $\Delta_1 \cap \Delta_2$ are mutually disjoint, the conclusion follows from the expansion of the right-hand side. □

5.8 Let N be a normal operator and E is the associated spectral family. Then for any Borel set $\Delta \subseteq \mathbb{C}$, $E(\Delta)$ is contained in the weak closure of the C^* algebra generated by N and N^*. Let $S \in L(\mathscr{X})$, $SN = NS$, show that $SE(\Delta) = E(\Delta)S$.

Lemma 1 The unit ball of a Banach space X is weak*-dense in the unit ball of X^{**}.

Proof of Lemma 1 Suppose there exists $x^{**} \in X^{**}$, $\|x\|_{X^{**}} \leq 1$, x is not in the weak*-closure of $B(X) \subseteq X^{**}$. Hence there exists a functional $f \in X^*$ and a real number c, by Hahn-Banach Theorem, such that $\Re\langle x, f\rangle < c < \Re\langle x^{**}, f\rangle$ for all x in the unit ball of X. Since 0 is contained in the unit ball of X, $c > 0$. We can then divide by c and replace f by $c^{-1}f$, and assume that there exists $f \in X^*$ such that $\Re\langle x, f\rangle < 1 < \Re\langle x^{**}, f\rangle$. Since $-ix$ is in the unit ball of X whenever x is in it, this implies that $\Im\langle x, f\rangle < 1$ and thus $|\langle x, f\rangle| \leq 1$ for all $\|x\| \leq 1$. Hence $\|f\|_{X^*} \leq 1$, then $1 < |\langle x^{**}, f\rangle| \leq \|x^{**}\| \, \|f\| \leq 1$. Contradiction. □

Lemma 2 Let X be a compact space and $\phi \in B(X)$. Then there exists a sequence of continuous functions $\{u_n\}$ on X such that $\|u_n\| \leq \|\phi\|$ for all n and $\int u_n \, dm \to \int \phi \, dm$ for all $m \in M(X)$.

Proof of Lemma 2 Note that $C(X)$ is a Banach space. The previous lemma tells us that the unit ball of $C(X)$ is weak*-dense in the unit ball of $C(X)^{**} = M(X)^*$. Identify $B(X)$ with a subspace of $M(X)^*$, and we are done. □

Proof. By Lemma 2 and Theorem 5.5.14, there exist $\{f_n\} \subseteq C(\sigma(N))$ such that

$$(f_n(N)x, y) = \int f_n(z)\, d(E(z)x, y) \to \int \chi_\Delta(z)\, d(E(z)x, y) = (\chi_\Delta(N)x, y)$$

for all x and y. Since $f_n(N) \in \mathscr{A}_N$, the result above shows that χ_Δ is in the weak closure of $\mathscr{A}_N$. Finally note that $\chi_\Delta(N) = E(\Delta \cap \sigma(N)) = E(\Delta)$. Also we have $\|f_n\| \le \|\chi_\Delta\| = 1$, thus by Theorem 5.5.12(4), $SE(\Delta) = E(\Delta)S$. ☐

5.9 Let N be a normal operator. Prove that

(1) N is unitary $\Leftrightarrow \sigma(N) \subset S^1$;

(2) N is self-adjoint $\Leftrightarrow \sigma(N) \subset \mathbb{R}^1$;

(3) N is positive $\Leftrightarrow \sigma(N) \subset \mathbb{R}^1_+$;

Proof. (1) N is unitary $\Rightarrow 1 = \Gamma(NN^*) = (\Gamma N)(\Gamma N^*) = \Gamma N \overline{\Gamma N} = |\Gamma N|^2 \Rightarrow \sigma(N) \subseteq S^1$. Reversing the procedure above, $\sigma(N) \subseteq S^1 \Rightarrow \Gamma(NN^*) = 1$ and similarly $\sigma(N) \subseteq S^1 \Rightarrow \Gamma(N^*N) = 1$. It follows from the injectivity of Γ that $NN^* = N^*N = I$, i.e., N is unitary.

