1 Concept of Linear Operators

($\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces in this section)

1.1 Prove that $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ iff T is a linear operator and maps bounded set in $\mathscr{X}$ into bounded set in $\mathscr{Y}$.

Proof. `Only if' part is obvious. Now we show the `if' part. If T is not a bounded operator, then we can find $\{x_n\} \subseteq X$ such that $\|T(x_n)\| > n\|x_n\|$. Let $y_n = x_n/\|x_n\|$, then $\|y_n\| = 1$ for all n while $\|Ty_n\| > n$. Contradiction. □

1.2 Let $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ and show that
(1) $\|A\| = \sup\limits_{\|x\| \le 1} \|Ax\|$; (2) $\|A\| = \sup\limits_{\|x\|<1} \|Ax\|$.

Proof. (1) On the one hand, we have

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \le \sup_{\|x\| \le 1} \|Ax\|,$$

On the other hand, it holds that for all x with $\|x\| \le 1$,

$$\|Ax\| \le \|A\| \, \|x\| \le \|A\|.$$

Taking supremum of both sides yields that

$$\sup_{\|x\| \le 1} \|Ax\| \le \|A\|.$$

(2) Let $M = \sup\limits_{\|x\|<1} \|Ax\|$ and $N = \sup\limits_{\|x\| \le 1} \|Ax\|$. It suffices to show that $M = N$. It is clear that $M \le N$. If $M < N$ then there exists $\|x_0\| = 1$ such that $\|Ax_0\| = s > M$, thus there exists $r \in (0, 1)$ such that $\|A(rx_0)\| = rs > M$. Note that $\|rx_0\| < 1$, we have met a contradiction. Therefore M and N must be the same. □

1.3 Let $f \in \mathscr{L}(\mathscr{X}, \mathbb{R}^1)$, show that
(1) $\|f\| = \sup\limits_{\|x\|=1} f(x)$; (2) $\sup\limits_{\|x\|<\delta} |f(x)| = \delta\|f\| (\forall \delta > 0)$.

Proof. (1) First of all it is obvious that $\|f\| = \sup\limits_{\|x\|=1} |f(x)|$. Note that f is an linear operator thus $f(-x) = -f(x)$, hence the absolute value symbol can be removed, yielding $\|f\| = \sup\limits_{\|x\|=1} f(x)$.

(2)

$$\frac{1}{\delta} \sup_{\|x\|<\delta} |f(x)| = \sup_{\|x\|<\delta} \left| f\left(\frac{x}{\delta}\right) \right| = \sup_{\|x\|<1} |f(x)| = \|f\|.$$

□

1.4 Let $y(t) \in C[0, 1]$, and we define a functional f over $C[0, 1]$ as

$$f(x) = \int_0^1 x(t)y(t)dt, \quad \forall x \in C[0, 1].$$

Find $\|f\|$.

Proof. Observe that

$$|f(x)| = \left|\int_0^1 x(t)y(t)dt\right| \le \int_0^1 |x(t)|\,|y(t)|dt \le \|x\| \int_0^1 |y(t)|dt,$$

thus

$$\|f\| = \sup \frac{|f(x)|}{\|x\|} \le \int_0^1 |y(t)|dt.$$

Note that $|y(t)| = \text{sgn}(y(t))y(t)$. Using Luzin's theorem, it is not difficult to show that there exists $x_n(t) \in C[0,1]$ with $\|x_n\| = 1$ such that

$$\int_0^1 |x(t)y(t) - \text{sgn}(y(t))y(t)|\,dt < \frac{1}{n}.$$

Then it follows that

$$\begin{aligned}\int_0^1 |y(t)|dt &= \left|\int_0^1 \text{sgn}(y(t))y(t)dt\right| < \left|\int_0^1 x_n(t)y(t)dt\right| + \frac{1}{n} \\ &= |f(x_n)| + \frac{1}{n} \le \|f\|\,\|x_n\| + \frac{1}{n} \le \|f\| + \frac{1}{n}.\end{aligned}$$

Letting $n \to \infty$ we obtain that

$$\|f\| \ge \int_0^1 |y(t)|dt.$$

□

1.5 Let f be a non-zero bounded linear functional over $\mathscr{X}$. Let $d = \inf\{\|x\| : f(x) = 1\}$, show that $\|f\| = 1/d$.

Proof. From the continuity of f we know that $d > 0$ and $f(x_0) = 1$ for some $\|x_0\| = d$. So it suffices to show that $\|f\| \le 1/d$, that is, $|f(x)| \le \|x\|/d$. This is obvious true for those x such that $f(x) = 0$. Assume $f(x) \neq 0$. Note that $f\left(x/f(x)\right) = 1$, then $\|x/f(x)\| \ge d$, yielding $|f(x)| \le \|x\|/d$. □

1.6 Let $f \in \mathscr{X}^*$, show that for any $\varepsilon > 0$, there exists $x_0 \in \mathscr{X}$ with $\|x_0\| < 1 + \varepsilon$ such that $f(x_0) = \|f\|$.

Proof. Let $\varepsilon > 0$. From the definition of $\|f\|$, there exists x such that

$$\frac{\|f\|}{1+\varepsilon} < \frac{|f(x)|}{\|x\|}.$$

Take $x_0 = \frac{\|f\|}{|f(x)|}x$ as desired. □

1.7 Let $T : \mathscr{X} \to \mathscr{Y}$ be a linear map and define

$$N(T) = \{x \in \mathscr{X} : Tx = 0\}.$$

(1) If $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, show that $N(T)$ is a closed subspace of $\mathscr{X}$.

(2) Does the condition that $N(T)$ is a closed subspace of $\mathscr{X}$ imply that $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$?

(3) If f is a linear functional, show that

$$f \in \mathscr{X}^* \iff N(f) \text{ is a closed subspace.}$$

Proof. (1) Trivial, as T is continuous.

(2) No. Let

$$\mathscr{X} = \left\{ (x_1, x_2, \ldots, x_n) \,\middle|\, \sum_{n=1}^{\infty} |x_n| < \infty \right\}.$$

Define the norm of $x = (x_1, \ldots, x_n, \ldots) \in \mathscr{X}$ as $\|x\| = \sup_{n\geq 1} |x_n|$. It is clear that $\mathscr{X}$ is a linear space under the usual addition and scalar multiplication. It is also easy to verify that $\mathscr{X}$ is complete, thus a Banach space. Define $f(x) = \sum_{i=1}^{\infty} x_i$. Let $a = (1, -1, 0, 0, \ldots) \in \mathscr{X}$ and define $Tx = x - af(x)$. Obviously $N(T) = \{0\}$ is closed.

We shall prove that T is unbounded. Suppose that T is bounded, then we have (note that $\|a\| = 1$)

$$|f(x)| = \|af(x)\| = \|x - Tx\| \leq \|x\| + \|Tx\| \leq (1 + \|T\|)\|x\|,$$

which indicates that $|f(x)|$ is bounded. However, it is easy to see that f is unbounded. Contradiction. Therefore, T must be unbounded.

(3) "$\Rightarrow$" follows from (1). Now we show the "$\Leftarrow$" part. Suppose that f is unbounded, there exist $\{x_n\} \subset \mathscr{X}$ such that $\|f(x_n)\| > n\|x_n\|$. Let $y_n = x_n/\|x_n\|$ then $\|y_n\| = 1$ and $|f(y_n)| > n$. Define

$$z_n = \frac{y_n}{f(y_n)} - \frac{y_1}{f(y_1)},$$

it holds that $f(z_n) = 0$ and thus $z_n \in N(f)$. On the other hand, it follows from

$$\left\| \frac{y_n}{f(y_n)} \right\| = \frac{1}{|f(y_n)|} \leq \frac{1}{n} \to 0, \quad n \to \infty$$

that $z_n \to -y_1/f(y_1)$. But $f(-y_1/f(y_1)) = -1$, which contradicts with the closedness of $N(f)$.

□

1.8 Let f be a linear functional on $\mathscr{X}$ and denote

$$H_f^{\lambda} = \{x \in \mathscr{X} \mid f(x) = \lambda\}, \quad \forall \lambda \in \mathbb{K},$$

If $f \in \mathscr{X}^*$ and $\|f\| = 1$, show that

(1) $|f(x)| = \inf\{\|x - z\| \mid \forall z \in H_f^0\}, \forall x \in \mathscr{X}$;

(2) For any $\lambda \in \mathbb{K}$, the distance to H_f^0 from any point $x \in H_f^{\lambda}$ is a constant. Give geometric explanation of (1) and (2) for $\mathscr{X} = \mathbb{R}^2, \mathbb{K} = \mathbb{R}^1$.

Proof. (1) Let $d = \inf\{\|x-z\| \mid \forall z \in H_f^0\}$. For any $\varepsilon > 0$, there exists $z_\varepsilon \in H_f^0$ such that $\|x - z_\varepsilon\| < d + \varepsilon$. Then $|f(x)| = |f(x - z_\varepsilon)| \leq \|f\| \, \|x - z_\varepsilon\| < d + \varepsilon$, whence it holds that $|f(x)| \leq d$.

On the other hand, if $y \notin H_f^0$ then for all x it holds that $x' = x - f(x)y/f(y) \in H_f^0$. Take norm on the both sides of $f(y)(x - x') = f(x)y$, we obtain that (notice that $\|y\| \neq 0$)

$$\frac{|f(y)|}{\|y\|}\|x - x'\| = |f(x)|.$$

For any $\varepsilon > 0$, there exists y such that $|f(y)|/\|y\| > \|f\| - \varepsilon$, thus

$$(\|f\| - \varepsilon)d \leq (\|f\| - \varepsilon)\|x - x'\| < |f(x)|.$$

It follows that $\|f\|d \leq |f(x)|$, or $d \leq |f(x)|$ from the arbitrariness of ε.

(2) Let $x \in H_f^\lambda$. It is implied by (1) that $d(x, H_f^0) = |f(x)| = |\lambda|$.

Geometric demonstration($\mathscr{X} = \mathbb{R}^2, \mathbb{K} = \mathbb{R}^1$). Let $e_1 = (1, 0)$, $e_2 = (0, 1)$, $f_1 = f(e_1)$, $f_2 = f(e_2)$. Then for $x = (a, b)$, $f(x) = af_1 + bf_2$, $\|f\| = 1$ implies that $\sqrt{f_1^2 + f_2^2} = 1$. Hence $|f(x)| = |af_1 + bf_2|$ gives the distance from point $x = (a, b)$ to the line $f_1x + f_2y = 0$, or, $|f(x)| = d(x, H_f^0)$. Since H_f^0 and H_f^δ are parallel lines, the distance between them is $|\lambda|$. □

1.9 Let $\mathscr{X}$ be a real normed linear space and f a non-zero real-valued linear functional on $\mathscr{X}$. Show that there does not exist an open ball $B(x_0, \delta)$ such that $f(x_0)$ is the maximum or minimum value of $f(x)$ in $B(x_0, \delta)$.

