1 Contraction Mapping

1.1 Prove that a closed subset of a complete space is a complete subspace, and any complete subspace of a metric space is closed.

Proof. Trivial. □

1.2 (Newton's Method) Let $f \in C^2[a,b]$ and $\hat{x} \in (a,b)$ such that $f(\hat{x}) = 0$ and $f'(\hat{x}) \neq 0$. Show that there exists a neighbourhood $U(\hat{x})$ such that for any $x_0 \in U(\hat{x})$ the sequence defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

converges to $\hat{x}$.

Proof. Since $f''(x)$ is continuous on $[a,b]$, it is bounded by M_2 on $[a,b]$ for some M_2. Besides, it follows from $f'(\hat{x}) \neq 0$ that there exists $\delta_1 > 0$ such that $|f'(x)| \geq M_1$ for all $x \in [\hat{x} - \delta_1, \hat{x} + \delta_1]$. Since $f(\hat{x}) = 0$, we can find $\delta < \delta_1$ such that $|f(x)| < M_1^2/(2M_2)$ for all $x \in [\hat{x} - \delta, \hat{x} + \delta]$. More strongly, we can find δ small enough such that $f(x) \neq 0$ for all $x \in [\hat{x} - \delta, \hat{x} + \delta] \setminus \{\hat{x}\}$, because $f'(\hat{x}) \neq 0$. Let $U(\hat{x}) = (\hat{x} - \delta, \hat{x} + \delta)$.

Let $g(x) = x - f(x)/f'(x)$ and we prove that $g(x)$ is a contraction mapping, and this is clear, because $g'(x) = |f(x)f''(x)/f'(x)^2| < M_1^2/(2M_2) \cdot M_2/M_1^2 = 1/2$. Therefore, $\{x_n\}$ would converge and denote the limit point by $x_0 \in [\hat{x} - \delta, \hat{x} + \delta]$. Take limit in the both sides of the recursive formula, we obtain that $x_0 = x_0 - f(x_0)/f'(x_0)$, which implies that $f(x_0) = 0$. From the construction of $U(\hat{x})$ we have seen that x is the only zero of f in $[\hat{x} - \delta, \hat{x} + \delta]$, whence it must hold $x_0 = \hat{x}$. □

1.3 Let $(\mathscr{X}, \rho)$ be a metric space and $T : \mathscr{X} \to \mathscr{X}$ be a mapping which satisfies that $\rho(Tx, Ty) < \rho(x, y)$ for all $x \neq y$ and has a fixed point. Show that the fixed point is unique.

Proof. Suppose x_1 and x_2 are two distinct fixed points of T. Then it follows from the assumption that $\rho(Tx_1, Tx_2) < \rho(x_1, x_2)$. However, $\rho(Tx_1, Tx_2)$ is exactly $\rho(x_1, x_2)$, which leads us to a contradiction. □

1.4 Let T be a contraction mapping on a metric space. Show that T is continuous.

Proof. Trivial. □

1.5 Let T be a contraction mapping. Show that $T^n (n \in \mathbb{N})$ is also a contraction mapping and the converse proposition may not hold.

Proof. Prove by induction. The statement is true when $n = 1$ and suppose the contraction coefficient is $\alpha_1 \in (0,1)$. Suppose the statement holds for $n = k$ with contraction coefficient $\alpha_k \in (0,1)$, then $\rho(T^{k+1}x, T^{k+1}y) \leq \alpha_1 \rho(T^k x, T^k y) \leq \alpha_1 \alpha_k \rho(x, y)$, whence the statement holds for $n = k + 1$. Therefore, T^n is a contradiction mapping for all $n \geq 0$.

Define $T : \mathbb{R} \to \mathbb{R}$ as

$$T(x) = \begin{cases} 1/x, & 0 < |x| < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then $T(x)$ is not a conformal mapping while $T^2(x)$ is. □

1.6 Let M be a bounded closed set in $(\mathbb{R}^n, \rho)$ and $T : M \to M$ satisfy $\rho(Tx, Ty) < \rho(x, y)$ for all $x \neq y$. Show that T has a unique fixed point in M.

Proof. It suffices to show the existence of fixed point and the uniqueness follows from Exercise 1.1.3.

Define $g(x) = \rho(x, f(x))$. From $|g(x_1) - g(x_2)| \leq \rho(x_1, x_2) + \rho(f(x_1), f(x_2)) < 2\rho(x_1, x_2)$ it follows that $g(x)$ is continuous thus it attains its minimum value at some point, say x_0, in M (which is a compact set). Hence $g(x_0) \leq g(f(x_0))$. If $x_0 \neq f(x_0)$, then we have $g(f(x_0)) = \rho(f(x_0), f(f(x_0))) < \rho(x_0, f(x_0)) = g(x_0)$ which is a contradiction. Therefore, it must hold that $x_0 = f(x_0)$. □

1.7 Show that the integral equation

$$x(t) - \lambda \int_0^1 e^{t-s} x(s) ds = y(t)$$

has a unique solution $x(t) \in C[0,1]$, where $y(t) \in C[0,1]$ and $|\lambda| < 1$.

Proof. Multiply both sides by e^{-t}, it suffices to show that

$$x(t) - \lambda \int_0^1 x(s) ds = y(t)$$

has a unique solution. Define $T : C[0,1] \to C[0,1]$ as

$$(Tf)(t) = y(t) + \lambda \int_0^1 f(s) ds,$$

and it follows that T is a contraction mapping from

$$\|Tf - Tg\| = |\lambda| \left\| \int_0^1 (f(s) - g(s)) ds \right\| \leq |\lambda| \, \|f - g\|.$$

And the unique fixed point is exactly the solution. □

2 Completion

2.1 (Space S) Let S be the set consisting of all sequences

$$x = (\xi_1, \xi_2, \dots, \xi_n, \dots)$$

of real(complex) numbers. Define metric in S as

$$\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\xi_k - \eta_k|}{1 + |\xi_k - \eta_k|},$$

where $x = (\xi_1, \dots, x_k, \dots)$ and $y = (\eta_1, \dots, \eta_k, \dots)$. Show that S is a complete metric space.

Proof. It is easy to verify that $\rho(x, y)$ is a metric indeed. Let $\{x_n\}$ be a Cauchy sequence in S and $x_n = (x_{n_1}, \dots, x_{n_k}, \dots)$. Then for any $\epsilon > 0$, there exists N such that

$$\rho(x_m, x_n) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_{m_k} - x_{n_k}|}{1 + |x_{m_k} - x_{n_k}|} < \frac{\epsilon}{1 - \epsilon}$$

for all $m > n > N$, which implies that

$$\sup_k \frac{|x_{m_k} - x_{n_k}|}{1 + |x_{m_k} - x_{n_k}|} < \frac{\epsilon}{1 - \epsilon},$$

or,

$$|x_{m_k} - x_{n_k}| < \epsilon, \quad \forall k \geq 0, \forall m, n \geq N.$$

It follows that $\{x_{n_k}\}_{n=1}^{\infty}$ is a Cauchy sequence for a fixed k, thus it converges to some x_k^*, and the convergence is uniform with regard to k. Therefore, $x_n \to x^* = (x_1^*, \dots, x_k^*, \dots)$, and S is complete. □

2.2 Let $(\mathscr{X}, \rho)$ be a metric space. Show that a Cauchy sequence is convergent iff it has a convergent subsequence.

Proof. `Only if' part is obvious and we shall prove `if' part below. Let $\{x_n\}$ be a Cauchy sequence and $x_{n_k} \to x^*$ as $k \to \infty$, where $\{n_k\}$ is strictly increasing. Given $\epsilon > 0$, there exists K such that $|x_{n_K} - x^*| < \epsilon/2$. Since $\{x_n\}$ is a Cauchy sequence, there exists $N \geq n_K$ such that $|x_n - x_{n_K}| < \epsilon/2$ for all $n > N$. Therefore, $|x_n - x^*| \leq |x_n - x_{n_K}| + |x_{n_K} - x^*| < \epsilon$, which implies that $x_n \to x^*$ as $n \to \infty$. □

2.3 Let F be the set consisting of sequences of real numbers each of which has only finitely many nonzero items. In F the metric is defined as

$$\rho(x, y) = \sup_{k \geq 1} |\xi_k - \eta_k|,$$

where $x = \{\xi_k\} \in F$ and $y = \{\eta_k\} \in F$. Show that (F, ρ) is not complete and points out its completion.