(2) N is self-adjoint $\Leftrightarrow \Gamma N = \Gamma N^* = \overline{\Gamma N} \Leftrightarrow \Gamma N \in \mathbb{R} \Leftrightarrow \sigma(N) \subseteq \mathbb{R}^1$.

(3) See Theorem 5.5.5. ☐

5.10 Suppose that N is a normal operator and $\sigma(N)$ is countable. Then there is an orthonormal basis $B = \{y\} \subseteq \mathscr{X}$, where y's are eigenvectors of N, and Fourier expansion

$$x = \sum_{y \in B} (x, y) y, \forall x \in \mathscr{X},$$

where the Fourier coefficients $(x, y) = 0$ except for countably many ones.

Proof. We claim that if x and y belong to different eigenvalues, say λ and μ, then $\langle x, y\rangle = 0$. In fact, $E(\{\lambda\})\mathscr{X} = \ker(\lambda I - N)$ and $E(\{\lambda\})\mathscr{X}$ is orthogonal to $E(\{\mu\})\mathscr{X}$.

Since $\ker(\lambda I - T)$ is closed, so we can choose an orthonormal basis. Combining those bases of each $\lambda \in \sigma(N)$, we obtain an orthonormal set in $\mathscr{H}$. We shall show that it is complete, i.e., $x \perp \{y\}$, or, $x \perp \ker(\lambda I - T)$ for all λ implies that $x = 0$.

Let $P_\lambda = E(\{\lambda\})$. Suppose that $x \perp \ker(\lambda I - T) = \operatorname{im} P_\lambda$, then $x \in \operatorname{im} P_\lambda^\perp = \ker P_\lambda$, that is, $P_\lambda x = 0$. Suppose that $\sigma(N) = \{\lambda_1, \lambda_2, \dots\}$, then

$$x = Ix = E(\sigma(N)x = \lim_{n\to\infty} \sum_{k=1}^{n} P_{\lambda_k} x = 0$$

as desired. Fourier expansion follows from Theorem 1.6.23 and 1.6.25. ☐

5.11 Let N be a normal operator on $\mathscr{H}$. Show that N is compact if and only if all following three conditions hold:

(1) $\sigma(N)$ is countable;

(2) If $\sigma(N)$ has a limit point, it must be 0;

(3) If $\lambda \in \sigma(N)$, $\lambda \neq 0$, then $\dim E(\{\lambda\})\mathscr{H} < +\infty$.

Proof. `Only if': This is rather straightforward. Since N is compact, we have that $\sigma(N) \setminus \{0\} = \sigma_p(N) \setminus \{0\}$ and $\sigma_p(N)$ has at most one limit point 0 (Theorem 4.3.1). Hence (1) and (2) hold, while (3) is just Fredholm Theorem (Theorem 4.2.10(3)), i.e., $\dim\ker(\lambda I - N) < +\infty$.

`If': Suppose that $|\lambda_1| \geq |\lambda_2| \geq \cdots$. Denote $P_n = E(\{\lambda_1, \ldots, \lambda_n\})$ and P_0 be the projector along $\ker N$. Let $N_n = NP_n$. Then $\dim N_n\mathscr{H} \leq \sum E(\lambda_i)\mathscr{H} < +\infty$, which implies that N_n is finite-rank and thus compact. If $\sigma(N)$ is finite then $N_n = N$ for some n and N is therefore compact. Now assume that $\sigma(N)$ is infinite and thus $\lambda_n \to 0$. We want to show that $N_n \to N$.

Construct an orthonormal basis as in the previous problem, we have that $x = \sum(x, e_i)e_i$ for all x and $Nx = \sum \lambda_i(x, e_i)e_i$. Note that $N_n x$ is just a partial sum of Nx, containing all terms up to λ_n (inclusive). Then the argument in Remark 1 after Theorem 4.4.7 is valid, and we see that $N_n \to N$ and N is therefore compact. □

5.12 Let N be a compact normal operator, show that

(1) There exists λ, an eigenvalue of N, such that $\|N\| = |\lambda|$;

(2) If $\phi \in C(\sigma(N))$ and $\phi(0) = 0$, then $\phi(N)$ is compact.