Proof. Notice that $f(\lambda x_0) = \lambda f(x_0)$. Given δ, there exists ϵ such that for all $\lambda \in (1-\epsilon, 1+\epsilon)$ it holds that $\lambda x_0 \in B(x_0, \delta)$. Hence, if $f(x_0) \neq 0$, $f(x_0)$ can not be maximum or minimum value of f in $B(x_0, \delta)$. □

2 Riesz Representation Theorem and Its Applications

(H refers to Hilbert space in this section)

2.1 Let $f_1, \ldots, f_n$ be bounded linear functional over H, let

$$M = \bigcap_{k=1}^{n} N(f_k), \quad N(f_k) = \{x \in H : f_k(x) = 0\}, \quad k = 1, \ldots, n.$$

Let $x_0 \in H$ and denote by y_0 the orthogonal projection of x_0 onto M. Show that there exists $y_1, \ldots, y_k \in H$ and $a_1, \ldots, a_k \in \mathbb{K}$ such that

$$y_0 = x_0 - \sum_{k=1}^{n} a_k y_k.$$

Proof. By Riesz Representation Theorem, there exists y_k for each k such that $f_k(x) = (x, y_k)$ for all $x \in H$. Since f is continuous, $N(f_k)$ is closed and so is M. Thus x_0 has the unique decomposition $x_0 = y_0 + z_0$ where $y_0 \in M$ and $z_0 \in M^\perp$. Therefore it is sufficient to show that $M^\perp = \operatorname{span}\{y_1, \ldots, y_n\}$, i.e., $M = \operatorname{span}\{y_1, \ldots, y_n\}^\perp$ (See Exercise 1.6.5). This is straight forward since $N(f_k) = (\operatorname{span} y_k)^\perp$ and M is their intersection. □

2.2 Let l be a real-valued bounded linear functional and C is a closed convex set in H. Define

$$f(v) = \frac{1}{2}\|v\|^2 - l(v).$$

(1) There exists $u^* \in H$ such that

$$f(v) = \frac{1}{2}\|u^* - v\|^2 - \frac{1}{2}\|u^*\|^2.$$

(2) There exists unique $u_0 \in C$ such that $f(u_0) = \inf_{v \in C} f(v)$.

Proof. (1) By Riesz Representation Theorem, there exists u^* such that $(v, u^*) = l(v)$ for all $v \in C$. This u^* is exactly our desire.

(2) Let $C' = \{u^*\} - C$, then C' is a closed convex set too. By Theorem 1.6.31, there exists $u' \in C'$, thus $u_0 \in C$ such that $\|u'\| = \|u^* - u_0\| = \inf_{u \in C} \|u^* - u\|$. The conclusion follows immediately. □

2.3 Suppose the elements of H are complex-valued functions on S. For $x \in S$, the map $J_x(f) = f(x)$ ($\forall f \in H$) induces a continuous linear functional over H. Show that there exists $K : S \times S \to \mathbb{C}$ satisfying

(1) For any $y \in S$, $K(x, y)$ as a function of x belongs to H;

(2) $f(y) = (f, K(\cdot, y)),\ \forall f \in H,\ \forall y \in S$.

(A function $K(x, y)$ satisfying the two conditions above is called the *reproducing kernel* of H; and the second condition is called *reproducing property*)

Proof. By Riesz Representation Theorem, there exists $f_x \in H$ such that $J_x(f) = (f, f_x) = f(x)$ for all $x \in X$. Define $K(x, y) = f_y(x)$ and the two conditions are satisfied. □

2.4 Prove that the reproducing kernel of $H^2(D)$ (See Example 1.6.28 for the definition) is

$$K(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}, \quad z, w \in D.$$

Proof. First we verify that $K(z, w) = 1/(\pi(1 - z\bar{w})^2)$ is a reproducing kernel of $H^2(D)$. Since $|1 - z\bar{w}| \geq 1 - |z\bar{w}| \geq 1 - |w|$ (note that $|z| < 1$), hence $K(z, w)$ is bounded over D and thus $K(z, w) \in H^2(D)$ as a function of z. On the other hand, let $f(z) \in H^2(D)$ with Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

it follows from

$$K(z, w) = \sum_{n=0}^{\infty} \frac{n+1}{\pi} \bar{w}^n z^n,$$

and Exercise 1.6.11(b) that

$$(f, K(\cdot, w)) = \sum_{n=0}^{\infty} a_k w^n = f(w),$$

which is exactly the reproducing property. Therefore, $K(z, w) = 1/(\pi(1 - z\bar{w})^2)$ is a reproducing kernel of $H^2(D)$.

Now we prove that the reproducing kernel is unique. Suppose K and K' are two reproducing kernels of H consisting of functions on S. Then $(K - K', K - K') = (K - K', K) - (K - K', K') = 0$ due to the reproducing property. □

2.5 Let L and M be two closed subspaces of H. Show that

(1) $L \perp M$ iff $P_L P_M = 0$;

(2) $L = M^\perp$ iff $P_L + P_M = I$;

(3) $P_L P_M = P_{L \cap M}$ iff $P_L P_M = P_M P_L$.

Proof. (1) `Only if' is obvious. Now we prove the `if' part. It follows from $P_L P_M = 0$ that $P_L x = 0$ for all $x \in M$. Writing $x = y + z$, where $y \in L$ and $z \in L^\perp$, we see that $x \in L^\perp$ for all $x \in M$. Therefore $L \perp M$.

(2) `Only if' is obvious. Now prove the `if' part. Write $x = y + z$, where $y \in M$ and $z \in M^\perp$. Then $x = P_L x + P_M x = P_L x + y$, thus $P_L x = z \in M^\perp$ and $L \subseteq M^\perp$. On the other hand, take $x \in M^\perp$, then $x = P_L x + P_M x = P_L x \in L$, which implies that $M^\perp \subseteq L$.

(3) `If': Noticing that $P_L P_M x \in L$ and $P_M P_L x \in M$, we know that $P_L P_M x \in L \cap M$. And it holds that

$$x = P_L x + P_{L^\perp} x = P_L(P_M x + P_{M^\perp} x) + P_{L^\perp} x = P_L P_M x + P_L P_{M^\perp} x + P_{L^\perp} x. \quad (1)$$

It is obvious that $P_{L^\perp} x \in (L \cap M)^\perp$. Observing that $P_L P_{M^\perp} = P_{M^\perp} - P_{L^\perp} P_{M^\perp}$, it follows that $P_L P_{M^\perp} x \in (L \cap M)^\perp$. Hence, $P_L P_{M^\perp} x + P_{L^\perp} x$ is in $(L \cap M)^\perp$ and $P_L P_M x = P_{L\cap M} x$.

`Only if': It is well-known that if P is a projector then it holds $(Px, y) = (x, Py)$ for all $x, y \in H$. Since $P_L P_M = P_{L\cap M}$ we have

$$(P_M P_L x, y) = (P_L x, P_M y) = (x, P_L P_M y) = (P_L P_M x, y) = (P_M x, P_L y) = (x, P_M P_L y).$$

Write $x = y + z$, where $y \in L \cap M$ and $z \in (L \cap M)^\perp$. It is clear that $P_M P_L y = y$, and $(P_M P_L z, P_M P_L z) = (z, P_M(P_L P_M)P_L u) = (z, P_{L\cap M} z) = 0$, hence $P_M P_L z = 0$. Therefore $P_M P_L x = y = P_{L\cap M} x$. □

3 Category and Open Mapping Theorem

3.1 Let $\mathscr{X}$ be a Banach space and $\mathscr{X}_0$ a closed subspace of $\mathscr{X}$. The map $\phi : \mathscr{X} \to \mathscr{X}/\mathscr{X}_0$ is defined as

$$\phi : x \mapsto [x], \quad \forall x \in \mathscr{X},$$

where $[x]$ is the quotient class of x (see Exercise 1.4.17). Show that ϕ is an open map.

Proof. From Exercise 1.4.17(6), $\mathscr{X}/\mathscr{X}_0$ is a Banach space. From (4), ϕ is continuous, thus $\phi \in \mathscr{L}(\mathscr{X}, \mathscr{X}/\mathscr{X}_0)$. It is obvious that ϕ is surjective. From open mapping theorem we know that ϕ is an open map. □

3.2 Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces. The equations $Ux = y$ has solution $x \in \mathscr{X}$ for all $y \in \mathscr{Y}$, where $U \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, and there exists $m > 0$ such that

$$\|Ux\| \geq m\|x\| \quad \forall x \in \mathscr{X}.$$

Show that U has continuous inverse U^{-1} and $\|U^{-1}\| \leq 1/m$.

Proof. It is clear that U is surjective, and we show that U is injective. Consider the equation $Ux = 0$. It has a solution $x = x_0$, thus $0 = \|Ux_0\| \geq m\|x_0\|$, which implies that $x_0 = 0$. Therefore, by Banach Inverse Mapping Theorem, we know that U^{-1} exists and $U^{-1} \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$.

Let $y \in \mathscr{Y}$, $\|y\| = \|U(U^{-1}y)\| \geq m\|U^{-1}y\|$, it follows that $\|U^{-1}y\|/\|y\| \leq 1/m$. Therefore, $\|U^{-1}\| \leq 1/m$. □

3.3 Let H be a Hilbert space, $A \in \mathscr{L}(H)$ and $\exists m > 0$ such that

$$|(Ax, x)| \geq m\|x\|^2, \quad \forall x \in H$$

Show that $\exists A^{-1} \in \mathscr{L}(H)$.

Proof. Clearly, $Ax = 0$ implies that $x = 0$, thus A is injective. Note that $\|Ax\|\,\|x\| \geq |(Ax, x)| \geq m\|x\|^2$, or, $\|Ax\| \geq m\|x\|$, we have that $R(A)$ is a closed subspace of H. If $R(A) \subsetneq H$, there exists $x_0 \neq 0$ such that $x_0 \perp R(A)$, resulting in $0 = |(Ax_0, x_0)| \geq m\|x_0\|^2$, which contradicts with $x_0 \neq 0$. Hence $R(A) = H$ and A is surjective. It follows from Inverse Mapping Theorem that A^{-1} is continuous. □

3.4 Let X, Y be two normed linear spaces, D is a linear subspace of $\mathscr{X}$ and $A : D \to \mathscr{Y}$ is a linear map. Show that

(1) If A is continuous and D is closed, then A is closed;

(2) If A is continuous and closed, then the completeness of $\mathscr{Y}$ implies the closedness of D;

(3) If A is an injective closed map, then A^{-1} is also a closed map;

(4) If $\mathscr{X}$ is complete, A is an injective closed operator, $R(A)$ is dense in $\mathscr{Y}$ and A^{-1} is continuous, show that $R(A) = \mathscr{Y}$.

Proof. (1) Let $\{x_n\}$ be a sequence of points in D converging to x in norm, and $Ax_n \to y$. Since D is closed, it holds that $x \in D$. As A is continuous we have that $Ax_n \to Ax$. Thus $Ax = y$, and A is a closed map.

(2) Let $x \in \bar{D}$, there exists a sequence of points x_n converging to x in norm. Since A is continuous, A is bounded, thus $\|Ax_m - Ax_n\| = \|A(x_m - x_n)\| \leq \|A\| \|x_m - x_n\|$, which implies that Ax_n is a Cauchy sequence in $\mathscr{Y}$. Since $\mathscr{Y}$ is complete, $Ax_n \to y$ for some $y \in \mathscr{Y}$. From the definition of a closed map, we have $x \in D$ and $y = Ax$. Therefore D is closed.

(3) It is clear that A^{-1} does exist. Suppose that $y_n \to y$ and $x_n = A^{-1}y_n \to x$. Since A is closed, we know that $x \in D$ and $y = Ax$, thus $y \in R(A)$ and $x = A^{-1}y$, implying that A^{-1} is a closed map.

(4) Let $y \in \mathscr{Y}$ we have from the density of $R(A)$ in $\mathscr{Y}$ that there exists $\{y_n\} \subset R(A)$ such that $y_n \to y$. Let $x_n = A^{-1}y_n$. Since A^{-1} is continuous and thus bounded, we have that

$$\|x_m - x_n\| = \|A^{-1}y_m - A^{-1}y_n\| = \|A^{-1}(y_m - y_n)\| \leq \|A^{-1}\| \|y_m - y_n\|,$$

which implies that $\{x_n\}$ is a Cauchy sequence, thus converging to some $x \in \mathscr{X}$ from the completeness of $\mathscr{X}$. Since A is a closed map, we have that $x \in D$ and $y = Ax \in R(A)$. Therefore, $R(A) = \mathscr{Y}$. □

3.5 Using Corollary 2.3.13, show that $(C[0,1], \|\cdot\|_1)$ is not a Banach space, where $\|f\|_1 = \int_0^1 |f(t)|dt (\forall f \in C[0,1])$.