Proof. It is easy to verify that (F, ρ) is metric space. Let $x_n = (1, 1/2, \ldots, 1/n, 0, 0, \ldots)$, then $\rho(x_n, x_m) = 1/(n+1)$ and $\{x_n\}$ is a Cauchy sequence, but it does not converge in F, because for any $y \in F$ with $y_k = 0$ for $k > N_y$ we have $\rho(x_n, y) \geq 1/N_y$ for all $n > N_y$, indicating that y is not the limit point of $\{x_n\}$. The completion of F is the space consisting of all sequences of the reals. □

2.4 Prove that the space of all polynomials on $[0, 1]$ under the metric

$$\rho(p, q) = \int_0^1 |p(x) - q(x)| dx$$

is not complete. Points out its completion.

Proof. Because any continuous function on $[0, 1]$ can be uniformly approximated by a sequence of polynomials, hence the space said above is incomplete. Its completion is $L^1[0, 1]$. □

2.5 Let $\{x_n\}$ be a sequence of points in metric space $(\mathscr{X}, \rho)$. If for any $\epsilon > 0$, there exists Cauchy sequence $\{y_n\}$ such that

$$\rho(x_n, y_n) < \epsilon,$$

then $\{x_n\}$ is convergent.

Proof. It suffices to show that $\{x_n\}$ is a Cauchy sequence. Given $\epsilon > 0$, there exists $\{y_n\}$ such that $\rho(x_n, y_n) < \epsilon/3$ and N such that $\rho(y_n, y_m) < \epsilon/3$ for all $n, m > N$. Hence for $n, m > N$ it holds that $\rho(x_n, x_m) \leq \rho(x_n, y_n) + \rho(y_n, y_m) < \rho(y_m, x_m) < \epsilon$. □

3 Sequentially Compact Sets

3.1 Show that a subset A of a complete metric space is sequentially compact iff for any $\epsilon > 0$ there exists a sequentially compact ϵ-net of A.

Proof. The `only if' part is obvious, because the sequentially compactness of $A$ implies that there exist a finite $\epsilon$-net, thus the net must be sequentially compact. Now we prove the `if' part.

Given $\epsilon > 0$, and A has a sequentially compact $\epsilon/2$-net B and B is totally bounded, thus has a finite $\epsilon/2$-net C. Let $a \in A$, there exists $b \in B$ such that $\rho(a, b) < \epsilon/2$ and $c \in C$ such that $\rho(b, c) < \epsilon/2$, so $\rho(a, c) \leq \rho(a, b) + \rho(b, c)\epsilon$, which implies that C is a finite ϵ-net of A. Hence A is totally bounded.

□

3.2 Show that a continuous function over a compact set in a metric space is bounded and it can attain its supremum and infimum.

Proof. See the proof of Proposition 1.3.12. □

3.3 Prove a totally bounded set in a metric space is bounded. Consider a subset $E = \{e_k\}$ of l^2 with

$$e_k = \{0, 0, \dots, \underbrace{1}_{k^{\text{th}}\text{item}}, 0, \dots\}$$

to show that the converse proposition may not be true.

Proof. Let A be a totally bounded set, then it has a finite 1-net $\{a_1, \dots, a_n\}$. Let $d = \max_j \rho(a_1, a_j)$ Obviously A is contained in $(a_1, d+1)$ thus bounded.

It is clear that $\|e_k\| = 1$ thus $\{e_k\}$ is bounded. However,

$$\|e_i - e_j\| = \sqrt{2}$$

for all $i \neq j$, implying that $\{e_k\}$ is not totally bounded. □

3.4 Let $(\mathscr{X}, \rho)$ be a metric space and F_1, F_2 be two compact subsets. Show that there exist $x_1 \in F_1$ and $x_2 \in F_2$ such that $\rho(F_1, F_2) = \rho(x_1, x_2)$, where

$$\rho(F_1, F_2) = \inf\{\rho(x, y) : x \in F_1, y \in F_2\}.$$

Proof. Let $f(x) = \rho(x, F_1)$ then f is continuous, from Exercise 1.3.2, it follows that there exists $x_2 \in F_2$ such that $f(x_2) = \min_{x \in F_2} f(x)$. Now we define $g(x) = \rho(x_2, x)$ on F_1. Since $g(x)$ is continuous, there exists $x_1 \in F_1$ such that $g(x_1) = \rho(x_2, F_1)$. We claim that x_1, x_2 satisfy the condition.

Let $x \in F_1$ and $y \in F_2$, we have $\rho(x_1, x_2) = \rho(x_2, F_1) \leq \rho(y, F_1) \leq \rho(y, x)$, so $\rho(x_1, x_2) \leq \rho(F_1, F_2)$. Also it is obvious that $\rho(F_1, F_2) \leq \rho(x_1, x_2)$ and the conclusion follows. □

3.5 Let M be a bounded set in $C[a, b]$. Show that the set

$$E = \left\{ F(x) = \int_a^x f(x) dt \middle| f \in M \right\}$$

is sequentially compact.

Proof. It is sufficient to show that E is uniformly bounded and uniformly equicontinuous. Suppose $M \subset B(0, r)$, then for any $F(x) \in E$, it holds that $\|F\| \leq (b-a)r$, indicating that F is uniformly bounded. We have also $|F(x_1) - F(x_2)| \leq r(x_2 - x_1)$, whence it follows that F is uniformly equicontinuous. □

3.6 Let $E = \{\sin nt\}_{n=1}^{\infty}$, show that E is not sequentially compact in $C[0, \pi]$.

Proof. Take $\epsilon = 1/2$, for any n, we have $|\sin(n \cdot \pi/(2n)) - \sin 0| = 1 > \epsilon$, and $\pi/(2n) \to 0$ as $n \to \infty$. Therefore, E is not uniformly equicontinuous. □

3.7 Prove that a subset A of S (see definition in Exercise 1.2.1) is sequentially compact iff for any n there exists $C_n > 0$ such that for every $x = (\xi_1, \dots, \xi_n, \dots) \in A$ it holds that $|\xi_n| \leq C_n$.

Proof. 'Only if': Since S is complete, A must be totally bounded. Let $x_1, \dots, x_m$ be a finite $1/2$-net of A ($x_i = \{x_{i1}, \dots, x_{in}, \dots\}$). For any $x = (\xi_1, \dots, \xi_n, \dots) \in A$, there exists x_i for some i such that $\rho(x_i, x) < 1/2$, which implies that $|x_{ik} - \xi_k|/(1 + |x_{ik} - \xi_k|) < 1/2$, or, $|x_{ik} - \xi_k| < 1$ for all k. Hence $|\xi_k| \leq \max\{|x_{1k}|, \dots, |x_{nk}|\} + 1$.

'If': Let $\{x_n\} \subseteq S$. Since $x_{n,1}$ is bounded, hence it has a convergent subsequence, say $\{x_{n_{1i}}\}_{i=1}^{\infty}$ converging to x_1. We also have $x_{n_{1i},2}$ bounded, so it has a subsequence $\{x_{n_{2i}}\}$ converging to x_2. In this way, we have that $\{x_{n_{m,i}}\} \subseteq \{x_{n_{m-1,i}}\}$ and $\{x_{n_{m,i},i}\}$ is convergent to x_m for each m. It is easy to see that $\{x_{n_{mm}}\}_{m=1}^{\infty}$ is a subsequence of $\{x_n\}$ and we shall show that it is convergent. It is clear that $x_{n_{mm}k} \to x_k$ as $m \to \infty$ for each k, because $\{x_{n_{mm}}\}_{m=k}^{\infty}$ is a subsequence of $\{x_{n_{ki}}\}_{i=1}^{\infty}$. Then it is not difficult to show that $x_{n_{mm}} \to x = (x_1, \dots, x_k, \dots)$ in norm. □

3.8 Let $(\mathscr{X}, \rho)$ be a metric space and M be a sequentially compact set in $\mathscr{X}$. If map $f : \mathscr{X} \to M$ satisfies

$$\rho(f(x_1), f(x_2)) < \rho(x_1, x_2), \quad x_1 \neq x_2$$

then it has a unique fixed point in $\mathscr{X}$.

Proof. Consider the closure of M and see Exercise 1.1.6. □

3.9 Let (M, ρ) be a compact metric space and $E \subseteq C(M)$ is uniformly bounded and satisfies Hölder condition

$$|x(t_1) - x(t_2)| \leq C\rho(t_1, t_2)^{\alpha}, \quad \forall x \in E, \forall t_1, t_2 \in M$$

where $0 < a \leq 1$ and $C > 0$, show that E is sequentially compact in $C(M)$.