Proof. (1) Suppose that $|\lambda_1| \geq |\lambda_2| \geq \cdots$. We claim that $\|N\| = |\lambda_1|$. Problem 5.5.10 and Problem 5.5.11 tells us that N is diagonalisable, $Nx = \sum \lambda_i(x, e_i)e_i$. Hence

$$\|Nx\| = \left\|\sum \lambda_i(x, e_i)e_i\right\| = \left(\sum |\lambda_i|^2 |(x, e_i)|^2\right)^{\frac{1}{2}} \leq |\lambda_1| \left(\sum |(x, e_i)|^2\right)^{\frac{1}{2}} \leq |\lambda_1|\,\|x\|,$$

hence $\|N\| \leq |\lambda_1|$. For $x \in \ker(\lambda_1 I - N)$, it holds that $\|Nx\| = \|\lambda_1\|, \|x\|$. Therefore $\|N\| = |\lambda_1|$.

(2) It is clear that $\phi(N)$ is normal. Then we shall verify the three conditions in the previous problem are satisfied. Firstly, $\sigma(\phi(N)) = \phi(\sigma(N))$ is countable because $\sigma(N)$ is countable. Secondly, suppose y_0 is a limit point of $\sigma(\phi(N)) = \phi(\sigma(N))$, then there exists $\{x_n\} \subseteq \sigma(N)$ (x_n mutually different) such that $\phi(x_n) \to y_0$. Since $\sigma(N)$ is compact, we can find a subsequence of x_n, still denoted by x_n, such that $x_n \to x_0$ for some $x_0 \in \sigma(N)$. We know that $x_0 = 0$, and thus $y_0 = \phi(x_0) = \phi(0) = 0$. Thirdly, note that $E_{\phi(N)}(\lambda) = E_N(\phi^{-1}(\lambda))$ and $\phi^{-1}(\lambda)$ is finite when $\lambda \neq 0$. It follows immediately that $\dim E_{\phi(N)}(\lambda)\mathscr{H} < \infty$ when $\lambda \neq 0$. □

5.13 Let N be a normal operator and E the spectral family corresponding to N. Let $\phi \in C(\sigma(N))$ and $\omega = \ker\phi$. Show that

$$\ker\phi(N) = \operatorname{im} E(\omega).$$

Proof. $\ker\phi(N) = \operatorname{im} E_{\phi(N)}(\{0\}) = \operatorname{im} E_N(\phi^{-1}(\{0\})) = \operatorname{im} E_N(\omega)$, as it is not difficult to show that $E_{\phi(N)}(\Omega) = E_N(\phi^{-1}(\Omega))$ using Problem 5.5.1. □

5.14 Let N be a normal operator, O an open set containing $\sigma(N)$ with a Jordan boundary. Suppose that ϕ is analytic on a neighbourhood of $\sigma(N)$ and O is contained in the analytic domain of ϕ. Show that

$$\phi(N) = \frac{1}{2\pi i} \int_{\partial O} \phi(z)(zI - N)^{-1} dz.$$

Proof. It is known in Theorem 2.6.9 that $(zI - N)^{-1}$ is analytic on its domain. Next, we show that if F is an operator-valued analytic function over a domain Ω then $\int_{\partial\Omega} F dz = 0$. In fact, let ϕ be any continuous functional, $\phi \circ F$ is analytic. It follows from continuity that $\phi(\int_{\partial\Omega} F dz) = \int_{\partial\Omega} \phi \circ F dz$, which equals to 0 by Cauchy's Theorem. Note that it holds for any continuous functional ϕ, by Hahn-Banach Theorem, it must hold that $\int_{\partial\Omega} F dz = 0$. Hence for polynomial ϕ, we can replace ∂O by a possibly larger circle outside $|z| \leq \|N\|$. It is therefore easy to verify the desired equation, using expansion $(zI - N)^{-1} = z^{-1} \sum_{i=0}^{\infty} z^{-i} A^i$. Having established the equation for polynomial ϕ, we can approximate a general ϕ by polynomials, completing the proof. □