Proof. Suppose that $(C[0,1], \|\cdot\|_1)$ is a Banach space, then the norm $\|\cdot\|$ defined in Example 1.4.5 is stronger than $\|\cdot\|_1$. Corollary 2.3.13 says that they are equivalent and thus there exists c such that $\|\cdot\| \leq c\|\cdot\|_1$. Take $f_n = x^n$, $\|f_n\| = 1$ but $\|f\|_1 = \frac{1}{n+1} \to 0 (n \to \infty)$. Contradiction. Therefore $(C[0,1], \|\cdot\|_1)$ is not a Banach space. □

3.6 (Gelfand Lemma) Let $\mathscr{X}$ be a Banach space, $p : \mathscr{X} \to \mathbb{R}^1$ satisfies

(1) $p(x) \geq 0 \quad (\forall x \in \mathscr{X})$

(2) $p(\lambda x) = \lambda p(x) \quad (\forall \lambda > 0, \forall x \in \mathscr{X})$.

(3) $p(x_1 + x_2) \leq p(x_1) + p(x_2) \quad (\forall x_1, x_2 \in \mathscr{X})$

(4) $\varliminf_{n\to\infty} p(x_n) \geq p(x)$ as $x_n \to x$.

Show that $\exists M > 0$, such that $p(x) \leq M\|x\| \ (\forall x \in \mathscr{X})$.

Proof. Define $\|x\|_1 = \|x\| + p(x) + p(-x)$. It is easy to verify that $p(0) = 0$ and $\|\cdot\|$ is a norm actually. Let $\{x_n\}$ be a Cauchy sequence with respect to $\|\cdot\|_1$, then it is also a Cauchy sequence with respect to $\|\cdot\|$, thus converges to some $x \in \mathscr{X}$ in norm $\|\cdot\|$. Also we have that $p(x_n - x_m) \leq \epsilon$ for all $n, m > N(\epsilon)$. It follows from (4) that $p(x - x_m) \leq \epsilon$, thus $p(x - x_m) \to 0$ as $m \to \infty$ and similarly $p(x_m - x) \to 0$ too. Therefore $\{x_n\}$ converges to x in norm $\|\cdot\|_1$. It is obvious that $\|\cdot\|_1$ is stronger than $\|\cdot\|$, thus by Corollary 2.3.13 there exists M such that

$$\|x\|_1 = \|x\| + p(x) + p(-x) \leq M\|x\|,$$

whence we obtain that $p(x) \leq (M-1)\|x\|$. □

3.7 Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces. $A_n \in \mathscr{L}(\mathscr{X}, \mathscr{Y})(n = 1, 2, \dots)$, and for all $x \in \mathscr{X}$, $\{A_n x\}$ is convergent in $\mathscr{Y}$. Show that $\exists A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, such that

$$A_n x \to Ax (\forall x \in \mathscr{X}), \text{ and} \|A\| \le \varliminf_{n\to\infty} \|A_n\|.$$

Proof. Since $\{A_n x\}$ converges in $\mathscr{Y}$ for all $x \in \mathscr{X}$. we can define $A : \mathscr{X} \to \mathscr{Y}$ as $Ax = \lim_{n\to\infty} A_n x$. It is easy to verify that A is linear. Suppose that $\{A_{n_k}\}$ satisfies $\lim_{k\to\infty} \|A_{n_k}\| = \varliminf_{n\to\infty} \|A_n\|$. Then for all $\|x\| \neq 0$, we have $\|Ax\| = \lim_{k\to\infty} \|A_{n_k} x\| \le \lim_{k\to\infty} \|A_{n_k}\| \|x\| = \varliminf_{n\to\infty} \|A_n\| \|x\|$. It follows from uniform boundedness theorem that $\{\|A_n\|\}$ is bounded thus A is bounded. □

3.8 Let $1 < p < \infty$ with $1/p+1/q = 1$. If the sequence $\{a_k\}$ makes $\sum_{k=1}^{\infty} a_k \xi_k$ convergent for all $x = \{\xi_k\} \in l^p$, show that $\{a_k\} \in l^q$. Define $f : l^p \to \mathbb{R}$ as $f(x) \sum_{k=1}^{\infty} a_k \xi_k$, show that

$$\|f\| = \left(\sum_{k=1}^{\infty} |a_k|^q \right)^{\frac{1}{q}}.$$

We need the following lemma.

Lemma Suppose that $\{a_k\}_{k=1}^n$ satisfies $\sum |a_k|^q = 1$, then there exists $\{b_k\}_{k=1}^n$ such that $\sum |b_k|^p = 1$ and $\sum |a_k b_k| = 1$.

Proof of Lemma Take $b_k = |a_k|^q / a_k$, and notice that $p(q-1) = q$.

Proof. Suppose that $\{a_k\} \notin l^q$, or $\sum_{k=1}^{\infty} |a_k|^q$ does not converge, then there exist $0 = i_1 < i_2 < \dots$ such that $S_n = \sum_{k=i_n+1}^{i_{n+1}} |a_k|^q > n^2$ for all n. Since

$$\sum_{k=i_n+1}^{i_{n+1}} \left| \frac{a_k}{S_n^{1/q}} \right|^q = 1,$$

according to the lemma there exists $b_k (k = i_n + 1, \dots, i_{n+1})$ such that

$$\sum_{k=i_n+1}^{i_{n+1}} |b_n|^p = 1, \quad \sum_{k=i_n+1}^{i_{n+1}} \left| \frac{a_n b_n}{S_n^{1/q}} \right| = 1.$$

Let $b'_k = b_n / n^{2/p}$ we have

$$\sum_{k=i_n+1}^{i_{n+1}} |b'_n|^p = \frac{1}{n^2}, \quad \sum_{k=i_n+1}^{i_{n+1}} |a_n b'_n| = \frac{S_n^{1/q}}{n^2} > n^{\frac{2}{q}-1}$$

We have obtained a sequence $\{b_n\}' \subseteq l^p$, however,

$$\sum_{k=1}^{\infty} |a_k b_k| > \sum_{n=1}^{\infty} n^{\frac{2}{q}-1} = \infty \quad (\text{since} \frac{2}{q} - 1 < 1),$$

which is a contradiction, and therefore $\{a_k\}$ must be in l^q.

It is easy to verify that f is linear. By Hölder's inequality,

$$|f(x)| \le \sum_{k=1}^{\infty} |a_k \xi_k| = \left(\sum_{k=1}^{\infty} |a_k|^q\right)^{\frac{1}{q}} \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}} = \|a\|_q \|x\|_p,$$

whence we see that $\|f\| \le \|a\|_q$. If $f = 0$, then $a_k = 0$ for all k and it holds automatically that $\|f\| = \|a\|_q$. Assume $f \ne 0$, then a_k are not all zeroes. Define $x_n = (\xi_1^{(n)}, \xi_2^{(n)}, \xi_3^{(n)}, \dots)$, where

$$x_k^{(n)} = \begin{cases} |a_n|^q / a_n, & k \le n, a_k \ne 0; \\ 0, & k > n, \text{ or } a_k = 0. \end{cases}$$

Notice that $p(q-1) = q$, it holds that

$$\|x_n\|_p = \left(\sum_{k=1}^{n} (|a_k|^{q-1})^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{n} |a_k|^q\right)^{\frac{1}{p}}$$

and

$$f(x_n) = \sum_{k=1}^{\infty} a_k \xi_k^{(n)} = \sum_{k=1}^{n} |a_k|^q,$$

Hence

$$\|f\| \ge \lim_{n\to\infty} \frac{|f(x_n)|}{\|x_n\|_p} = \lim_{n\to\infty} \left(\sum_{k=1}^{n} |a_k|^q\right)^{1-\frac{1}{p}} = \|a\|_q.$$

□

3.9 Let $\{a_k\}$ be a sequence such that $\sum\limits_{k=1}^{\infty} a_k \xi_k$ exists for every $x = \{\xi_k\} \in l^1$. Show that $\{a_k\} \in l^\infty$. Define $f : l^1 \to \mathbb{R}$ as $f(x) = \sum\limits_{k=1}^{\infty} a_k \xi_k$, show that $\|f\| = \sup\limits_{k\ge 1} |a_k|$.

Proof. If $\{a_k\}$ is unbounded, we can find $i_1 < i_2 < \dots$ such that $|a_{i_k}| > k$. Let $x = \{\xi_k\}$ where $\xi_{i_k} = \operatorname{sgn} a_{i_k} / k^2$ and $\xi_n = 0$ otherwise. It is clear that $x \in l^1$, however, $\sum a_k \xi_k = \sum |a_{i_k}| / k^2 > \sum 1/k$, which is a contradiction. Therefore it must hold that $\{a_k\} \in l^\infty$.
Let $x \in l^1$, it holds that

$$|f(x)| \le \sum_{k=1}^{\infty} |a_k \xi_k| \le \sup_{k\ge 1} |a_k| \sum_{k=1}^{\infty} |\xi_k| = \|a\|_\infty \|x\|_1.$$

So $\|f\| \le \|a\|_\infty$.
Let $x_n = (x_{n1}, x_{n2}, \dots)$ where $x_{nn} = 1$ and $x_{nk} = 0$ for all $k \ne n$, then $\|x_n\|_1 = 1$ and $|f(x)| = |a_n|$. Hence

$$\|f\| \ge \sup_n \frac{|f(x_n)|}{\|x_n\|} = \sup_n |a_n|.$$

□

3.10 Prove uniform boundedness theorem by Gelfand lemma.

Proof. Suppose that $W \subset \mathscr{L}(\mathscr{X}, \mathscr{Y})$ and $\sup\limits_{A\in W} \|Ax\| < \infty$. Define $p(x) = \sup\limits_{A\in W} \|Ax\|$, and $p(x)$ satisfies all the conditions in the Gelfand lemma. Here we show fourth condition.

Suppose that $x_n \to x$. Let $q_n = \inf\limits_{m\geq n} p(x_m)$. For every $A \in W$ we have $q_n \geq \inf\limits_{m\geq n} \|Ax_m\|$. Taking limits on both sides we have that $\lim\limits_{n\to\infty} q_n \geq \varliminf\limits_{n\to\infty} |Ax_n| = |Ax|$. Therefore, $\varliminf\limits_{n\to\infty} p(x_n) \geq \sup\limits_{A\in W} |Ax|$.

Now we apply Gelfand's lemma that there exists M such that $p(x) = \sup\limits_{A\in W} \|Ax\| \leq M\|x\|$ for all $x \in \mathscr{X}$, indicating that $\|A\| \leq M$. □

3.11 Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces, $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ is a surjection. Show that if $y_n \to y_0$ in $\mathscr{Y}$, then there exist $C > 0$ and $x_n \to x_0$ such that $Ax_n = y_n$ and $\|x_n\| \leq C\|y_n\|$.

Proof. Define $f : \mathscr{X}/N(A) \to \mathscr{Y}$ as $f : [x] \to Ax$. This is well-defined and bijective. Exercise 1.4.17(5) claims that for all $[x] \in \mathscr{X}/N(A)$, there exists $x' \in [x]$ such that $\|x'\| \leq 2\|[x]\|$. It follows that

$$\|f[x]\| = \|Ax'\| \leq \|A\|\,\|x'\| \leq 2\|A\|\,\|[x]\|$$

which implies that f is bounded. Thus f^{-1} is bounded too from Banach Inverse Mapping Theorem. Let $y_n \to y_0$ and $[z_n] = f^{-1}(y_n)$. And we can find $a_n \in N(A)$ such that $\|z_n - a_n\| \leq 2\|[z_n - a_n]\| = 2\|[z_n]\|$. Let $x_n = z_n - a_n$, we have that $Ax_n = Az_n = y_n$, $x_n \to x_0$ and $\|x_n\| \leq 2\|f^{-1}\|\,\|A\|\,\|y_n\|$. □

3.12 Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces and T a closed linear map with $D(T) \subset \mathscr{X}$ and $R(T) \subset Y$. Let $N(T) = \{x \in \mathscr{X} : Tx = 0\}$.