Proof. It is obvious that E is uniformly equicontinuous. Together with the uniform boundedness, it follows that E is sequentially compact. □

4 Normed Linear Space

4.1 For $z = (x, y) \in \mathbb{R}^2$, define

$$\|z\|_1 = |x| + |y|;\ \|z\|_2 = \sqrt{x^2 + y^2}; \|z\|_3 = \max(|x|, |y|); \|z\|_4 = (x^4 + y^4)^{1/4}.$$

(1) Show that all of $\|\cdot\|_i$ are norms of $\mathbb{R}^2$ for $i = 1, 2, 3, 4$;

(2) Draw the unit sphere in $(\mathbb{R}^2, \|\cdot\|_i)$.

(3) Find the lengths of three sides of the triangle with vertices $O(0, 0), A(1, 0), B(0, 1)$ under the four different norms.

4.2 Let $c(0, 1]$ denote the set of continuous and bounded functions on $(0, 1]$. Let $\|x\| = \sup\limits_{0<t\leq 1} |x(t)|$. Show that

(1) $\|\cdot\|$ is a norm on $c(0, 1]$;

(2) l^{∞} is isometric to $c(0, 1]$.

Proof. (1) Trivial.

(2) Define $F : l^{\infty} \to c(0, 1]$ as

$$x = (x_1, \ldots, x_k, \ldots) \mapsto f(x) = \begin{cases} x_k, & x = \frac{1}{k}; \\ x_{k+1} + (x_k - x_{k+1})k((k+1)x - 1), & \frac{1}{k+1} < x < \frac{1}{k} \end{cases}.$$

Then F is an isometry. □

4.3 In $C^1[a, b]$ define

$$\|f\|_1 = \left(\int_a^b (|f|^2 + |f'|^2)\, dx \right)^{\frac{1}{2}}, \quad \forall f \in C^1[a, b].$$

(1) Prove that $\|\cdot\|_1$ is a norm on $C^1[a, b]$;

(2) Is $(C^1[a, b], \|\cdot\|_1)$ complete?

Proof. (1) Trivial.

(2) No. Take $[a,b]=[-1,1]$ and $f_n=\sqrt{x^2+n^2}$, we have

$$\|f_n\|_1=\frac{8}{3}+\frac{2}{n^2}-\frac{2\arctan n}{n}\to 0$$

as $n\to\infty$, hence $\{f_n\}$ is a Cauchy sequence in $(C^1[a,b],\|\cdot\|_1)$. It is clear that $f_n\to|x|$ and we shall prove that if $\{f_n\}$ converges to some continuous f in norm, then it must hold that $f(x)=|x|$. If $f_n\to f$ in norm, then $\int|f_n-f|^2\to 0$ thus $f_n\to f$ in measure. Also $f_n(x)\to|x|$ in measure hence $f(x)$ and $|x|$ differ at most in a set of measure zero, since f is continuous, it follows that $f(x)=|x|\notin C^1[a,b]$, indicating that $\{f_n\}$ does not converge in $(C^1[a,b],\|\cdot\|_1)$. □

4.4 In $C[0,1]$ define

$$\|f\|_1=\left(\int_0^1|f|^2\,dx\right)^{\frac{1}{2}};\ \|f\|_2=\left(\int_0^1(1+x)|f(x)|^2\,dx\right)^{\frac{1}{2}}$$

for every $f\in C[0,1]$. Show that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$.

Proof. It is easy to verify that $\|f\|_1\le\|f\|_2\le\sqrt{2}\|f\|_1$. □

4.5 Let $BC[0,\infty)$ be the set of continuous and bounded functions on $[0,\infty)$ and $a>0$. Define

$$\|f\|_a=\left(\int_0^\infty e^{-ax}|f(x)|^2\,dx\right)^{\frac{1}{2}}$$

(1) Show that $\|\cdot\|_a$ is a norm on $BC[0,\infty)$.

(2) Prove that $\|\cdot\|_a$ is not equivalent to $\|\cdot\|_b$ for all $a\neq b$.

Proof. (1) Trivial.

(2) Suppose $a<b$. For any $C>1$, take $f=e^{-\lambda x}$ where $\lambda=(b-aC^2)/(C^2-1)$, we have $\|f\|_a=1/\sqrt{a+\lambda}$ and thus $\|f\|_a=C\|f\|_b$. □

4.6 Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be two B^* spaces and $\mathscr{X}=\mathscr{X}_1\times\mathscr{X}_2$ with norm

$$\|x\|=\max\{\|x_1\|_1,\|x_2\|_2\},$$

where $x=(x_1,x_2)$, $x_i\in\mathscr{X}_i$ and $\|x_i\|$ is the norm of $\mathscr{X}_i$ $(i=1,2)$. Show that if $\mathscr{X}_1$ and $\mathscr{X}_2$ are Banach spaces, so is $\mathscr{X}$.

Proof. It suffices to show that $\mathscr{X}$ is complete. Let $\{x_n\}$ be a Cauchy sequence in $\mathscr{X}$ and $x_n=(x_{n1},x_{n2})$. If follows immediately that $\{x_{n1}\}$ and $\{x_{n2}\}$ are both Cauchy sequences in $\mathscr{X}_1$ and $\mathscr{X}_2$, respectively. Since $\mathscr{X}_i$ is complete, there exists $x_i^*\in\mathscr{X}_i$ such that $x_{ni}\to x_i^*$. Therefore, $x_n\to(x_1^*,x_2^*)\in\mathscr{X}$ and $\mathscr{X}$ is complete. □

4.7 Let $\mathscr{X}$ be a B^* space. Show that $\mathscr{X}$ is a Banach space iff for any $\{x_n\}\subset\mathscr{X}$, $\sum_{n=1}^\infty x_n$ is convergent whenever $\sum_{n=1}^\infty\|x_n\|$ is convergent.

Proof. `Only if': Let $\{x_n\}\subset\mathscr{X}$ satisfying that $\sum_{n=1}^\infty\|x_n\|$ is convergent. Let $y_n=\sum_{k=1}^n x_k$, we shall prove that $\{y_n\}$ is a Cauchy sequence. This is because

$$\|y_n-y_m\|\le\sum_{k=m}^n\|x_k\|,\quad\forall n>m.$$

`If': Let $\{x_n\}$ be a Cauchy sequence in $\mathscr{X}$. We can choose $1 \le n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$\|x_m - x_{n_k}\| < \frac{1}{2^k}$$

for all $m > n_k$. Let $y_n = x_{n_{k+1}} - x_{n_k} (n \ge 1)$ and $y_1 = x_{n_1}$ then $\sum \|y_n\|$ is convergent, thus $x = \sum y_n$ exists. From

$$\|x_n - x\| \le \|x_n - x_{n_k}\| + \|x_{n_k} - x\| = \|x_n - x_{n_k}\| + \left\|\sum_{k=1}^{n-1} y_k - x\right\|$$

it follows that $x_n \to x$. □

4.8 Let P_n be the set of polynomials on $[a, b]$ with degree less than or equal to n. Prove that for any $f(x) \in C[a, b]$ there exists $P_0(x) \in P_n$ such that

$$\max_{a \le x \le b} |f(x) - P_0(x)| = \min_{P \in P_n} \max_{a \le x \le b} |f(x) - P(x)|.$$

That is, if we use elements in P_n to approximate $f(x)$ uniformly, $P_0(x)$ is the optimal one.

Proof. This is a direct corollary of Theorem 1.4.23, where $e_i = x^i$. □

4.9 In $\mathbb{R}^2$ we define $\|x\| = \max\{|x_1|, \|x_2|\}$ for $x = (x_1, x_2)$. Let $e_1 = (1, 0)$ and $x_0 = (0, 1)$. Find $a \in \mathbb{R}$ such that

$$\|x_0 - ae_1\| = \min_{\lambda \in \mathbb{R}} \|x_0 - \lambda e_1\|,$$

and is such a unique? Give a geometric explanation.

Proof. We have that $x_0 - \lambda e_1 = (-\lambda, 1)$, so $\|x_0 - \lambda e_1\|$ reaches the minimum value 1 when $|\lambda| \le 1$. □

4.10 Prove the strict convexity of norm is equivalent to

$$\|x + y\| = \|x\| + \|y\| (\forall x \ne 0, y \ne 0) \Rightarrow x = cy (c > 0).$$

Proof. Assume the convexity (as in Definition 1.4.24) first. Let $x \ne 0$ and $y \ne 0$, $x' = x/\|x\|$ and $y' = y/\|x\|$, $\alpha = \|x\|/(\|x\| + \|y\|)$ and $\beta = \|y\|/(\|x\| + \|y\|)$ then $\|x'\| = \|y'\| = 1$. Thus from the convexity of the norm it holds that $\|\alpha x' + \beta y'\| < 1$, i.e., $\|x + y\| < \|x\| + \|y\|$, if $x' \ne y'$. Therefore, we must have $x' = y'$, and $x = cy$ for some $c > 0$.