5.15 Let N be a normal operator and C a connected component of $\sigma(N)$. Suppose that $\Gamma \subseteq \rho(N)$ is a Jordan curve, Γ encloses C and contains no other spectrum inside itself besides C. Show that

$$E(C) = \frac{1}{2\pi i} \oint_\Gamma (zI - N)^{-1} dz.$$

Proof. Let $P = \frac{1}{2\pi i} \oint_\Gamma (zI - N)^{-1} dz$. First we shall show that $P^2 = P$. Choose Γ_1 inside Γ such that Γ_1 encloses C also. Similar to the argument in the previous problem, we have that $\oint_{\Gamma_1} (zI - N)^{-1} dz = \oint_\Gamma (zI - N)^{-1} dz$.

$$\begin{aligned}
P^2 &= \frac{1}{2\pi i} \oint_\Gamma (zI - N)^{-1} dz \frac{1}{2\pi i} \oint_{\Gamma_1} (wI - N)^{-1} dw \\
&= -\frac{1}{4\pi^2} \oint_\Gamma \int_{\Gamma_1} (zI - N)^{-1} (wI - N)^{-1} dw dz \\
&= -\frac{1}{4\pi^2} \oint_\Gamma \int_{\Gamma_1} \frac{(zI - N)^{-1} - (wI - N)^{-1}}{w - z} dw dz \\
&= -\frac{1}{4\pi^2} \oint_\Gamma (zI - N)^{-1} \int_{\Gamma_1} \frac{dw}{w - z} dz - \frac{1}{4\pi^2} \oint_{\Gamma_1} (wI - N)^{-1} \oint_\Gamma \frac{dz}{z - w} dw \\
&= 0 + \frac{1}{4\pi^2} \int_{\Gamma_1} 2\pi i (wI - N)^{-1} dw \\
&= \frac{1}{2\pi i} \oint_{\Gamma_1} (wI - N)^{-1} dw = P.
\end{aligned}$$

It is clear from the definition that P commutes with any bounded operator that commutes with N. Now we shall show that the spectrum of N restricted to $\operatorname{im} P$ is contained in C, that is, $\lambda I - N$ is invertible on $\operatorname{im} P$ for $\lambda \in \sigma(N) \setminus C$. Consider

$$Q = \frac{1}{2\pi i} \oint_\Gamma \frac{1}{\lambda - z} (zI - N)^{-1} dz$$

It is clear that $QN = NQ$ and thus $QP = PQ$. In fact,

$$\begin{aligned}
Q(\lambda I - N) = (\lambda I - N)Q &= \frac{1}{2\pi i} \oint_\Gamma \frac{(\lambda - z)I + (zI - N)}{\lambda - z} (zI - N)^{-1} dz \\
&= \frac{1}{2\pi i} \oint_\Gamma (zI - N)^{-1} dz + \frac{I}{2\pi i} \oint_\Gamma \frac{dz}{\lambda - z} \\
&= P + 0 = P.
\end{aligned}$$

Note that P is the identity map on $\operatorname{im} P$, hence Q is the the inverse of $(\lambda I - N)$ on $\operatorname{im} P$. Also, the spectrum of N restricted to $\ker P$ is contained in $\sigma(N) \setminus C$, which can be shown similarly by replacing C by $\sigma(N) \setminus C$ (the connectedness of C is not essential, separation of two compact sets is).

Now we look at $E(C)$, which satisfies all the properties that we have proved for P. As a consequence, it is easy to prove that $PE(C) = P$ and $E(C)P = E(C)$. Therefore $P = E(C)$. □

6 Applications to Singular Integral Operators

No Exercises.