Extreme Power Dispersion Profiles for Nakagami-m Fading Channels with Maximal-Ratio Diversity

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***Abstract*— Using nonlinear optimization theory, we derive extreme power dispersion profiles (PDPs) for wireless channels with maximal-ratio diversity. Specifically, we obtain the best and worst PDPs that respectively minimize and maximize the symbol error probability in Nakagami-m fading channels having not necessarily identical means.**

***Index Terms*— Best and worst power dispersion profiles, diversity combining, maximal-ratio diversity, Nakagami-m fading, nonlinear optimization.**

I. INTRODUCTION

THE PERFORMANCE of diversity combining receivers depends on the relative powers or signal-to-noise ratios (SNRs) of the diversity branches.[1] As such, relative powers among the diversity branches, referred to as power dispersion profiles (PDPs), have been used to classify different wireless environments [1], [2]. Among them, the uniform PDP gives rise to the best performance for maximal-ratio combining (MRC) of Rayleigh fading diversity branches [3]. This implies that the performance of MRC based on a uniform PDP can serve as a benchmark in Rayleigh fading environments. Note that performance based on a uniform PDP has been investigated to study different aspects of wireless systems (for example, see [4]–[8]).

Here, we consider the Nakagami-m fading channels as these channels have received considerable attention in the study for various aspects of wireless systems [9]–[12]. In particular, it was shown recently that the amplitude distribution of the resolved multipaths in ultra-wide bandwidth (UWB) indoor channels can be well-modeled by the Nakagami-m distribution [13]. The Nakagami-m family of distributions, also known as the "m-distribution," contains Rayleigh fading ($m = 1$) as a special case; along with cases of fading that are more severe than Rayleigh ($1/2 \leq m < 1$) as well as cases less severe than Rayleigh ($m > 1$).

In this letter, we derive the best and worst PDPs that respectively minimize and maximize the symbol error probability (SEP) for coherent detection of two-dimensional signaling constellation with polygonal decision boundaries using MRC. Specifically, we formulate this problem as a simple application of nonlinear optimization theory and obtain the extreme PDPs for Nakagami-m fading channels. We obtain a proof that uniform PDPs give the minimum SEP in a broader class of Nakagami-m fading channels, spanning from the one-sided Gaussian distribution ($m = 1/2$) to the non-fading channel case ($m = \infty$). In the context of UWB systems, this implies that the SEP performance based on a uniform PDP serves as a lower bound for the performance in various environments.

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[1]Unless otherwise stated, the terms power and SNR will be used interchangeably in the following to denote the mean power and the mean SNR (averaged over the fast fading).

II. POWER DISPERSION PROFILES

As the performance of wireless systems depends on PDPs, they have been used to characterize different wireless environments. The PDP is typically defined as follows:

Definition 1: The power dispersion profile (PDP) of a wireless environment $\boldsymbol{e}$ is defined by[2]

$$\mathbf{e} \triangleq (e_1, e_2, \ldots, e_N) \in \mathbb{R}_+^N, \quad (1)$$

where the quantity e_k represents the normalized SNR of the $k^{\underline{\text{th}}}$ diversity branch, i.e., $e_k \triangleq \Gamma_k/\Gamma_{\text{tot}}$ with Γ_k denoting the SNR of the $k^{\underline{\text{th}}}$ diversity branch and $\Gamma_{\text{tot}} = \sum_{k=1}^{N} \Gamma_k$ is the total SNR among all diversity branches.[3]

The SEP for coherent detection of M-ary phase-shift keying (MPSK) in Nakagami-m fading channels was derived in [14], [15], and can be written explicitly in terms of PDP and Γ_{tot} as

$$P_e(\mathbf{e}, \Gamma_{\text{tot}}) = \frac{1}{\pi} \int_0^{\Theta} \prod_{k=1}^{N} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\text{MPSK}} \Gamma_{\text{tot}} e_k / m} \right]^m d\theta \quad (2)$$

where $c_{\text{MPSK}} = \sin^2(\pi/M)$, and $\Theta = \pi(M-1)/M$.