(1) Show that $N(T)$ is a closed subspace of $\mathscr{X}$;

(2) Show that $N(T) = \{0\}$ and $R(T)$ is closed in $\mathscr{Y}$ if and only if $\exists a > 0$ such that $\|x\| \leq a\|Tx\|$ for all $x \in D(T)$;

(3) Denote by $d(x, N(T))$ the distance between $x \in \mathscr{X}$ and $N(T)$. Show that $R(T)$ is closed in $\mathscr{Y}$ if and only if $\exists a > 0$ such that $d(x, N(T)) \leq a\|Tx\|$ for all $x \in D(T)$.

Proof. (1) Suppose that $\{x_n\} \subset N(T)$ with $x_n \to x_0$. Then $Tx_n = 0 \to 0$. Since T is closed, we have that $x_0 \in D(T)$ and $Tx_0 = 0$, thus $x_0 \in N(T)$. $N(T)$ is closed.

(2) `If': Let $y_n = Tx_n \to y \in \mathscr{Y}$. If follows from $\|x_m - x_n\| \leq a\|y_m - y_n\|$ that $\{x_n\}$ is a Cauchy sequence, thus converges to some point $x \in \mathscr{X}$ since $\mathscr{X}$ is complete. Since T is a closed map, we have that $x \in D(T)$ and $y = Tx \in R(T)$, which implies that $R(T)$ is closed. Let $x \in N(T)$, then $\|x\| \leq a\|Tx\| = 0$, so $x = 0$, and $N(T) = \{0\}$.

`Only if': Since $R(T)$ is closed, $(R(T), \|\cdot\|)$ is a Banach space. Since $N(T) = 0$, T is injective and there exists $T^{-1} : R(T) \to X$. According to Exercise 2.3.4(3), T^{-1} is a closed map. The closed graph theorem tells us that T^{-1} is continuous thus bounded, $\|T^{-1}x\| \leq a\|x\|$ for some a and $x \in D(T^{-1}) = R(T)$, so $\|x\| \leq a\|Tx\|$ for all $x \in D(T)$.

(3) From Exercise 1.4.17(6), we know that $\mathscr{X}/N(T)$ is a Banach space. Define $\widetilde{T} : \mathscr{X}/N(T) \to \mathscr{Y}$ as $\widetilde{T}[x] = Tx$. We have $N(\widetilde{T}) = [0]$ and $R(\widetilde{T}) = R(T)$. We shall show that $\widetilde{T}$ is a closed map.

Suppose that $[x_n] \to [x_0]$ and $\widetilde{T}[x_n] \to y$. According to Exercise 1.4.17(5), we can choose $x'_n \in [x_n]$ such that $x_n \to x'_0$, and thus $Tx'_n \to y$. Since T is a closed map, it holds that $x'_0 \in D(T)$ and $y = Tx'_0$. Hence $[x'_0]$ is well-defined, $y = \widetilde{T}[x'_0]$ and $\widetilde{T}$ is a closed map. It follows from (2) that $R(T)$ is closed iff $d(x, N(T)) = \|[x]\| \leq a\|\widetilde{T}[x]\| = a\|Tx\|$ for some a. □

3.13 Let $a(x, y)$ be a sesquilinear functional over a Hilbert space H, which satisfies that

(1) $\exists M > 0$, such that $|a(x, y)| \leq M\|x\|\,\|y\| \quad \forall x, y \in H$;

(2) $\exists\delta > 0$, such that $|a(x,x)| \geq \delta\|x\|^2 \quad \forall x \in H$.

Prove that $\forall f \in H^*$, $\exists \mid y_f \in H$, such that

$$a(x, y_f) = f(x) \quad (\forall x \in H)$$

and y_f depends on f continuously.

Proof. Fix $y \in H$, define $T_y(x) = a(x,y)$ for $x \in H$. It is clear that T_y is a linear functional and $|T_y(x)| = |a(x,y)| \leq M\|x\|\|y\|$, which implies that T_y is continuous and $\|T_y\| \leq M\|y\|$. By Riesz's Representation Theorem, there exists unique $Y(f)$ for every f such that

$$(x, Y(f)) = f(x), \forall x \in H.$$

We want to find unique y such that $T_y(x) = f(x)$ for all $x \in H$, or (because $Y : H^* \to H$ is injective), $Y(T_y) = Y(f)$.

Let $0 < \rho < \delta/M^2$. Define $A : H \to H$ as

$$Ay = y - \rho(Y(T_y) - Y(f)), \quad y \in H,$$

then for all $y_1, y_2 \in H$, denote $y = y_1 - y_2$, it holds that

$$\begin{aligned}
\|Ay_1 - Ay_2\| &= \|y_1 - y_2 - \rho(Y(T_{y_1}) - Y(T_{y_2}))\\
&= \|y - \rho Y(T_y)\|\\
&= \|y\|^2 - 2\rho(y, Y(T_y)) + \rho^2\|Y(T_y)\|^2\\
&= \|y\|^2 - 2\rho T_y(y) + \rho^2 T_y(Y(T_y))\\
&= \|y\|^2 - 2\rho a(y,y) + \rho^2 a(y, Y(T_y))\\
&\leq \|y\|^2 - 2\rho\delta\|y\|^2 + \rho^2 M\|y\|\,\|Y(T_y)\|\\
&\leq (1 - 2\rho\delta + \rho^2 M^2)\|y\|^2 \quad (\text{because } \|Y(T_y)\| = \|T_y\|)
\end{aligned}$$

Since $1 - 2\rho\delta + \rho^2 M^2 < 1$, A is a contraction map, thus A has a unique fixed point, say y, and $\rho(Y(T_y) - Y(f)) = 0$ or $Y(T_y) = Y(f)$. This y is exactly the y_f we desire.

Now we show the continuity. Let $f, g \in H^*$, then

$$\begin{aligned}
\delta\|y_f - y_g\|^2 &\leq |a(y_f - y_g, y_f - y_g)|\\
&= |(f-g)(y_f - y_g)| \leq \|f - g\|\,\|y_f - y_g\|
\end{aligned}$$

hence $\|y_f - y_g\| \leq \|f - g\|/\delta$, which indicates the continuity. □

3.14 Suppose that Ω be a bounded open region with smooth boundary in $\mathbb{R}^2$. The map $\alpha : \Omega \to \mathbb{R}$ is bounded and measurable such that $0 < \alpha_0 \leq \alpha$. Let $f \in L^2(\Omega)$. Define

$$\alpha(u,v) = \int_\Omega (\nabla u \cdot \nabla v + \alpha uv)dxdy, \forall u, v \in H^1(\Omega);$$

$$F(v) = \int_\Omega fvdxdy, \forall v \in L^2(\Omega).$$

Show that there exists unique $u \in H^1(\Omega)$ such that

$$a(u,v) = F(v), \forall v \in H^1(\Omega).$$

Proof. We verify the conditions in the previous problem are satisfied and the conclusion follows immediately. It is clear that $a(u, v)$ is bilinear. Suppose that $|\alpha(x, y)| \le M$ ($M \ge 1$), then it holds that

$$\begin{aligned} |a(u,v)| &\le M \int_\Omega (\|\nabla u\| \cdot \|\nabla v\| + |u| \cdot |v|) \\ &\le M \int_\Omega \left(\|\nabla u\|^2 + |u|^2\right)^{\frac{1}{2}} \left(\|\nabla v\|^2 + |v|^2\right)^{\frac{1}{2}} \\ &\le M \left[\int_\Omega \left(\|\nabla u\|^2 + |u|^2\right)\right]^{\frac{1}{2}} \left[\int_\Omega \left(\|\nabla v\|^2 + |v|^2\right)\right]^{\frac{1}{2}} \\ &= M\|u\|_{H^1}\|v\|_{H^1} \end{aligned}$$

and

$$|a(u,u)| = \int_\Omega (\|\nabla u\|^2 + \alpha|u|^2) \ge \min\{1, \alpha_0\} \int_\Omega \left(\|\nabla u\|^2 + |u|^2\right) = \min\{1, \alpha_0\}\|u\|^2_{H^1}.$$

□

4 Hahn-Banach Theorem

4.1 Let p be a sublinear functional on a real linear space $\mathscr{X}$, show that

(1) $p(0) = 0$;

(2) $p(-x) \ge -p(x)$;

(3) Given $x_0 \in \mathscr{X}$, there exists a real functional f on $\mathscr{X}$ such that $f(x_0) = p(x_0)$ and $f(x) \le p(x)$.

Proof. (1) $p(a \cdot 0) = a \cdot p(0)$ for $a > 0$, hence $p(0) = 0$.

(2) $0 = p(0) = p(x + (-x)) \le p(x) + p(-x)$, hence $p(-x) \ge -p(x)$.

(3) Consider $\mathscr{X}_0 = \{\lambda x_0 \mid \lambda \in \mathbb{R}\}$. It is a closed subspace, and we define $f_0(\lambda x_0) = \lambda p(x_0)$ for $\lambda > 0$ and $f_0(\lambda x_0) = -\lambda f(-x_0)$ for $\lambda \le 0$ on it. From real Hahn-Banach Theorem, we can extend f_0 to f over entire $\mathscr{X}$ such that $f(x) \le p(x)$ and $f(x_0) = f_0(x_0) = p(x_0)$. □

4.2 Let $\mathscr{X}$ be a real linear space consisting of all real sequence $x = \{a_n\}$. The equality and operations over $\mathscr{X}$ are defined coordinate-wise, and we define

$$p(x) = \overline{\lim_{n\to\infty}}\, a_n \quad \forall x = \{a_n\} \in \mathscr{X}$$

Show that $p(x)$ is a sublinear functional on $\mathscr{X}$.

Proof. Trivial. □

4.3 Let X be a complex linear space and p a seminorm over $\mathscr{X}$ satisfying that $p(x_0) \ne 0$ for all $x_0 \in \mathscr{X}$. Show that there exists linear functional f on $\mathscr{X}$ such that

(1) $f(x_0) = 1$;

(2) $|f(x)| \le p(x)/p(x_0), \quad \forall x \in \mathscr{X}$.

Proof. Consider $\mathscr{X}_0 = \{\lambda x_0 \mid \lambda \in \mathbb{R}\}$, which is a closed subspace. We define $f_0(x) = |\lambda|$ for $x = \lambda x_0$, then $|f_0(x)| \le p(x)/p(x_0)$. According to Hahn-Banach Theorem, we can extend f_0 to f over the entire $\mathscr{X}$ such that $f(x) \le p(x)/p(x_0)$ and $f(x_0) = f_0(x_0) = 1$. □

4.4 Let $\mathscr{X}$ be a normed linear space and $\{x_n\} \subseteq \mathscr{X}$. If $\{f(x_n)\}$ is bounded for all $f \in \mathscr{X}^*$, show that $\{x_n\}$ is bounded in $\mathscr{X}$.

Proof. Let $g_n(f) = f(x_n)$, hence for every $f \in X^*$, $\{g_n(f)\}$ is bounded, it follows from uniform boundedness theorem that $\{\|g_n\|\}$ is bounded.

$$\|g_n\| = \sup_{f \in X^*} \frac{\|g_n(f)\|}{\|f\|} = \sup_{f \in X^*} \frac{|f(x_n)|}{\|f\|}.$$

Since $f \in X^*$, the right-hand side $\le \|x_n\|$, however, from Corollary 2.4.6 there exists $f \in X^*$ such that $f(x_n) = \|x_n\|$ and $\|f\| = 1$, it follows that the right-hand side is exactly $\|x_n\|$. Therefore $\{x_n\}$ is bounded. □

4.5 Let $\mathscr{X}_0$ be a closed subspace of normed linear space $\mathscr{X}$. Show that

$$\rho(x, \mathscr{X}_0) = \sup\{|f(x)| : f \in X^*, \|f\| = 1, f(\mathscr{X}_0) = 0\},$$

where $\rho(x, \mathscr{X}_0) = \inf_{y \in \mathscr{X}_0} \|x - y\|$.