On the contrary, let $\|x\| = \|y\| = 1 (x \ne y)$ and $\alpha + \beta = 1$. It is clear that $\|\alpha x + \beta y\| \le \alpha\|x\| + \beta\|y\| = 1$, but the equality holds iff $\alpha x = C\beta y$ for some $C > 0$, or, $x = (C\beta/\alpha)y$. But $\|x\| = \|y\|$ hence $C\beta/\alpha = 1$, resulting in $x = y$ which is a contradiction. Hence the equality cannot hold and it must hold the strict inequality. □

4.11 Let $\mathscr{X}$ be a normed linear space. A function $\phi : \mathscr{X} \to \mathbb{R}$ is said convex if it holds

$$\phi(\lambda x + (1 - \lambda)y) \le \lambda\phi(x) + (1 - \lambda)\phi(y)$$

for all $0 \le \lambda \le 1$. Prove that a local minimum of a convex function is also a global minimum.

Proof. Suppose x is a local minimum of a convex function f. For any y, let λ be close to 1 enough, we have $\lambda x + (1 - \lambda)y$ close to x enough, thus

$$f(x) \le \phi(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y),$$

or

$$f(x) \le f(y),$$

which implies that x is also a global minimum. □

4.12 Let $(\mathscr{X}, \|\cdot\|)$ be a normed linear space and M is a finite dimensional subspace of $\mathscr{X}$ with a basis $\{e_1, \dots, e_n\}$. Given $g \in \mathscr{X}$, define $F : \mathbb{K}^n \to \mathbb{R}$ as

$$F(c_1, \dots, c_n) = \left\| \sum_{i=1}^{n} c_i e_i - g \right\|.$$

(1) F is convex;

(2) If F attains minimum value at $c = (c_1, \dots, c_n)$ then

$$f = \sum_{i=1}^{n} c_i e_i$$

is the best approximation of g in M.

Proof. (1) Trivial.

(2) This is exactly the definition of 'best approximation'. □

4.13 Let $\mathscr{X}$ be a B^* space and $\mathscr{X}_0$ a linear subspace of $\mathscr{X}$. Suppose there exists $c \in (0, 1)$ such that

$$\inf_{x \in \mathscr{X}_0} \|y - x\| \le c\|y\|, \forall y \in \mathscr{X},$$

show that $\mathscr{X}_0$ is dense in $\mathscr{X}$.

Proof. Let $y \in \mathscr{X} \setminus \mathscr{X}_0$ and $\epsilon > 0$. There exists $x_1 \in \mathscr{X}_0$ such that $\|y - x_1\| \le c\|y\| + \epsilon/2$, then there exists $x_2 \in X_0$ such that $\|y - x_1 - x_2\| \le c\|y - x_1\| + \epsilon/4 \le c^2\|y\| + c\epsilon/2 + \epsilon/4 < c^2\|y\| + \epsilon/2 + \epsilon/4$. Continue this process, for each n we have $x_n \in \mathscr{X}_0$ such that

$$\|y - x_1 - \cdots - x_n\| \le c^n\|y\| + \frac{\epsilon}{2} + \frac{\epsilon}{2^2} + \cdots + \frac{\epsilon}{2^n} < c^n\|y\| + \epsilon.$$

It follows that $\{\sum_{k=1}^{n} x_k\} \subseteq \mathscr{X}_0$ is a sequence of points converging to y, and hence $\mathscr{X}_0$ is dense. □

4.14 Let C_0 be the set of sequences of real numbers converging to 0, and define the norm in C_0 as $\|x\| = \max_{n \ge 1} |\xi_n| \quad \forall x = (\xi_1, \dots, \xi_n, \dots) \in C_0$. Let $M = \{x = \{\xi_n\} \in C_0 : \sum_{n=1}^{\infty} \xi_n/2^n = 0\}$.

(1) Show that M is a closed sybspace of C_0;

(2) Let $x_0 = (2, 0, \dots, 0, \dots)$, show that $\inf_{Z \in M} \|x_0 - Z\| = 1$ but $\|x_0 - y\| > 1$ for all $y \in M$.

(Remark: This problem provides an example indicating that the best approximation may not exist for infinite-dimensional subspace)

Proof. (1) Let $x \in M$, it is clear that $\sum |\xi_k|/2^k$ converges, since $\|\xi_k\| \le \|x\|$. Then it is trivial to verify that M is a subspace. Let $\{x_n\}$ be a Cauchy sequence in M, $x_n = \{\xi_{n1}, \dots, \xi_{nk}, \dots\}$. Then for any $\epsilon > 0$, there exists N such that $\|x_n - x_m\| < \epsilon$ for all $n, m > N$, thus for all k, $\|\xi_{nk} - \xi_{mk}\| < \epsilon$, and therefore $\{\xi_{nk}\}$ (with respect to n) is a Cauchy sequence and suppose it converges to ξ_k uniformly. Therefore, $\{x_n\}$ converges to some $x \in C_0$. Note that $|\xi_k - \xi_{nk}| < \epsilon$ for n large enough, we have $\left|\sum \xi_k/2^k\right| < \sum |\xi_k - \xi_{nk}|/2^k + \left|\sum \xi_{nk}/2^k\right| < \epsilon$. Therefore we know that $x \in M$ and M is closed.

(2) First we show that $\|x_0 - y\| > 1$ for all $y \in M$. Let $y = (y_1, \dots, y_k)$. If $y_1 < 1$ or $|y_k| > 1$ for some $k > 1$ then $\|x_0 - y\| > 1$. Now assume $y_1 \ge 1$ and $|y_k| \ge 1$ for $k \ge 2$, then $\sum y_n/2^n > 1/2 + \sum_{n=2}^{\infty} y_n/2^n$. Note that $\sum_{n=2}^{\infty} |y_n|/2^n \le 1/2$, hence $\sum y_n/2^n \ge 0$, and the equality holds only if $y_1 = 1$ and $y_k = -1$ for all $k > 1$, resulting in that $\{y_k\}$ does not converge to 0, which contradicts with $y \in C_0$.

Now we show that $\inf \|x_0 - y\| = 1$. We shall prove that for any $\epsilon > 0$, we can find $y \in M$ such that $\|x_0 - y\| \le 1 + \epsilon$. Let $\delta = 2(\epsilon + 1)/(\epsilon + 2)$, $y_1 = 1 - \epsilon$ and $y_k = -\delta^{k-1}$ for $k > 1$. Then $\|x_0 - y\| \le 1 + \epsilon$, $y_n \to 0$ and $\sum y_i/2^i = 0$. □

4.15 Let $\mathscr{X}$ be a B^* space and M a finite-dimensional proper subspace of $\mathscr{X}$. Show that there exists $y \in \mathscr{X}$ with $\|y\| = 1$ such that $\|y - x\| \geq 1$ for all $x \in M$.

Proof. Let $z \in \mathscr{X} \setminus M$ and $z' \in M$ be the best approximation of z. Let $d = \|z - z'\|$ and $y = (z - z')/d$ then $\|y\| = 1$, and $\|y - x\| = \|z - (z' + dx)\|/d \geq d/d = 1$ for all $x \in M$. □

4.16 Let f be a complex-valued function defined on $[0, 1]$ and define

$$\omega_\delta(f) = \sup\{|f(x) - f(y)| : \forall x, y \in [0, 1], |x - y| \leq \delta\}.$$

Let $0 < \alpha \leq 1$. Define Lipschitz space $\operatorname{Lip} \alpha$ as the set of all functions f such that

$$\|f\| = |f(0)| + \sup_{\delta > 0}\{\delta^{-\alpha}\omega_\delta(f)\} < \infty$$

and

$$\operatorname{lip} \alpha = \{f \in \operatorname{Lip} \alpha : \lim_{\delta \to 0} \delta^{-\alpha}\omega_\delta(f) = 0.\}$$

Show that $\operatorname{Lip} \alpha$ is a Banach space and $\operatorname{lip} \alpha$ a closed subspace of $\operatorname{Lip} \alpha$.

Proof. It is trivial that $\operatorname{Lip} \alpha$ is a normed linear space. Now we prove its completeness. Let $\{f_n\}$ be a Cauchy sequence in $\operatorname{Lip} \alpha$, then for any $\epsilon > 0$ there exists N such that $|f_n(0) - f_m(0)| < \epsilon$ and $\sup_{\delta > 0} \delta^{-\alpha}\omega_\delta(f_n - f_m) < \epsilon$ for all $n, m > N$. Hence $|f_n(x) - f_m(x)| < \epsilon\delta^\alpha + |f_n(0) - f_m(0)| < \epsilon(\delta^\alpha + 1)$ for all $n, m > N$, which implies that $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$ thus it converges to $f(x)$ for some f. We shall show that $f \in \operatorname{Lip} \alpha$. This is because

$$|f(x) - f(y)| \leq |(f(x) - f_n(x)) - (f(y) - f_n(y))| + |f_n(x) - f_n(y)| \tag{1}$$

and the uniform pointwise convergence of $f_n \to f$.