III. EXTREME POWER DISPERSION PROFILES

In this section, we formulate our problem in the framework of nonlinear optimization theory and obtain explicit expressions for extreme PDPs for Nakagami-m channels.

A. Best Power Dispersion Profile

Definition 2: The best PDP, denoted by $\mathbf{e}_\text{b}$, is the value of $\mathbf{e}$ that results in the minimum SEP under the total SNR

[2]The notation $\mathbb{R}_+$ denotes the set of non-negative real numbers.

[3]Another common characterization is the un-normalized PDP. However the definitions are consistent up to a scale factor as the performance of diversity combining receivers depends on the shape of the PDP. In this paper, we will use the normalized PDP given in (1).

constraint $\sum_{i=1}^{N} e_k \leq 1$. Mathematically, $\mathbf{e}_{\rm b}$ is the solution to the following nonlinear optimization problem:

$$\begin{aligned} \min_{\mathbf{e}} \quad & \frac{1}{\pi}\int_0^{\Theta} \prod_{k=1}^{N} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\rm MPSK}\Gamma_{\rm tot} e_k/m}\right]^m d\theta \\ \text{s.t.} \quad & \sum_{k=1}^{N} e_k \leq 1\,. \end{aligned} \tag{3}$$

Theorem 1: The best PDP for Nakagami-m fading channel with MRC is given by,

$$\mathbf{e}_{\rm b} = (\bar{e}, \bar{e}, \ldots, \bar{e})\,, \tag{4}$$

where $\bar{e} = \frac{1}{N}$, i.e., the best PDP exhibits equal average SNR among the diversity branches.

Proof: In order to obtain the best PDP for Nakagami-m fading channels with MRC, we will solve the optimization problem given by (3). Since the integrand of (3) is non-negative, minimizing it for each $\theta \in [0, \Theta]$ is equivalent to minimizing the SEP. Moreover, the contribution of the point $\theta = 0$ to the integral in (3) is zero since the integrand is finite at that point. Thus, the point-wise minimization can be relaxed to the region $\theta \in (0, \Theta]$. Thus, (3) is equivalent to

$$\begin{aligned} \min_{\mathbf{e}} \quad & \left[\prod_{k=1}^{N} \frac{\sin^2\theta}{\sin^2\theta + c_{\rm MPSK}\Gamma_{\rm tot} e_k/m}\right]^m \\ \text{s.t.} \quad & \sum_{k=1}^{N} e_k \leq 1\,, \end{aligned} \tag{5}$$

for each $\theta \in (0, \Theta]$.

Note that, for $x > 0$ and for each $m > 0$, the function $f(x) = x^m$ is monotonically increasing in x, and therefore minimizing x^m is equivalent to maximizing $1/x$. Hence, (3) is equivalent to

$$\begin{aligned} \max_{\mathbf{e}} \quad & \prod_{k=1}^{N} \left[1 + \frac{c_{\rm MPSK}\Gamma_{\rm tot}}{m\sin^2\theta} e_k\right] \\ \text{s.t.} \quad & \sum_{k=1}^{N} e_k \leq 1\,, \end{aligned} \tag{6}$$

for each $\theta \in (0, \Theta]$.