Proof. The statement holds trivially for $\rho(x, \mathscr{X}_0) = 0$. Assume $\rho(x, \mathscr{X}_0) > 0$, for f with $\|f\| = 1$ and $f(\mathscr{X}_0) = 0$, it holds that

$$|f(x)| = |f(x - y)| \le \|f\| \, \|x - y\| = \|x - y\|, \quad y \in \mathscr{X}_0$$

hence

$$\sup\{|f(x)| \mid f \in X^*, \|f\| = 1, f(\mathscr{X}_0) = 0\} \le \inf_{y \in \mathscr{X}_0} \|x - y\| = \rho(x, \mathscr{X}_0).$$

It follows from Theorem 2.4.7 that the equality does hold. □

4.6 Let $\mathscr{X}$ be a normed linear space and $x_1, \dots, x_n$ be n linearly independent vectors in $\mathscr{X}$ and $C_1, \dots, C_n$ be n constants in $\mathbb{K}$. Suppose that $M > 0$, show that there exists $f \in \mathscr{X}^*$ such that $f(x_k) = C_k$ and $\|f\| \le M$ if and only if it holds that

$$\left|\sum_{k=1}^n a_k C_k\right| \le M \left\|\sum_{k=1}^n a_k x_k\right\|$$

fir all $a_1, \dots, a_n \in \mathbb{K}$.

Proof. The `only if' part is trivial. We shall prove the `if' part. Assume that for all $a_1, \dots, a_n \in \mathbb{K}$ it holds that $|\sum_{k=1}^n a_k C_k| \le M \|\sum_{k=1}^n a_k x_k\|$, then we can define a linear functional f_0 on span $\{x_1, \dots, x_n\}$ such that $f_0(x_k) = C_k$ and it is clear that $\|f_0\| \le M$. According to Hahn-Banach Theorem, we can extend f_0 to a functional f over entire $\mathscr{X}$ with $\|f\| = \|f_0\| \le M$. □

4.7 Given n linearly independent vectors $x_1, \dots, x_n$ in a normed linear space $\mathscr{X}$, show that there exist $f_1, \dots, f_n \in \mathscr{X}^*$ such that

$$\langle f_i, x_j \rangle = \delta_{ij}, \quad i, j = 1, 2, \dots, n.$$

Proof. Given j, we can define a linear functional f_0 over span$\{x_1, \dots, x_n\}$ with $f_0(x_k) = 1$ iff $k = j$. We shall show that $f_0 \in X^*$ then Hahn-Banach Theorem states the existence of f_j as required. Let $x = \sum_{i=1}^n \lambda_i x_i$ then $f_0(x) = |\lambda_j|$. Thus

$$\|f_0\| = \sup \frac{|\lambda_j|}{\|\sum \lambda_i x_i\|} = \sup \frac{1}{\left\|\sum \frac{\lambda_i}{|\lambda_j|} x_i\right\|},$$

it suffices to show that the norm of $\sum \lambda_i x_i$ (where $\lambda_j = 1$) is not less than d for some $d > 0$, or, $\rho(x_j, \mathscr{X}_0) > 0$, where $\mathscr{X}_0 = \text{span}\{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}$. This is obviously true, as $\mathscr{X}_0$ is closed and does not contain x_j. □

4.8 Let $\mathscr{X}$ be a linear space, show that M is the maximal linear subspace of $\mathscr{X}$ iff $\dim(\mathscr{X}/M) = 1$.

Proof. `If': Assume that $\dim(\mathscr{X}/M) = 1$, then $\mathscr{X}/M = \{[ax_0] : a \in \mathbb{K}\}$, where $x_0 \notin M$. Suppose that there is a linear space $S \supsetneq M$. Take $x_0 \in S \setminus M$, we have that $ax_0 \in S$ and $ax_0 + M \subseteq S$ for all $a \in \mathbb{K}$. Notice that the union of $ax_0 + M$ is exactly $\mathscr{X}$, hence $S = \mathscr{X}$, which implies that M is the maximal subspace.

`Only if': Suppose that M is a maximal subspace. Take $x_0 \notin M$. Since $\bigcup_{a\in\mathbb{K}}(ax_0 + M)$ is a linear subspace and it must be $\mathscr{X}$ due to the maximality of M. So $\dim(\mathscr{X}/M) = 1$. □

4.9 Let $\mathscr{X}$ be a complex linear space, E an nonempty balanced set and f a linear functional on $\mathscr{X}$. Show that

$$|f(x)| \le \sup_{y\in E} \Re f(y), \quad \forall x \in E.$$

Proof. Let θ be the argument of $f(x)$, then $|f(x)| = e^{-i\theta} f(x) = f(e^{-i\theta}x) = \Re f(e^{-i\theta}x)$, and $e^{-i\theta}x \in E$ since E is balanced. The conclusion follows immediately. □

4.10 Let $\mathscr{X}$ be a normed linear space, E an nonempty balanced closed convex set and $x_0 \in \mathscr{X} \setminus E$. Prove that there exists $f \in X^*$ and $a > 0$ such that

$$|f(x)| < a < |f(x_0)|, \quad \forall x \in E.$$

Proof. Since $x_0 \notin E$ and E is closed, there exists δ such that $B(x_0, \delta) \subseteq E^c$. View $\mathscr{X}$ as a real normed linear space and apply Theorem 2.4.15, there exists $s \in \mathbb{R}$ and a nonzero continuous linear functional g such that $g(x) \le s$ for all $x \in E$ and $g(x) \ge s$ for all $x \in B(x_0, \delta)$. Define $f(x) = g(x) - ig(ix)$ then f is a complex linear functional, and $|f(x)| = g(e^{-i\theta}x)$, where $\theta = \arg f(x)$. Hence $|f(x)| \le s$ for all $x \in E$, and $|f(x_0)| \ge \Re f(x_0) = g(x_0) > s$, because $g(x_0)$ can not be the minimum or maximum value of g in $B(x_0, \delta)$ (see Exercise 2.1.9). □

4.11 Let $\mathscr{X}$ be a normed linear space, E, F are two disjoint nonempty convex set in $\mathscr{X}$, where E is open and balanced. Show that there exists $f \in X^*$ such that

$$|f(x)| < \inf_{y\in F} |f(y)|, \quad \forall x \in E.$$

Proof. View $\mathscr{X}$ as real linear space and apply Theorem 2.4.15, we obtain $s \in \mathbb{R}$ and a nonzero continuous linear functional g such that $g(x) \le s$ for all $x \in E$ and $g(x) \ge s$ for all $x \in F$. Let $f(x) = g(x) - ig(ix)$ then f is a complex linear functional, and $|f(x)| = g(e^{-i\theta}x) \le s$, where $\theta = \arg f(x)$. Actually the inequality holds strictly because E is open. And for $y \in F$, $|f(y)| \ge \Re f(y) = g(y) \ge s$, hence $\inf |f(y)| \ge s$. □

4.12 Let C be a convex set in a real normed linear space B^* and $x_0 \in C^\circ$, $x_1 \in \partial C$, $x_2 = m(x_1 - x_0) + x_0$ $(m > 1)$. Show that $x_2 \notin C$.

Proof. Suppose that $x_2 \in C$. Since x_0 is an interior point, there exists δ such that $B(x_0, \delta) \subseteq C$. Let $d = \delta(1 - \frac{1}{m})$. We shall show that $B(x_1, d) \subseteq C$ to meet a contradiction.

Let $z \in B(x_1, d)$ and $u = (z - \frac{1}{m}x_2)/(1 - \frac{1}{m})$, then $z = \frac{1}{m}x_2 + (1 - \frac{1}{m})u$. It suffices to show that $\|u - x_0\| < \delta$, and in fact, we have

$$\|u - x_0\| = \left\| \frac{z - \frac{1}{m}x_2}{1 - \frac{1}{m}} - x_0 \right\| = \left\| \frac{z - (\frac{1}{m}x_2 + (1 - \frac{1}{m})x_0}{1 - \frac{1}{m}} \right\| = \left\| \frac{z - x_1}{1 - \frac{1}{m}} \right\| < \frac{d}{1 - \frac{1}{m}} = \delta.$$

□

4.13 Let M be a closed convex set in a normed linear space $\mathscr{X}$, show that for all $x \in \mathscr{X} \setminus M$ there exists $f_1 \in \mathscr{X}^*$ satisfying $\|f_1\| = 1$ and

$$\sup_{y \in M} f_1(y) \leq f_1(x) - d(x),$$

where $d(x) = \inf_{z \in M} \|x - z\|$.

Proof. Let $x_0 \in M^c$, we see that $B(x_0, d(x)) \cap M = \emptyset$, then applying Theorem 2.4.15, there exists s and $g \in \mathscr{X}^*$ such that $g(x) \leq s$ for $x \in M$ and $g(x) \geq s$ for $x \in B(x_0, \delta)$. Let $f_1 = g/\|g\|$, we have $f_1(x) \leq s/\|g\|$ for $x \in M$ and $f_1(x) \geq s/\|g\|$ for all $x \in B(x_0, \delta)$. It suffices to show that $\inf_{x \in B(x_0, \delta)} f_1(x) = f_1(x) - d(x)$. This is true, because

$$\begin{aligned} \inf_{x \in B(x_0, d(x))} f_1(x) &= \inf_{y \in B(0,1)} f_1(x_0 - d(x) y) \\ &= f_1(x_0) - d(x) \sup_{y \in B(0,1)} f_1(y) \\ &= f_1(x_0) - d(x)\|f_1\|. \end{aligned}$$

□

4.14 Let M be a closed convex set of real normed linear space $\mathscr{X}$. Show that

$$\inf_{z \in M} \|x - z\| = \sup_{\substack{f \in \mathscr{X}^* \\ \|f\|=1}} \left\{ f(x) - \sup_{z \in M} f(z) \right\}, \quad \forall x \in \mathscr{X}.$$

Proof. Since $\|f\| = 1$, we have that $f(x) - f(z) \leq \|x - z\|$. Take infimum on both sides, we obtain that $f(x) - \sup_{z \in M} f(z) \leq \inf_{z \in M} \|x - z\|$, and hence the right-hand side is not greater than the left one. If $x \notin M$, combining with the previous problem, we know the equality holds.

If $x \in M$, the proposition is false. Take the closed unit ball as M and $x = 0$. The left-hand side is obviously 0. Since $\|f\| = 1$, that is, $\sup_{\|x\|=1} |f(x)| = 1$, we must have that $\sup_{z \in M} f(z) = 1$. Thus the right-hand side is -1. □

4.15 Let $\mathscr{X}$ be a Banach space and $f : \mathscr{X} \to \overline{\mathbb{R}}$ is a continuous convex functional with $f(x) \not\equiv \infty$. Define $f^* : \mathscr{X}^* \to \overline{\mathbb{R}}$ as

$$f^*(x^*) = \sup_{x \in \mathscr{X}} \{\langle x^*, x \rangle - f(x)\}, \quad \forall x^* \in \mathscr{X}^*,$$

show that $f^*(x^*) \not\equiv \infty$.

4.16 Let $\mathscr{X}$ be a Banach space and $x(t) : [a, b] \to \mathscr{X}$ is a continuous function. Denote by Δ a partition of $[a, b]$:

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b.$$

Define $\|\Delta\| = \max_{0 \leq i \leq n-1} \{|t_{i+1} - t_i|\}$. Show that the limit

$$\lim_{\|\Delta\| \to 0} \sum_{i=0}^{n-1} x(t_i)(t_{i+1} - t_i)$$

exists in $\mathscr{X}$. (The limit is called the Riemann integral of function $x(t)$ over $[a, b]$)

4.17 Let $\mathscr{X}$ be a Banach space and G an open domain with boundary (simple curve) L. If $x(z) : \bar{G} \to \mathscr{X}$ is analytic within G and continuous on $\bar{G}$. Show that $\int_L x(z) dz = 0$.

4.18 Prove that

(1) $|x|$ is convex on $\mathbb{R}$;

(2) The subderivative of $|x|$ at $x = 0$, namely, $\partial|x|(0)$, is $[-1, 1]$.