It is also trivial to see that $\operatorname{lip} \alpha$ is a subspace. Let $\{f_n\}$ be a Cauchy sequence in $\operatorname{lip} \alpha$ and thus $f_n \to f$ uniformly pointwise for some $f \in \operatorname{Lip} \alpha$. It follows easily that $f \in \operatorname{lip} \alpha$ by (1). □

4.17 (Quotient space) Let $\mathscr{X}$ be a normed linear space and $\mathscr{X}_0$ a closed linear subspace of $\mathscr{X}$. Define a equivalence relation $\sim$ by $x \sim y$ iff $x - y \in \mathscr{X}_0$. The quotient space is X/X_0.

(1) Let $[x] \in \mathscr{X}/\mathscr{X}_0$ and $x \in \mathscr{X}$. Show that $x \in [x]$ iff $[x] = x + \mathscr{X}_0$.

(2) Define addition and scalar multiplication in $\mathscr{X}/\mathscr{X}_0$ as follows.

$$[x] + [y] = x + y + \mathscr{X}_0, \quad \forall [x], [y] \in \mathscr{X}/\mathscr{X}_0;$$

$$\lambda[x] = \lambda x + \mathscr{X}_0, \quad \forall [x] \in \mathscr{X}/\mathscr{X}_0, \forall \lambda \in \mathbb{K},$$

where x, y are arbitrary chosen from $[x], [y]$ respectively. Define the norm

$$\|[x]\|_0 = \inf_{x \in [x]} \|x\|, \quad \forall [x] \in \mathscr{X}/\mathscr{X}_0,$$

Show that $(\mathscr{X}/\mathscr{X}_0, \|\cdot\|_0)$ is a B^* space.

(3) Let $[x] \in \mathscr{X}/\mathscr{X}_0$ show that for any $x \in [x]$ it holds that

$$\inf_{Z \in \mathscr{X}_0} \|x - Z\| = \|[x]\|_0$$

(4) Define the mapping $\phi : \mathscr{X} \to \mathscr{X}/\mathscr{X}_0$ as

$$\phi(x) = [x] = x + \mathscr{X}_0, \forall x \in \mathscr{X},$$

show that ϕ is a continuous linear mapping.

(5) Let $[x] \in \mathscr{X}/\mathscr{X}_0$. Show that there exist $x \in \mathscr{X}_0$ such that $\phi(x) = [x]$ and $\|x\| \le 2\|[x]\|_0$.

(6) Suppose that $(\mathscr{X}, \|\cdot\|)$ is complete. Show that $(\mathscr{X}/\mathscr{X}_0, \|\cdot\|_0)$ is complete too.

(7) Let $\mathscr{X} = C[0,1]$, $\mathscr{X}_0 = \{f \in \mathscr{X} : f(0) = 0\}$, show that $\mathscr{X}/\mathscr{X}_0$ is isometric to $\mathbb{K}$.

Proof. (1) Trivial. (Note that $0 \in \mathscr{X}_0$).

(2) Trivial.

(3) Let $x \in [x]$, then $[x] = x + \mathscr{X}_0$, hence $\|[x]\|_0 = \inf_{z\in\mathscr{X}_0} \|x+z\|$. Since $\mathscr{X}_0$ is a subspace, $\inf_{z\in\mathscr{X}_0} \|x+z\| = \inf_{z\in\mathscr{X}_0} \|x-z\|$.

(4) It is trivial that f is linear. Now suppose that $x_n \to x$, then $\|[x_n] - [x]\|_0 = \|[x_n - x]\|_0 = \inf_{z\in\mathscr{X}_0} \|x_n - x + z\| \le \|x_n - x\|$, which implies that $f(x_n) \to f(x)$. Thus f is continuous.

(5) If $\|[x]\|_0 = 0$ then $x \in \mathscr{X}_0$ then we are done. Assume that $\|[x]\|_0 > 0$. Consider $x + \mathscr{X}_0 = \{x - z : z \in \mathscr{X}_0\}$. Note that $\|[x]\|_0 = \inf_{z\in\mathscr{X}_0} \|x-z\|$, there exists $z \in X_0$ such that $\|x - z\| \le 2\inf_{z\in\mathscr{X}_0} \|x-z\|$, and $x - z$ is what we desire.

(6) Use $\|[a] - [b]\|_0 = \|[a-b]\|_0 \le \|a - b\|$.

(7) Let $f_c(x) = c$ denoting the constant function with value c. Then we claim that $g : x \mapsto [f_x]$ is an isometry between $\mathbb{K}$ and $\mathscr{X}/\mathscr{X}_0$. Notice that $\|[f_x] - [f_y]\| = \|[f_x - f_y]\| = \|[f_{x-y}]\|$, it suffices to show that $\|[f_a]\| = |a|$. On the one hand, $\|[f_a]\| \le \|f_a\| = |a|$; on the other hand, for any $f \in [f_a]$, $\|f\| \ge \|f(0)\| = |a|$. □

5 Convex Sets and Fixed Points

5.1 Let $\mathscr{X}$ be a B^* space and E be a convex proper set with an interior point 0. Denote by P the Minkowski functional corresponding to E. Show that

(1) $x \in E^\circ \iff P(x) < 1$;

(2) $\overline{E^\circ} = \overline{E}$.

Proof. (1) ``$\Rightarrow$" is obvious. Now we prove ``$\Leftarrow$". Suppose that $r \in (1, 1/P(x))$ and $B(0,\delta) \subseteq C$. Let $d = \delta(1 - 1/r)$ and we claim that $B(x,d) \subseteq C$. Suppose that $\|y - x\| < d$, then

$$\frac{1}{r} \cdot rx + \left(1 - \frac{1}{r}\right) \cdot \frac{r(y-x)}{r-1} = y.$$

Note that $r(y-x)/(r-1)$ is in $B(0,\delta)$ and thus belongs to C, we have $y \in C$ and x is therefore an interior point of C.

(2) It suffices to show every $x \in E$ can be approximated by a sequence of points in E°. This is easy, because if $x \in E$ and $rx \in E$ for all $r \in (0,1)$ and rx belongs to E^c since $P(rx) < 1$ for $r \in (0,1)$. □

5.2 Show that the convex hull of a sequentially compact set in a Banach space is also sequentially compact.

Proof. Let E be a sequentially compact set in a Banach space $\mathscr{X}$ thus it is totally bounded. We shall show that $\operatorname{co} E$ is totally bounded too. Given $\epsilon > 0$ and suppose that $\{a_1, \ldots, a_n\}$ is an ϵ-net of E. Let $x \in \operatorname{co} E$ and $x = \sum \lambda_i x_i$, where $\sum \lambda_i = 1$, $x_i \in E$, thus we can find $a_{x,i}$ such that $\|a_{x,i} - x_i\| < \epsilon$. Hence it holds that $\|x - \sum \lambda_i a_{x,i}\| \le \sum \lambda_i \|x - a_{x,i}\| < \epsilon$, which implies that $A = \{\sum \lambda_i a_i : \sum \lambda_i = 1\}$ constitutes an ϵ-net of $\operatorname{co} E$. Now we claim that A is sequentially compact, and the conclusion would follow from Exercise 1.3.1.

Given a sequence $\{x_n\}$, where $x_n = \sum \lambda_{ni} a_i$. Since $0 \le \lambda_{k1} \le 1$, there exists a convergent subsequence $\lambda_{n_{1j}1} \subseteq \{\lambda_{n1}\}$. Then from $x_{n_{1j}2} \in [0,1]$, there exists a convergent subsequence $x_{n_{2j}2}$. Continue this for finitely many steps (note that $\{a_i\}$ is a finite set) and we will get a subsequence $\{x_{m_k}\}$ of $\{x_n\}$ with $\{\lambda_{m_k} i\}$ convergent for all i. Therefore, $\{x_n\}$ contains a convergent subsequence and A is sequentially compact. □

5.3 Let C be a compact convex set in a B^* space $\mathscr{X}$ and the mapping $T : C \to C$ be continuous. Show that T has a fixed point in C.

Proof. This is a direct corollary of Schauder's Fixed Point Theorem. $\square$

5.4 [1] Let C be a closed bounded convex set in Banach space $\mathscr{X}$. The maps $T_i : C \to \mathscr{X}$ $(i = 1, 2)$ satisfies

(1) For all $x, y \in C$ we have $T_1 x + T_2 y \in C$;

(2) T_1 is a contraction mapping and T_2 is a compact mapping.

Show that $T_1 + T_2$ has at least one fixed point in C.

Proof. Fix $y \in C$, it is not difficult to see that the map $x \mapsto T_1 x + T_2 y$ is a contraction mapping thus it has a unique fixed point $x_y \in C$. Define $T : C \to C$ as $y \mapsto x_y$, then $Tx = T_1(Tx) + T_2 x$. We shall prove that T is continuous and compact, then a fixed point of T is also a fixed point of $T_1 + T_2$.

It holds that

$$\begin{aligned}\|Tx_1 - Tx_2\| &= \|(T_1(Tx_1) + T_2x_1) - (T_1(Tx_2) + T_2x_2)\| \\ &\le \|(T_1(Tx_1 - Tx_2))\| + \|T_2(x_1 - x_2)\| \\ &\le \alpha\|Tx_1 - Tx_2\| + \|T_2(x_1 - x_2)\|,\end{aligned}$$

thus

$$\|Tx_1 - Tx_2\| \le \frac{1}{1-\alpha}\|T_2(x_1 - x_2)\|.$$

Since T_2 is continuous and compact, it follows immediately that T is continuous and compact too. $\square$

5.5 Let A be n-by-n matrix with positive elements. Show that there exists λ and a vector $x \in \mathbb{R}^n$ such that $Ax = \lambda x$ and the elements of x are all non-negative but not all zeroes.

Proof. Consider $C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1,\ x_i \ge 0\ (i = 1, \ldots, n)\}$ then C is a compact convex set. Define a map $f : C \to C$ as $f(x) = (Ax)/\sum_{i=1}^n (Ax)_i$. Now it suffices to show that $f(x)$ is continuous, which would imply that $T(C)$ is compact and the conclusion would follow from Schauder's Fixed Point Theorem.

Let $x \in C$, and we write $\sum_{i=1}^n (Ax)_i = a^T x$, where $a \in \mathbb{R}^n$. Then from

$$\left|\frac{Ax}{a^T x} - \frac{Ay}{a^T y}\right| = \frac{|a^T y Ax - a^T x Ay|}{a^T x a^T y} \le \frac{|a^T(y-x)| \cdot |Ax|}{a^T x a^T y} + \frac{a^T x |A(x-y)|}{a^T x a^T y}$$

and

$$a^T x a^T y = (a^T x)^2 + (a^T x)(a^T(y-x))$$

we can see that when $|y - x|$ is small enough, $|f(x) - f(y)|$ is small. $\square$

5.6 Let $K(x, y)$ be positive-valued continuous function on $[0, 1] \times [0, 1]$. Define

$$(Tu)(x) = \int_0^1 K(x, y)u(y)dy, \quad \forall u \in C[0, 1]$$

Prove that there exists $\lambda > 0$ and continuous non-negative function $u \ne 0$ such that $Tu = \lambda u$.

[1] M. A. Krasnoselskii. Two Remarks on the Methods of Successive Approximations. *Uspeckhi Mat. Nauk.* 10(1955), 123--127.

Proof. Consider the set

$$C = \left\{ u \in C[0,1] : u \geq 0, \ \int_0^1 u(t)dt = 1 \right\}.$$

Then C is a closed convex set. Define $S : C \to C$ as $Su = Tu / \int_0^1 Tu(t)dt$, and it is not difficult to see that S is continuous. We shall prove that $S(C)$ is sequentially compact, or, $S(C)$ is uniformly bounded and uniformly equicontinuous.

Suppose $0 < m \leq K(x,y) \leq M$ on $[0,1] \times [0,1]$, then we have $\|Su\| = \|Tu\| / \int_0^1 Tu(t)dt \leq M/m$. Thus $S(C)$ is uniformly bounded. On the other hand, we have

$$\|(Su)(y) - (Su)(x)\| = \frac{\|(Tu)(y) - (Tu)(x)\|}{\int_0^1 Tu(t)dt} \leq \frac{\int_0^1 |K(y,t) - K(x,t)| u(t)dt}{m},$$

together with the uniform continuous of K the uniform equicontinuity follows. □

6 Inner Product Space

6.1 (Polarization Identity) Let a be a sesquilinear function on a complex linear space $\mathscr{X}$ and q the quadratic form induced by a. Show that for any $x, y \in \mathscr{X}$ it holds that

$$a(x,y) = \frac{1}{4}\{q(x+y) - q(x-y) + iq(x+iy) - iq(x-iy)\}.$$

Proof.

$$\begin{aligned}
&\frac{1}{4}\{q(x+y) - q(x-y) + iq(x+iy) - iq(x-iy)\} \\
=& \frac{1}{4}\{a(x+y,x+y) - a(x-y,x-y) + ia(x+iy,x+iy) - ia(x-iy,x-iy)\} \\
=& \frac{1}{4}\{2a(x,y) + 2a(y,x) + 2i(a(x,iy) + a(iy,x))\}\} \\
=& \frac{1}{4}\{2a(x,y) + 2a(y,x) + 2i(-ia(x,y) + ia(y,x))\}\} \\
=& \frac{1}{4} \cdot 4a(x,y) = a(x,y)
\end{aligned}$$

□

6.2 Show that it is impossible to introduce an inner product $(\cdot,\cdot)$ in $C[a,b]$ such that

$$(f,f)^{\frac{1}{2}} = \max_{a \leq x \leq b} |f(x)|, \quad \forall f \in C[a,b].$$

Proof. It is suffices that the parallelogram equality does not hold. Assume that $[a,b] = [0,1]$. Let $f(x) = x$ and $g(x) = x^2$ we have that $\|f+g\|^2 + \|f-g\|^2 = 4 + 1/4$ while $2(\|f\|^2 + \|g\|^2) = 4$. □

6.3 In $L^2[0,T]$ show that the function

$$x \mapsto \left| \int_0^T e^{-(T-\tau)} x(\tau) d\tau \right|, \quad \forall x \in L^2[0,T]$$

reaches its maximum value on the unit sphere, and find the maximum value with the point x at which it attains the maximum value.

Proof. Assume that $\int_0^T |x(\tau)|^2\, d\tau = 1$, by Cauchy-Schwarz Inequality we have

$$\left|\int_0^T e^{-(T-\tau)}x(\tau)d\tau\right| \leq \left(\int_0^T e^{-2(T-\tau)}d\tau\right)^{\frac{1}{2}} \left(\int_0^T |x^2(\tau)|dx\right)^{\frac{1}{2}} = \sqrt{\frac{1-e^{-2T}}{2}},$$

where the equality holds iff $x(\tau) = \lambda e^{-(T-\tau)}$ for some λ. Combining with $\int_0^T |x(\tau)|^2\, d\tau = 1$, we can obtain that $\lambda = \pm\sqrt{2}e^T/\sqrt{e^{2T}-1}$. Therefore, the function attains the maximum value $\sqrt{1-e^{-2T}}/\sqrt{2}$ at $x(\tau) = \pm\sqrt{2}e^{\tau}/\sqrt{e^T - e^{-T}}$. □

6.4 Let M, N be two subsets in an inner-product space. Prove that

$$M \subseteq N \Rightarrow N^\perp \subseteq M^\perp.$$

Proof. Trivial. □

6.5 Let M be a subset of Hilbert space $\mathscr{X}$, show that

$$(M^\perp)^\perp = \overline{\operatorname{span} M}.$$

Proof. Firstly we prove that $M^\perp = \overline{\operatorname{span} M}^\perp$ and it suffices to show that $M^\perp \subseteq \overline{\operatorname{span} M}^\perp$. Let $x \in M^\perp$ and $y \in \overline{\operatorname{span} M}$. If $y \in \operatorname{span} M$, then $y = \sum a_n x_n$ with $x_n \in M$. Since $x \perp x_n$, we know that $x \perp y$. If $y \notin \operatorname{span} M$, then there exists $\{y_n\} \subseteq \operatorname{span} M$ such that $y_n \to y$. We have that $x \perp y_n$, so $x \perp y$. Therefore, $M^\perp \subseteq \overline{\operatorname{span} M}^\perp$. Now we show that if A is a closed subspace of $\mathscr{X}$ then $(A^\perp)^\perp = A$. It is clear that $A \subseteq (A^\perp)^\perp$. Now we shall prove that $(A^\perp)^\perp \subseteq A$. Suppose that $x \perp A^\perp$. Write $x = y + z$, where $y \in A$ and $z \in A^\perp$, hence $(x, z) = (y, z) + (z, z)$ with $(x, z) = (y, z) = 0$, yielding that $(z, z) = 0$ and $z = 0$. Thus $x = y \in A$. □

6.6 In $L^2[-1, 1]$ what is the orthogonal complement of the set of even functions? Prove your result.

Proof. The orthogonal complement consists of such function which differs from some odd function on a set of measure zero. It is such a function is orthogonal to any even function, and we shall prove the converse, that is, if $\int_{-1}^1 f\bar{g} = 0$ for all even function g, then f differs from an odd function on a set of measure zero. Write $\int_{-1}^1 f\bar{g} = \int_0^1 (f(x) + f(-x))\overline{g(x)}dx$ and let $g(x) = f(x) + f(-x)$ on $[0, 1]$, hence $\int_0^1 |f(x) + f(-x)|^2 = 0$, which indicates that $f(x) + f(-x) = 0$ almost everywhere on $[0, 1]$ and thus on $[-1, 1]$. □

6.7 In $L^2[a, b]$ consider the set $S = \{e^{2\pi inx}\}_{n=-\infty}^{\infty}$.

(1) If $|b - a| \leq 1$ then $S^\perp = \{0\}$.
(2) If $|b - a| > 1$ then $S^\perp \neq \{0\}$.