Proof. (1) Trivial.

(2) From definition, we have that $\partial|x|(0) = \{x^* \in \mathbb{R}^* : \langle x^*, x\rangle \leq |x|(\forall x \in \mathbb{R})\}$. Let $x^* \in \partial|x|(0)$ and $\lambda = \langle x^*, 1\rangle$, then $|\lambda| \leq 1$ and $\langle x^*, y\rangle = \lambda y$. On the other hand, it is easy to verify that such x^* are subgradient of $|x|$. The conclusion follows immediately. □

5 Conjugate Space, Weak Convergence and Reflexive Space

5.1 Show that $(l^p)^* = l^q$, $1 \leq p < \infty, 1/p + 1/q = 1$.

Proof. Let $b = \{b_k\} \in l^q$, define $f_b(a) = \sum a_k b_k$ for $a \in l^p$ which is a linear functional. From Hölder's Inequality we have that $|\sum a_k b_k| \leq \|a\| \|b\|$ and thus the map $b \mapsto f_b$ is a map from l^q to $(l^p)^*$, denote it by F.

We show that F is a surjective isometry. Given $f \in (l^p)^*$ we shall find $b \in l^q$ such that $f_b = f$ and $\|b\| = \|f\|$. Let $z_n = (z_{n1}, z_{n2}, \dots)$ where $z_{nk} = \delta_{nk}$, and $b_k = f(z_k)$. We shall prove that $b \in l^q$ based on the following two cases.

(1) Case $p > 1$. Let $c_k = |b_k|^{q-2} b_k$, and $y_n = (c_1, \dots, c_n, 0, \dots)$, then we have $(\sum_{k=1}^n |b_k|^q) = \sum_{k=1}^n c_k b_k = f(y_n) \leq \|f\| \|y_n\| = \|f\| (\sum_{k=1}^n |b_k|^q)^{\frac{1}{p}}$, hence $(\sum_{k=1}^n |b_k|^q)^{\frac{1}{q}} \leq \|f\|$ for all n and $b \in l^q$.

(2) Case $p = 1$. $|b_k| = |g(z_n)| \leq \|g\| \|z_n\| = \|g\|$, thus $b \in l^\infty$.

Then it is not difficult to show that $f(a) = \sum a_k b_k$ for all $a \in l^p$. □

5.2 Let C be the set of all convergent sequences of numbers, and define

$$\|\cdot\| : \{\xi_k\} \in C \mapsto \sup_{k\geq 1} |\xi_k|,$$

show that $C^* = l^1$.

Proof. Let $a = \{a_k\} \in l^1$, define $f(a) = g_a : C \to \mathbb{R}$ as $g(x) = a_1\lambda + \sum_{k=1}^\infty a_{k+1} x_k$, where λ is the limit of $\{x_n\}$, then we have that $|g_a(x)| \leq |\lambda||a_1| + \|x\| \sum_{k=2}^\infty |a_k| \leq \|a\| \|x\|$, thus $g_A \in C^*$ Furthermore, take $x = (\operatorname{sgn} a_1, \operatorname{sgn} a_2, \dots)$ we have $|g_a(x)| = \|a\|$, hence $\|g_a\| = \|a\|$, and $f : l^1 \to C^*$ is an isometry (it is obvious that f is injective).

It suffices to show that f is surjective. Let $g \in C^*$, let $a_{k+1} = g(x_k)$, where $x_k = (x_{k1}, \dots, x_{kj}, \dots)$ and $x_{kj} = \delta_{kj}$. Let $z_n = (\operatorname{sgn} a_2, \operatorname{sgn} a_3, \dots, \operatorname{sgn} a_n, 0, \dots)$, then $\sum_{k=2}^n |a_k| = |g(z_n)| \leq \|g\|\|z_n\| = \|g\|$ for all n, hence $\sum_{k=2}^\infty |a_k|$ exists and we can define $a_1 = g((1, 1, \dots)) - \sum a_{k+1}$. It is easy to verify that $a = (a_1, a_2, \dots)$ is what we desire. Therefore f is surjective. □

5.3 Let C_0 be the set of all sequences converging to 0, and define

$$\|\cdot\| : \{\xi_k\} \in C \mapsto \sup_{k\geq 1} |\xi_k|,$$

show that $C_0^* = l^1$.

Proof. Let $a = \{a_k\} \in l^1$, define $f(a) = g_a : C \to \mathbb{R}$ as $g(x) = \sum a_k \xi_k$. It is clear that $g_a(x)$ exists since $\{\xi_k\}$ is bounded. And $|g_a(x)| \le \|a\| \, \|x\|$, which implies that $\|g_a\| \le \|a\|$. Let $y_n = (\operatorname{sgn} a_1, \dots, \operatorname{sgn} a_n, 0, 0, \dots)$, then $g_a(y_n) = \sum_{i=1}^n |a_k| \to \|a\|$ as $n \to \infty$, and $\|y_n\| = 1$ for all n, hence $\|g_a\| = \|a\|$, and $f : l^1 \to C^*$ is an isometry.

Now we shall show that f is surjective. Let $g \in C^*$, let $a_k = g(x_k)$, where $x_k = (x_{k1}, \dots, x_{kj}, \dots)$ and $x_{kj} = \delta_{kj}$. Notice that if $x = (\lambda_1, \lambda_2, \dots) \in C$ then $\sum \lambda_i x_i$ converges to x in C_0, hence $g(x) = \sum a_k \lambda_k$. The last thing is to show that a is in l^1 indeed. Take $z_n = (\operatorname{sgn} a_1, \dots, \operatorname{sgn} a_n, 0, \dots)$, then $\sum_{k=1}^n |a_k| = |g(z_n)| \le \|g\| \, \|z_n\| = \|g\|$ for all n. Hence $\sum |a_k|$ exists and f is therefore surjective. □

5.4 Prove that a finite dimensional normed linear space is reflexive.

Proof. Suppose that $e_1, \dots, e_n$ be a basis of normed linear space $\mathscr{X}$. Since the natural map $T : \mathscr{X} \to \mathscr{X}^{**}$ is injective, $Te_1, \dots, Te_n$ is a basis of $T(\mathscr{X})$, thus $\dim T(\mathscr{X}) = n$. Now it suffices to show that the conjugate space of a n-dimensional space is also n-dimensional. Based on this, we would have that $\dim \mathscr{X}^{**} = \dim \mathscr{X}^* = \dim \mathscr{X} = n$, whence it must hold that $T(\mathscr{X}) = \mathscr{X}^{**}$ and $\mathscr{X}$ is reflexive.

To prove that $\dim \mathscr{X}^* = n$, we shall find a basis of $\mathscr{X}^*$. Take $f_1, \dots, f_n$ defined by $f_i(e_j) = \delta_{ij}$, it is easy to verify that $\{f_i\}$ is a basis. □

5.5 Prove that a Banach space is reflexive iff its conjugate space is reflexive.

Proof. Denote the Banach space by $\mathscr{X}$, the natural map from $\mathscr{X}$ to $\mathscr{X}^{**}$ by T and the natural map from $\mathscr{X}^*$ to $\mathscr{X}^{***}$ by U.

\`Only if': Assume that $\mathscr{X}$ is reflexive, or T is bijective. Let $y \in \mathscr{X}^{***}$ we need to find $f \in \mathscr{X}^*$ such that $Uf = y$, or $y(X) = X(f)$ for all $X \in \mathscr{X}^{**}$. Define $f : \mathscr{X} \to \mathbb{K}$ as $f(x) = y(Tx)$, we shall show that f is exactly desired.

Let $X \in \mathscr{X}^{**}$, then we have $X(f) = f(T^{-1}X) = y(X)$. And we also know that $f \in \mathscr{X}^*$, because $|f(x)| = |y(Tx)| \le \|y\| \, \|Tx\| = \|y\| \, \|x\|$.

\`If': Assume that $\mathscr{X}^*$ is reflexive, hence $\mathscr{X}^{**}$ is reflexive (this is the \`only-if' part). Since $\mathscr{X}$ is complete, $T\mathscr{X}$ is a closed subspace of $\mathscr{X}^{**}$ thus reflexive (Pettis Theorem). □

5.6 Let X be a normed linear space and T the natural map from X to X^{**}. Show that $R(T)$ is closed iff $\mathscr{X}$ is complete.

Proof. Trivial, as T is an isometry. □

5.7 Define T over l^1 as

$$T : (x_1, x_2, \dots, x_n, \dots) \mapsto (0, x_1, x_2, \dots, x_n, \dots).$$

Show that $T \in \mathscr{L}(l^1)$ and find T^*.

Proof. It is clear that $\|Tx\| = \|x\|$ hence $T \in \mathscr{L}(l^1)$. For all $f \in (l^1)^*$, $T^* f(x) = f(Tx)$. Note that f corresponds to $(b_1, b_2, \dots)$ in l^∞ as presented in Exercise 2.5.1. Thus $T^* f$ corresponds to $(b_2, b_3, \dots)$. So $T^* f$ is the left shift operator in l^∞. □

5.8 Define T over l^2 as

$$T : (x_1, x_2, \dots, x_n, \dots) \mapsto (x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, \dots).$$

Show that $T \in \mathscr{L}(l^2)$ and find T^*.

Proof. $\|Tx\| = \sum |x_k|/k^2 \le (\sum |x_k|^2)^{\frac{1}{2}} (\sum 1/k^2)^{\frac{1}{2}} \le \sqrt{2}\|x\|$, hence T is a bounded operator. Suppose $f \in (l^2)^* \simeq l^2$ corresponds to $(b_1, b_2, \dots) \in l^2$. Then $T^* f$ corresponds to $(b_1, b_2/2, \dots)$. Therefore $T^* f = f$. □

5.9 Let H be a Hilbert space, $A \in \mathscr{L}(H)$ satisfies

$$(Ax, y) = (x, Ay), \quad \forall x, y \in H.$$

Show that

(1) $A^* = A$;

(2) If $R(A)$ is dense in H, then function $Ax = y$ has a unique solution for all $y \in R(A)$.

Proof. (1) Let $f \in H^*$, from Riesz Representation Theorem, there exists $y_f \in H$ such that $f(x) = (x, y_f)$ and $\|f\| = \|y_f\|$. So $H^* \simeq H$, under the isometry $f \mapsto y_f$. Since it holds that $(x, y_{A^*f}) = (A^*f)(x) = f(Ax) = (Ax, y_f) = (x, Ay_f)$ for all x, we must have $y_{A^*f} = Ay_f$ for all $f \in H^*$. A^* maps f to A^*f, corresponding with y_f to $y_{A^*f} = Ay_f$ in H, therefore $A^* = A$.

(2) It suffices to show that $Ax = 0$ has unique solution $x = 0$. Suppose y is a solution to $Ax = 0$ then $0 = (x, Ay) = (Ax, y)$ for all x, hence y is orthogonal to every element in $R(A)$. Since $R(A)$ is dense, we have that y is orthogonal to every element in $\mathscr{X}$, and y must be 0. □

5.10 Let $\mathscr{X}$ and $\mathscr{Y}$ be normed linear spaces and $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$. Suppose that A^{-1} exists and $A^{-1} \in \mathscr{L}(\mathscr{Y}, \mathscr{X})$, show that

(1) $(A^*)^{-1}$ exists and $(A^*)^{-1} \in \mathscr{L}(\mathscr{X}^*, \mathscr{Y}^*)$;

(2) $(A^*)^{-1} = (A^{-1})^*$.

Proof. First we show that A^* is bijective. Suppose that $A^*f = 0$, where $f \in \mathscr{Y}^*$. Then $f(Ax) = A^*f(x) = 0$ for all x. Since A is bijective and $A\mathscr{X} = \mathscr{X}$, we have that $f(x) = 0$ for all $x \in \mathscr{X}$, i.e., $f = 0$, which implies that A is injective. Let $f \in \mathscr{X}^*$, define g as $g(x) = f(A^{-1}x)$, then $g(Ax) = f(x)$, indicating that $A^*g = f$, thus A^* is surjective. Therefore, $(A^*)^{-1}$ exists.