Proof. (1) If $|b - a| = 1$, it is well-known that $S^\perp = \{0\}$. If $|b - a| < 1$, if $u \in S^\perp$, we can extend u to some u' on $[a, a + 1]$ such that $\int_a^b u' e^{2\pi inx}dx = 0$ for all n. Thus $u' = 0$ on $[a, a + 1]$ and accordingly $u = 0$.
(2) Note that $\{e^{2\pi inx}\}$ is an orthonormal basis on $[b - 1, b]$. So for any $u \in L^2[a, b - 1]$ ($u \neq 0$), we can extend it to a function u' in $L^2[a, b]$ such that $u' \in S^\perp$. □

6.8 Denote by $\mathscr{X}$ the set of all analytic functions on the closed unit circle. The inner product is defined as

$$(f, g) = \frac{1}{i}\oint_{|z|=1} \frac{f(z)\overline{g(z)}}{z}dz, \quad \forall f, g \in \mathscr{X}.$$

Show that $\{z^n/\sqrt{2\pi}\}_{n=0}^{\infty}$ is an orthonormal set.

Proof. Let $z_n = z^n/\sqrt{2\pi}$. First we show that $(z_n, z_n) = 1$.

$$(z_n, z_n) = \frac{1}{2\pi i}\oint_{|z|=1} \frac{z^n \bar{z}^n}{z} dz = \frac{1}{2\pi i}\oint_{|z|=1} \frac{|z|^{2n}}{z} dz = \frac{1}{2\pi i}\oint_{|z|=1} \frac{dz}{z} = 1$$

Next we show that z_n and z_m $(n > m)$ are orthogonal.

$$(z_n, z_m) = \frac{1}{2\pi i}\oint_{|z|=1} \frac{z^n \bar{z}^m}{z} dz = \frac{1}{2\pi i}\oint_{|z|=1} \frac{|z|^{2m} z^{n-m}}{z} dz = \frac{1}{2\pi i}\oint_{|z|=1} z^{n-m-1} dz = 0. \quad \square$$

6.9 Let $\{e_n\}_1^\infty$ and $\{f_n\}_1^\infty$ be two orthonormal sets in Hilbert space $\mathscr{X}$ and they satisfy that

$$\sum_{i=1}^{\infty} \|e_n - f_n\|^2 < 1.$$

Show that the completeness of one of $\{e_n\}$ and $\{f_n\}$ implies that of the other.

Proof. Assume that $\{e_n\}$ is complete. If $\{f_n\}$ is not complete then there exists $x \neq 0$ such that $x \perp f_n$ for all n. It follows that

$$\|x\|^2 = \sum |(x, e_n)|^2 = \sum |(x, e_n - f_n)|^2 \leq \|x\|^2 \sum \|e_n - f_n\|^2 < \|x\|^2,$$

which is a contradiction. Therefore, $\{f_n\}$ must be complete. $\square$

6.10 Suppose that $\mathscr{X}$ be a Hilbert space and $\mathscr{X}_0$ a closed subspace of $\mathscr{X}$. Let $\{e_n\}$ and $\{f_n\}$ be orthonormal bases of $\mathscr{X}_0$ and $\mathscr{X}_0^\perp$, respectively. Show that $\{e_n\} \cup \{f_n\}$ is an orthonormal basis of $\mathscr{X}$.

Proof. It is clear that $\{e_n\} \cup \{f_n\}$ is an orthonormal set. From the unique decomposition theorem this set is a basis of $\mathscr{X}$. $\square$

6.11 Let $H^2(D)$ an inner-product as defined in Example 1.6.28.

(1) Suppose the Taylor expansion of $u(z)$ is $u(z) = \sum_{k=0}^\infty b_k z^k$, show that

$$\sum_{k=0}^{\infty} \frac{|b_k|^2}{k+1} < \infty;$$

(2) Let $u(z), v(z) \in H^2(D)$ and

$$u(z) = \sum_{k=0}^{\infty} a_k z^k, \quad v(z) = \sum_{k=0}^{\infty} b_k z^k,$$

show that

$$(u, v) = \pi \sum_{k=0}^{\infty} \frac{a_k \overline{b_k}}{k+1};$$

(3) Let $u(z) \in H^2(D)$, show that

$$|u(z)| \leq \frac{\|u\|}{\sqrt{\pi}(1 - |z|)}, \quad \forall\, |z| < 1;$$

(4) Verify that $H^2(D)$ is a Hilbert space.

Proof. (1) From the definition of $H^2(D)$ we have

$$\iint_D \left|\sum_{k=0}^{\infty} b_k z^k\right|^2 dxdy = \iint_D \left(\sum_{k=0}^{\infty} b_k z^k\right)\left(\sum_{k=0}^{\infty} \overline{b_k}\bar{z}^k\right) dxdy = \iint_D \sum_{k\geq 0, l\geq 0} b_k\overline{b_l} z^k \bar{z}^l \, dxdy < \infty.$$

Since $\sum b_k z^k$ converges uniformly on $B(0,r)$ $(0 < r < 1)$, it holds that

$$\begin{aligned}
&\iint_{|z|\leq r} \sum_{k\geq 0, l\geq 0} b_k\overline{b_l} z^k \bar{z}^l \, dxdy \\
=& \sum_{k\geq 0, l\geq 0} \iint_{|z|\leq r} b_k\overline{b_l} z^k \bar{z}^l \, dxdy \\
=& \sum_{k\geq 0, l\geq 0} b_k\overline{b_l} \int_0^r s^{k+l+1} ds \int_0^{2\pi} (\cos k\theta + i\sin k\theta)(\cos l\theta - i\sin l\theta) d\theta \\
=& \sum_{k\geq 0} b_k\overline{b_k} \cdot \frac{r^{2k+2}}{2(k+1)} \cdot 2\pi = \pi \sum_{k=0}^{\infty} \frac{|b_k|^2}{k+1} r^{2k+2}.
\end{aligned}$$

Since $\iint_D |u(z)|^2 dxdy < \infty$, it holds that $\lim_{r\to 1^-} \iint_{|z|\leq r} |u(z)|^2 dxdy = \iint_D |u(z)|^2 dxdy$. We also have $\lim_{r\to 1^-} \sum |b_k|^2 r^{2k+2}/(k+1) = \sum |b_k|^2/(k+1)$, the conclusion follows immediately.

(2) The proof is very similar to the previous one.

(3) Let $r = 1 - |z|$ then $B(z,r) \subseteq D$. Note that $f(x,y) = u(x+iy)$ is harmonic, hence we have

$$|u(z)| = \frac{1}{\pi r^2}\left|\iint_{B(z,r)} u(x+iy) dxdy\right| \leq \frac{1}{\sqrt{\pi} r}\left(\iint_{B(z,r)} |u(x+iy)|^2 dxdy\right)^{\frac{1}{2}} \leq \frac{\|u\|}{\sqrt{\pi}(1-|z|)}.$$

(4) Everything is clear except completeness. Let $\{u_n\}$ be a Cauchy sequence. Then for all z on a circle $|z| \leq r$ we have from (3) that $|u_n(z) - u_m(z)| \leq \|u_n - u_m\|/(\sqrt{\pi}(1-r))$, hence $\{u_n(z)\}$ uniformly converges within $|z| \leq r$ to some $u(z)$. We know that $u(z)$ is holomorphic and $\iint_{u(z)} dxdy < \infty$ from Minkowski's Inequality. □

6.12 Let $\mathscr{X}$ be an inner-product space and $\{e_n\}$ be an orthonormal set. Show that

$$\left|\sum_{n=1}^{\infty} (x, e_n)\overline{(y, e_n)}\right| \leq \|x\|\,\|y\|, \quad \forall x, y \in \mathscr{X}.$$

Proof. We have

$$\left|\sum_{n=1}^{\infty} (x, e_n)\overline{(y, e_n)}\right| \leq \sum_{n=1}^{\infty} |(x, e_n)|\,|\overline{(y, e_n)}| \leq \left(\sum_{n=1}^{\infty} |(x, e_n)|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |(y, e_n)|^2\right)^{\frac{1}{2}} \leq \|x\|\,\|y\|$$

using Cauchy-Schwarz and Bessel Inequalities. □

6.13 Let $\mathscr{X}$ be an inner-product space. For any $x_0 \in \mathscr{X}$ and any $r > 0$, define

$$C = \{x \in \mathscr{X} : \|x - x_0\| < r\}.$$

(1) Show that C is a closed convex subset;

(2) For any $x \in \mathscr{X}$ define

$$y = \begin{cases} x_0 + r(x - x_0)/\|x - x_0\|, & x \notin C; \\ x, & x \in C, \end{cases}$$

Show that y is the best approximation of x in C.