Let $f \in \mathscr{X}^*$ and $g = (A^*)^{-1}f$, then $f = A^*g$, thus $g(Ax) = A^*g(x) = f(x)$, and $|g(x)| = |f(A^{-1}x)| \leq \|f\|\,\|A\|\,\|x\|$, hence $\|(A^*)^{-1}f\| \leq \|f\|\,\|A^{-1}\|$, and $\|(A^*)^{-1}\| \leq \|A^{-1}\|$. So $(A^*)^{-1} \in \mathscr{L}(\mathscr{X}^*, \mathscr{Y}^*)$.

It follows from $(A^*)^{-1}f(x) = f(A^{-1}x) = (A^{-1})^*f(x)$ that $(A^*)^{-1} = (A^{-1})^*$. □

Remark. It is known that $I^* = I$, hence $I = I^* = (AA^{-1})^* = (A^{-1})^*A^*$.

5.11 Let $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ be normed linear spaces, $B \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ and $A \in \mathscr{L}(\mathscr{Y}, \mathscr{Z})$. Show that $(AB)^* = B^*A^*$.

Proof. Let $f \in \mathscr{X}^*$. It follows from $(AB)^*f(x) = f(ABx) = A^*f(Bx) = B^*A^*f(x)$ that $(AB)^* = B^*A^*$. □

5.12 Let $\mathscr{X}, \mathscr{Y}$ be Banach space and $T : \mathscr{X} \to \mathscr{Y}$ is a linear operator. For all $g \in \mathscr{Y}^*$ it holds that the map $x \mapsto g(Tx)$ is in $\mathscr{X}^*$, show that T is continuous.

Proof. It suffices to show that T is a closed map and the conclusion follows from Closed Graph Theorem. Suppose that $x_n \to x$ and $Tx_n \to y$. From the assumption we know that $g(Tx_n) \to g(Tx)$ for all $g \in Y^*$, thus $Tx_n \rightharpoonup Tx$. Hence $Tx = y$. □

5.13 Suppose that $\{x_n\} \subseteq C[a,b]$, $x \in C[a,b]$ and $x_n \rightharpoonup x$. Show that $\lim x_n(t) = x(t)$ for all $x \in [a,b]$.

Proof. Fix t, define $f \in C[a,b]^*$ as $f(x) = x(t)$ ($|f(x)| \leq \|x\|$ thus $\|f\| \leq 1$). According to the definition of weak convergence, $\lim f(x_n) = f(x)$, which is exactly $\lim x_n(t) = x(t)$. □

5.14 In a normed linear space holds $x_n \rightharpoonup x_0$. Show that $\liminf_{n\to\infty} \|x_n\| \geq \|x_0\|$.

Proof. It holds obviously if $x_0 = 0$. Now assume that $x_0 \leq 0$. There exists a linear functional such that $f(x_0) = \|x_0\|$, and $\|f\| = 1$. From $\lim f(x_n) = f(x_0)$ it follows that for any $\epsilon > 0$, $\|x_n\| \geq |f(x_n)| > |f(x_0)| - \epsilon = \|x_0\| - \epsilon$ when n is large enough, which implies that $\liminf \|x_n\| \geq \|x_0\|$. □

5.15 Let H be a Hilbert space and $\{e_n\}$ an orthonormal basis. Show that $x_n \rightharpoonup x_0$ in H iff

(1) $\|x_n\|$ is bounded;

(2) $(x_n, e_k) \to (x_0, e_k)$ as $n \to \infty$ for all k.

Proof. `If': Let $f \in H^*$, by Riesz Representation Theorem we have y_f such that $f(x) = (x, y_f)$ for all $x \in H$. Write $y_f = \sum_k y_k e_k$. Let $y_f^n = \sum_{k=1}^n y_k e_k$. Given $\epsilon > 0$, there exists N such that $\|y_f - y_f^n\| < \epsilon$ for all $n > N$. For this y_f^n, we have $|(x_n - x_0, y_f^n)| \leq \sum_{k=1}^n |y_k||(x_n - x_0, e_k)| \to 0$ from (2), and thus there exists $N_1 > N$ such that $|(x_n - x_0, y_f^n)| < \epsilon$ for all $n > N_1$. Suppose that $\|x_n\| \leq M$ and from the previous problem we see that $\|x_0\| \leq M$. It follows that

$$\begin{aligned} |f(x_n) - f(x_0)| &= |(x_n - x_0, y_f)| \\ &\leq |(x_n - x_0, y_f^n)| + |(x_n - x_0, y_f - y_f^n)| \\ &\leq \epsilon + \|x_n - x_0\|\|y_f - y_f^n\| \\ &\leq (1 + 2M)\epsilon, \end{aligned}$$

which implies that $f(x_n) \to f(x_0)$.

`Only if': Assume that $x_n \rightharpoonup x_0$. Let $f \in H^*$ as $f(x) = (x, e_k)$, then (2) holds by the definition of weak convergence. For each n, define $T_n \in H^{**}$ as $T_n(f) = f(x_n)$, then it can be shown that $\|T_n\| = \|x_n\|$. Then (1) follows from Uniform Boundedness Theorem. □

5.16 Let T_n be a translation map in $L^p(\mathbb{R})$ as $(T_n u)(x) = u(x + n)$ for all $u \in L^p(\mathbb{R})$. Prove that $T_n \rightharpoonup 0$ but $\|T_n u\|_p = \|u\|_p$.

Proof. It is clear that $\|T_n u\|_p = \|u\|_p$. Let $f \in L^p(\mathbb{R})^*$, we shall show that $f(T_n u) \to f(0)$ for all u, or, equivalently, let $v \in L^q$, show that $\int_{\mathbb{R}} T_n u \cdot v \to 0$. Given $\epsilon > 0$. Since $u \in L^p$ there exists X such that $(\int_N^\infty |u|^p)^{\frac{1}{p}} < \epsilon$, and since $u \in L^q$ there exists A such that $(\int_{-\infty}^A |v|^q)^{\frac{1}{q}} < \epsilon$. Thus there exists N such that $A + n \geq X$ for all $n > N$. And for those n, it holds that

$$\begin{aligned} \left|\int_{\mathbb{R}} T_n u \cdot v\right| &= \left|\int_{-\infty}^A u(x+n)v(x)\right| + \left|\int_A^\infty u(x+n)v(x)\right| \\ &\leq \epsilon\|v\| + \|u\|\epsilon = (\|u\| + \|v\|)\epsilon, \end{aligned}$$

which implies that $\int T_n u \cdot v \to 0$. □

5.17 Let S_n be an operator on $L^p(\mathbb{R})$ defined as

$$(S_n u)(x) = \begin{cases} u(x), & |x| \leq n; \\ 0, & |x| > n. \end{cases}$$

Show that S_n converges to I strongly but not uniformly.

Proof. Let $u \in L^p(\mathbb{R})$, then $\|(S_n - I)(u)\| = (\int_n^\infty |u|^p)^{\frac{1}{p}} \to 0$, hence $S_n \to I$ strongly. For a fixed $n > 1$, take $u(x) = \chi_{|x|>n} 1/x^2$ then $\|(S_n - I)u\| = \|u\|$, so $\|S_n - I\| \geq 1$. Therefore S_n does not converge to I uniformly. □

5.18 Let H be a Hilbert space, $x_n \rightharpoonup x_0$ and $y_n \to y_0$. Show that $(x_n, y_n) \to (x_0, y_0)$.

Proof. The weak convergence of $\{x_n\}$ implies that for any $y \in H$ it holds that $(x_n, y) \to (x_0, y)$. Given ϵ, there exists N such that $|(x_n, y_0) - (x_0, y_0)| < \epsilon$ and $\|y_n - y_0\| < \epsilon$ for all $n > N$. And from the uniform boundedness priciple there exists M such that $\|x_n\| \leq M$ for all n. Therefore,

$$\begin{aligned} |(x_n, y_n) - (x_0, y_0)| &\leq |(x_n, y_n) - (x_n, y_0)| + |(x_n, y_0) - (x_0, y_0)| \\ &\leq \|x_n\| \|y_n - y_0\| + \epsilon \\ &\leq (M+1)\epsilon, \end{aligned}$$

implying that $(x_n, y_n) \to (x_0, y_0)$. □

5.19 Let $\{e_n\}$ be an orthonormal set in Hilbert space H. Prove that $e_n \rightharpoonup 0$ and $e_n \not\to 0$.

Proof. It follows from $\sum_{n=1}^{\infty} |(e_n, x)|^2 \leq \|x\|^2$ that $(e_n, x) \to 0$ for all $x \in H$, thus $e_n \rightharpoonup 0$. It is obvious that $e_n \not\to 0$ because $\|e_n\| = 1$ for all n. □

5.20 Let H be a Hilbert space. Show that $x_n \to x$ in H iff $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$.

Proof. `If': Given $\epsilon > 0$. Since $x_n \rightharpoonup x$, there exists N such that $|(x_n - x, x)| < \epsilon$ and $|(x_n, x) - (x, x)| < \epsilon$ for all $n > N$. We can also require that $|\|x_n\| - \|x\|| < \epsilon$ for $n > N$. Then we have

$$\begin{aligned} |(x_n - x, x_n - x)| &\leq |(x_n - x, x_n)| + |(x_n - x, x)| \\ &< |\|x_n\| - (x_n, x)| + \epsilon \\ &< |\|x_n\| - \|x\|| + |\|x\| - (x_n, x)| + \epsilon \\ &< 3\epsilon, \end{aligned}$$

which shows that $\|x_n - x\| \to 0$, or $x_n \to x$.

`Only if': Trivial. □

5.21 Show that the weak sequential compactness of a set in a reflexive Banach space is equivalent to the boundedness.

Proof. In the proof of Eberlein-Smulian Theorem, it has been proved that boundedness implies weak sequential compactness. Now we prove the converse. Assume that $A \subseteq \mathscr{X}$ is weak sequentially compact, we shall show that A is bounded.

If A is unbounded, then there exist $\{x_n\} \subseteq A$ such that $\|x_n\| > n$ and $x_n \rightharpoonup x_0$. View x_n as elements in $\mathscr{X}^{**}$ and $\langle x_n, f\rangle \to \langle x_0, f\rangle$ for all $f \in X^*$. From uniform boundedness principle it follows that $\|x_n\|$ is bounded. We meet a contradiction and therefore A must be bounded. □

5.22 Show that closed convex set in a normed linear space is weakly closed, that is, if M is a closed convex set, $\{x_n\} \subseteq M$ and $x_n \rightharpoonup x_0$ then $x_0 \in M$.

Proof. From Mazur's Theorem and the closedness of M, it follows immediately that $x_0 \in \overline{\mathrm{co}(\{x_n\})} \subseteq M$. □

5.23 Let $\mathscr{X}$ be a reflexive Banach space and M a bounded closed convex set in $\mathscr{X}$. Show that for every $f \in \mathscr{X}^*$, f attains maximum and minimum value on M.

Proof. It is clear that $\sup_{x \in M} f(x)$ exists, denote it by S. Then there exists $\{x_n\} \subseteq M$ such that $S \geq f(x_n) > S - \frac{1}{n}$. From Eberlein-Smulian Theorem and the previous problem we know that there exists $x_0 \in M$ such that $x_{n_k} \rightharpoonup x_0$, then $f(x_0) = \lim f(x_{n_k}) = S$. The proof of the minimum case is similar. □

5.24 Let $\mathscr{X}$ be a reflexive Banach space and M a nonempty closed convex set in $\mathscr{X}$. Show that there exists $x_0 \in M$ such that $\|x_0\| = \inf\{\|x\| : x \in M\}$.

Proof. Let $d = \inf\{\|x\| : x \in M\}$ then there exists $\{x_n\} \subseteq M$ such that $d \leq \|x_n\| < d + \frac{1}{n}$. Thus $\{x_n\}$ is bounded, it follows from Eberlein-Smulian Theorem that there exists x_0 such that $x_{n_k} \rightharpoonup x_0$. Moreover, from Exercise 2.5.22, we have that $x_0 \in M$, thus $\|x_0\| \geq d$. On the other hand, there exists $f \in \mathscr{X}^*$ that $f(x_0) = \|x_0\|$ and $\|f\| = 1$. Thus $\|x_0\| = |f(x_0)| = \lim |f(x_{n_k})| \leq \limsup \|f\| \, \|x_{n_k}\| = \limsup \|x_{n_k}\| = d$. Therefore $\|x_0\| = d$. □

6 Spectrum of Linear Operators

6.1 Let $\mathscr{X}$ be a Banach space. Show that the set of all continuously invertible operators is an open subset of $\mathscr{L}(\mathscr{X})$.

Proof. First we have this fact: If A, B are invertible then AB is invertible too and $(AB)^{-1} = B^{-1}A^{-1}$. Let A be an invertible operator in $\mathscr{L}(\mathscr{X})$. For any $B \in L(\mathscr{X})$ with $\|A - B\| \leq 1/\|A^{-1}\|$, it holds that $\|(A-B)A^{-1}\| < 1$ and thus from Lemma 2.6.6 we have that $I - ((A-B)A^{-1}) = BA^{-1}$ is invertible. Therefore $B = (BA)^{-1}A$ is invertible. □

6.2 Let A be a closed linear operator, $\lambda_1, \ldots, \lambda_n \in \sigma_p(A)$ are mutually distinct, x_i is an eigenvector of λ_i. Show that $\{x_1, \ldots, x_n\}$ are linearly independent.

Proof. Suppose that $\{x_1, \ldots, x_n\}$ is linearly dependent, then we can find a shortest equation (`shortest' means least number of items) with nontrivial coefficients as

$$c_1 x_{k_1} + c_2 x_{k_2} + \cdots c_n x_{k_m} = 0 \tag{2}$$

where $k_i \in \{1, \ldots, n\}$ and $c_i \neq 0$. Then we have

$$c_1 A x_{k_1} + c_2 A x_{k_1} + \cdots + c_n A x_{k_m} = 0,$$

or

$$c_1 \lambda_1 x_{k_1} + c_2 \lambda 2 x_{k_2} \cdots + c_n \lambda_n x_{k_m} = 0.$$

On the other hand, we have

$$c_1 \lambda 1 x_{k_1} + c_2 \lambda 1 x_{k_2} + \cdots + c_n \lambda_1 x_{k_m} = 0.$$

Thus

$$c_2(\lambda 1 - \lambda_2) x_{k_2} + \cdots + c_n(\lambda_1 - \lambda_n) x_{k_m} = 0,$$

which has nontrivial coefficients and less terms than has (2). This is a contradiction, and therefore $\{x_1, \ldots, x_n\}$ must be linearly independent. □

6.3 In two-sided l^2 space, the right shift operator A is defined as

$$x = (\ldots, \xi_{-n}, \xi_{-n+1}, \ldots, \xi_{-1}, \xi_0, \xi_1, \ldots, \xi_{n-1}, \xi_n, \ldots) \in l^2$$
$$\mapsto Ax = (\ldots, \eta_{-n}, \eta_{-n+1}, \ldots, \eta_{-1}, \eta_0, \eta_1, \ldots, \eta_{n-1}, \eta_n, \ldots)$$

where $\eta_m = \xi_{m-1}$ ($m \in \mathscr{Z}$). Prove that $\sigma_c(A) = \sigma(A) =$ unit circle.

Proof. Since $\|A\| = 1$, $\sigma(A)$ is contained within the unit disc. If $|\lambda| < 1$, then $(\lambda I - A)x = B(\frac{I}{\lambda} - A)Cx = \frac{1}{\lambda} \cdot B(\lambda I - \lambda A)Dx$, where B is the left-shift operator and D is the reverse operator ($(Dx)_n = x_{-n}$). All of the three operators on the right-hand side is invertible and so is their product. Therefore, $\sigma(A) \subseteq C$, where C denotes the unit circle.

Consider $\lambda = 1$. Since

$$y = (I - A)x \iff y_k = x_k - x_{k-1} \iff x_k = \begin{cases} x_0 + \sum_{j=1}^{k} y_j, & k > 0 \\ x_0 - \sum_{j=-k}^{-1} y_j, & k < 0 \end{cases},$$

it holds that

$$R(I - A) = \left\{ y \in l^2 : \sum_{k=1}^{\infty} \left(\left| x_0 + \sum_{j=1}^{k} y_j \right|^2 + \left| x_0 - \sum_{j=-k}^{-1} y_j \right|^2 \right) < \infty \text{ for some } x_0 \right\}.$$

It is obvious that $R(I - A) \neq \mathscr{X}$. Let $\xi \in l^2$, for any $\epsilon > 0$, there exists N such that

$$\sum_{k=N+1}^{\infty} |\xi_k|^2 + \sum_{k=-\infty}^{k=-N-1} |\xi_k|^2 < \frac{\epsilon^2}{6}.$$

Let $c = \sum_{k=-N}^{N} \xi_k$, find m such that $|c|^2/m < \epsilon^2/3$, let

$$y_j = \begin{cases} \xi_j, & |j| \leq N; \\ -c/m, & N + 1 \leq |j| \leq N + m; \\ 0, & |j| > N + m, \end{cases}$$

then $y = (\ldots, -y_n, \ldots, y_0, \ldots, y_n) \in R(I - A)$ and $\|\xi - y\| < \epsilon$. Hence $R(I - A)$ is dense in $\mathscr{X}$ and $1 \in \sigma_c(A)$. Moreover, the general case of $|\lambda| = 1$ can be reduced to $\lambda = 1$. In fact we have that

$$\begin{aligned} y = (\lambda I - A)x &\iff y_k = \lambda x_k - x_{k-1} \\ &\iff \lambda^{k-1} y_k = \lambda^k x_k - \lambda^{k-1} x_{k-1} \\ &\iff \eta_k = \xi_k - \xi_{k-1}, \end{aligned}$$

where $\eta_k = \lambda^{k-1} y_k$ and $\xi_k = \lambda^k y_k$, which reduces to $\lambda = 1$. As a result, $C \subseteq \sigma_c(A)$. Finally we conclude that $\sigma_c(A) = \sigma(A) = C$. □

6.4 Consider the left shift operator in l^2

$$A : (\xi_1, \xi_2, \ldots) \mapsto (\xi_2, \xi_3, \ldots).$$

Show that $\sigma_p(A) = \{\lambda : |\lambda| < 1\}$, $\sigma_c(A) = \{\lambda : |\lambda| = 1\}$, and $\sigma(A) = \sigma_p(A) \cup \sigma_c(A)$.

Proof. Since $\|A\| = 1$, $\sigma(A)$ is contained within the unit disk. We discuss the following two cases.

(1) $|\lambda| < 1$. Take $y = (1, \lambda, \lambda^2, \ldots)$ we have that $Ay = \lambda y$, therefore $\lambda \in \sigma_p(A)$.

(2) $|\lambda| = 1$. First it is clear that λ is not an eigenvalue.
First consider $\lambda = 1$. Since

$$y = (I - A)x \iff y_k = x_k - x_{k+1} \iff x_{k+1} = x_1 - \sum_{j=1}^{k} y_j,$$

it holds that

$$R(I-A)=\left\{y\in l^2:\sum_{k=1}^{\infty}\left|x_1-\sum_{j=1}^{k}y_j\right|^2<\infty \text{ for some } x_1\right\}$$

Obviously $R(I-A)\neq l^2$. Let $\xi\in l^2$, for any $\epsilon>0$, there exists N such that

$$\sum_{k=N+1}^{\infty}|\xi_k|^2<\frac{\epsilon^2}{6}.$$

Let $c=\sum_{k=1}^{N}\xi_k$, find m such that $|c|^2/m<\epsilon^2/6$, let

$$\begin{cases}\xi_j, & j\leq N;\\ -c/m, & N+1\leq j\leq N+m;\\ 0, & j>N+m.\end{cases}$$

Then we have that $y=(y_1,y_2,\dots)\in R(I-A)$ (where $x_1=0$) and $\|\xi-y\|<\epsilon$, so $\overline{R(I-A)}=l^2$ and $1\in\sigma_c(A)$. Moreover, the general case of $|\lambda|=1$ can be reduced to $\lambda=1$. In fact we have that

$$\begin{aligned}y=(\lambda I-A)x &\iff y_k=\lambda x_k-x_{k+1}\\ &\iff \lambda^{-k-1}y_k=\lambda^{-k}x_k-\lambda^{-k-1}x_{k+1}\\ &\iff \eta_k=\xi_k-\xi_{k+1},\end{aligned}$$

where $\eta_k=\lambda^{-k-1}y_k$ and $\xi_k=\lambda^{-k}y_k$, which reduces to $\lambda=1$. As a result, $C\subseteq\sigma_c(A)$.

Finally we conclude that $\sigma_c(A)=C$, $\sigma_p(A)=\{z:|z|<1\}$ and $\sigma(A)=\sigma_c(A)\cup\sigma_p(A)$. □

6.5 Consider the differential operator on $L^2(0,\infty)$

$$A:x(t)\mapsto\frac{dx}{dt},\quad D(A)=H^1(0,\infty).$$

Show that

(1) $\sigma_p(A)=\{\lambda\in\mathbb{C}:\Re\lambda<0\}$;
(2) $\sigma_c(A)=\{\lambda\in\mathbb{C}:\Re\lambda=0\}$;
(3) $\sigma_r(A)=\emptyset$.

Proof. Consider the differential equation $(\lambda I-A)x=0$, or $\lambda x(t)=\frac{dx}{dt}$. It has solution $x(t)=Ke^{\lambda t}$. Then it is easy to verity that $x\in H^1(0,\infty)$ when $\Re\lambda<0$ and in other cases $x(t)=0$ is the unique solution in $H^1(0,\infty)$. Therefore $\sigma_p(A)=\{\lambda\in\mathbb{C}:\Re\lambda<0\}$.

It is a well-known result (Paley–Wiener Theorem) that $L^2(0,\infty)$ is exactly the image of holomorphic fourier transform of Hardy space of the upper half-plane $H(\mathbb{C}^+)$, namely, $\mathcal{F}(H(\mathbb{C}^+))=L^2(0,\infty)$.

Consider the equation

$$(\lambda I-A)x=y,\quad x,y\in L^2(0,\infty)$$

Let $x=\hat{f}$ and $y=\hat{g}$, where f,g are holomorphic on upper half-plane. Applying inverse Fourier transform on both sides, the above equation becomes

$$\lambda f-izf=g,$$

or

$$f(z)=\frac{g(z)}{\lambda-iz}. \tag{3}$$

The right-hand side is well-defined over upper half-plane when $\Re\lambda > 0$ and f is holomorphic on upper-plane. It is clear that f is square-integrable (since $|\lambda - iz| \leq \Re\lambda$) and so is izf since $izf = \lambda f - g$, hence $\hat{f}$ admits weak derivative, that is, $x \in H^1(0, \infty)$. Therefore, $\{\lambda \in \mathbb{C} : \Re\lambda > 0\} \subseteq \rho(A)$.

The above argument still applies to the case $\Re\lambda = 0$, however, with further constraints. We see that $f(z)$ given by (3) may fail to be square-integrable (despite it is still holomorphic). Thus $R(\lambda I - A) \neq L^2(0, \infty)$. But for y in a dense subset of $L^2(0, \infty)$, such as span$\{s^n e^{-s}\}$, the $f(z)$ given by (3) falls in $D(A)$. Hence $\overline{R(\lambda I - A)} = L^2(0, \infty)$ and $\{\lambda \in \mathbb{C} : \Re\lambda = 0\} \subseteq \sigma_c(A)$. □