Proof. (1) Trivial.

(2) If $x \in C$ then $\|y - x\| = 0$ and thus y is obviously the best approximation. Now assume that $x \notin C$, then for any $c \in C$ we have

$$\|y - x\| = \left\| x_0 + r\frac{x - x_0}{\|x - x_0\|} - x \right\| = \|x - x_0\| - r \geq \|x - x_0\| - \|c - x_0\| \geq \|x - c\|. \quad \square$$

6.14 Find $(a_0, a_1, a_2) \in \mathbb{R}^3$ which minimizes $\int_0^1 |e^t - a_0 - a_1 t - a_2 t^2|^2 dt$.

Proof. This is to find the projection of e^t on span$\{1, t, t^2\}$ in $L^2[0,1]$. According to the system of equations (1.6.4), we obtain that $a_0 = 39e - 105$, $a_1 = -12(18e - 49)$ and $a_2 = 30(7e - 19)$. $\square$

6.15 Let $f(x) \in C^2[a,b]$ satisfying

$$f(a) = f(b) = 0, \quad f'(a) = 1, \quad f'(b) = 0.$$

Show that

$$\int_a^b |f''(x)|^2 dx \geq \frac{4}{b-a}.$$

Proof. The curve spline is a cubic function, say $g(x) = Ax^3 + Bx^2 + Cx + D$. Then from $g(a) = g(b) = 0$, $g'(a) = 1$, $g(b) = 0$, we can obtain that $A = 1/(a-b)^2$ and $B = -(a+2b)/(a-b)^2$. Then $\int_a^b |f''(x)|^2 \geq \int_a^b |g''(x)|^2 dx = 4/(a-b)^4 \cdot \int_a^b (a + 2b - 3x)^2 dx = 4/(b-a)$. $\square$

6.16 (Variational Inequality) Let $\mathscr{X}$ be a Hilbert space and $a(x,y)$ a Hermitian sesquilinear function on $\mathscr{X}$. Suppose that there exists $M > 0$ and $\delta > 0$ such that

$$\delta\|x\|^2 \leq a(x,x) \leq M\|x\|^2, \quad \forall x \in \mathscr{X}.$$

Let $u_0 \in \mathscr{X}$ and C be a closed convex subset on $\mathscr{X}$. Show that the function

$$x \mapsto a(x,x) - \Re(u_0, x)$$

attains minimum value at some x_0 on C and the point x_0 is unique and satisfies

$$\Re(2a(x_0, x - x_0) - (u_0, x - x_0)) \geq 0, \quad \forall x \in C.$$

Proof. Denote the function by $f(x)$. We have $f(x) \geq \delta\|x\|^2 - |(u_0, x)| \geq \delta\|x\|^2 - \|u_0\|\,\|x\|$, which is bounded below, so we can suppose that $d = \inf_{x \in C} f(x)$, and $d < f(x_n) < d + 1/n$. We can write

$$f(x) = a(x,x) + \frac{\|u_0 - x\|^2 - \|x\|^2 - \|u_0\|^2}{2}.$$

We shall show that $\{x_n\}$ is convergent, that is, it is a Cauchy sequence. From parallelogram equality we have

$$\|x_m - x_n\|^2 = 2(\|x_m\|^2 + \|x_n\|^2) - 4\left\|\frac{x_n + x_m}{2}\right\|^2 \tag{2}$$

and

$$
\begin{aligned}
\|x_m - x_n\|^2 &= \|(u_0 - x_m) - (u_0 - x_n)\|^2 \\
&= 2(\|u_0 - x_m\|^2 + \|u_0 - x_n\|^2) - 4\left\|u_0 - \frac{x_n + x_m}{2}\right\|^2 \\
&= 2\left(2f(x_m) - 2a(x_m, x_m) + \|x_m\|^2 + \|u_0\|^2 + 2f(x_n) - 2a(x_n, x_n) + \|x_n\|^2 + \|u_0\|^2\right) \\
&\quad - 4\left(2f\left(\frac{x_n + x_m}{2}\right) - 2a\left(\frac{x_n + x_m}{2}, \frac{x_n + x_m}{2}\right) + \left\|\frac{x_n + x_m}{2}\right\|^2 + \|u_0\|^2\right) \\
&= 4\left(f(x_m) + f(x_n) - 2f\left(\frac{x_m + x_n}{2}\right)\right) \\
&\quad + 4\left(2a\left(\frac{x_n + x_m}{2}, \frac{x_n + x_m}{2}\right) - a(x_m, x_m) - a(x_n, x_n)\right) + \|x_m - x_n\|^2 \quad \text{(use (2))}
\end{aligned}
$$

So it holds that

$$
\begin{aligned}
0 &= 4\left(f(x_m) + f(x_n) - 2f\left(\frac{x_m + x_n}{2}\right)\right) + 4\left(2a\left(\frac{x_n + x_m}{2}, \frac{x_n + x_m}{2}\right) - a(x_m, x_m) - a(x_n, x_n)\right) \\
&< 4\left(\frac{1}{m} + \frac{1}{n}\right) + 4\left(a\left(x_m + \frac{x_n - x_m}{2}, x_m + \frac{x_n - x_m}{2}\right) - a(x_m, x_m)\right) \\
&\quad + 4\left(a\left(x_n - \frac{x_n - x_m}{2}, x_n - \frac{x_n - x_m}{2}\right) - a(x_n, x_n)\right) \\
&= 4\left(\frac{1}{m} + \frac{1}{n}\right) + 4\left(\Re a(x_m, x_n - x_m) + a\left(\frac{x_n - x_m}{2}, \frac{x_n - x_m}{2}\right)\right) \\
&\quad + 4\left(-\Re a(x_n, x_n - x_m) + a\left(\frac{x_n - x_m}{2}, \frac{x_n - x_m}{2}\right)\right) \\
&= 4\left(\frac{1}{m} + \frac{1}{n}\right) - 4\Re a(x_m - x_n, x_m - x_n) + 2a(x_n - x_m, x_n - x_m) \\
&= 4\left(\frac{1}{m} + \frac{1}{n}\right) - 2a(x_n - x_m, x_n - x_m) \leq 4\left(\frac{1}{m} + \frac{1}{n}\right) - 2\delta\|x_n - x_m\|^2.
\end{aligned}
$$

Therefore,

$$
\|x_m - x_n\|^2 < \frac{2}{\delta}\left(\frac{1}{m} + \frac{1}{n}\right) \to 0, \quad m, n \to \infty.
$$

Now we prove that the uniqueness. If $f(x) = f(y) = d$ and $x \neq y$, then similar to the process above, we have that $\|x - y\|^2 < 0$, which is a contradiction. Hence it must hold that $x = y$.

Suppose that $f(x_0) = d$, let $g_x(t) = f(tx + (1-t)x_0)$, then $g_x(t) \geq g_x(0)$ for all $x \in C$ and $t \in [0, 1]$.

$$
\begin{aligned}
g_x(t) &= a(t(x - x_0) + x_0, t(x - x_0) + x_0) - \Re(u_0, t(x - x_0) + x_0) \\
&= t^2 a(x - x_0, x - x_0) + 2t\Re a(x_0, (x - x_0)) + a(x_0, x_0) - t\Re(u_0, x - x0) - \Re(u_0, x_0)
\end{aligned}
$$

Hence $g_x'(0) = 2\Re a(x_0, x - x_0) - \Re(u_0, x - x_0)$. Since $g_x(t) - g_x(0) = g_x'(0)t + a(x - x_0, x - x_0)t^2 \geq 0$, it follows that $g_x'(0) \geq -a(x - x_0, x - x_0)t$ for all $t \in (0, 1]$ and thus $g_x'(0) \geq 0$. □