Using the arithmetic and geometric mean inequality [16], [17] with $p_k = \frac{1}{N}$ and $x_k = 1 + \frac{c_{\rm MPSK}\Gamma_{\rm tot}}{m\sin^2\theta} e_k$ for $k = 1, 2, \ldots, N$, we have

$$\prod_{k=1}^{N} \left[1 + \frac{c_{\rm MPSK}\Gamma_{\rm tot}}{m\sin^2\theta} e_k\right] \leq \left[1 + \frac{c_{\rm MPSK}\Gamma_{\rm tot}}{Nm\sin^2\theta} \sum_{k=1}^{N} e_k\right]^N \tag{7}$$

$$\leq \left[1 + \frac{c_{\rm MPSK}\Gamma_{\rm tot}}{Nm\sin^2\theta}\right]^N. \tag{8}$$

The equality in (7) is achieved if and only if

$$e_i = e_j \quad \text{for} \quad 1 \leq i, j \leq N\,, \tag{9}$$

and the equality in (8) is achieved if and only if

$$\sum_{k=1}^{N} e_k = 1\,. \tag{10}$$

Equations (9) and (10) imply that the maximum in equation (6) is achieved if and only if $e_i = 1/N$ for all $i = 1, 2, \ldots, N$. Therefore, the best PDP is the one where all diversity branches have equal average SNR, i.e., $\mathbf{e}_{\rm b} = (\bar{e}, \bar{e}, \ldots, \bar{e})$, with $\bar{e} = 1/N$.

□

Under this best PDP, the minimum SEP is achieved and is given by

$$P_{e,\min}(\Gamma_{\rm tot}) = \frac{1}{\pi}\int_0^{\Theta} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\rm MPSK}\Gamma_{\rm tot}/(Nm)}\right]^{Nm} d\theta\,. \tag{11}$$

Note that (11) is the SEP for N-branch MRC in independent and identically distributed (i.i.d.) Nakagami-m channel with total SNR $\Gamma_{\rm tot}$.

B. Worst Power Dispersion Profile

Definition 3: The worst PDP, denoted by $\mathbf{e}_{\rm w}$, is the value of $\mathbf{e}$ that results in the maximum SEP under total SNR constraint $\sum_{k=1}^{N} e_k = 1$. Mathematically, $\mathbf{e}_{\rm w}$ is the solution to the following nonlinear optimization problem:

$$\begin{aligned} \max_{\mathbf{e}} \quad & \frac{1}{\pi}\int_0^{\Theta} \prod_{k=1}^{N} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\rm MPSK}\Gamma_{\rm tot} e_k/m}\right]^m d\theta \\ \text{s.t.} \quad & \sum_{k=1}^{N} e_k = 1\,. \end{aligned} \tag{12}$$

Theorem 2: The worst PDP for Nakagami-m fading channel with MRC is given by

$$\mathbf{e}_{\rm w} = (1, 0, \ldots, 0)\,. \tag{13}$$

In other words, the worst PDP is a degenerate one with all the energy concentrated in a single branch, i.e., there is no diversity.

Proof: Similar to the derivation for the best PDP, it can be shown that (12) is equivalent to

$$\begin{aligned} \min_{\mathbf{e}} \quad & \prod_{k=1}^{N} \left[1 + \frac{c_{\rm MPSK}\Gamma_{\rm tot}}{m\sin^2\theta} e_k\right] \\ \text{s.t.} \quad & \sum_{k=1}^{N} e_k = 1\,, \end{aligned} \tag{14}$$

for $\theta \in (0, \Theta]$. Using the polynomial expansion, we have

$$\prod_{k=1}^{N} \left[1 + \frac{c_{\rm MPSK}\Gamma_{\rm tot}}{m\sin^2\theta} e_k\right] = 1 + \sum_{k=1}^{N} \left[\frac{c_{\rm MPSK}\Gamma_{\rm tot}}{m\sin^2\theta}\right]^k \mathfrak{E}_k(\mathbf{e})\,, \tag{15}$$

where $\mathfrak{E}_k(\mathbf{e})$, the $k^{\underline{\rm th}}$ *elementary symmetric function* (ESF) of $\mathbf{e}$, defined as the sum of all possible products (k at a time) of the elements of $\mathbf{e}$ [16]. Mathematically,

$$\mathfrak{E}_k(\mathbf{e}) \triangleq \sum_{S \in \mathcal{S}_k} \prod_{n \in S} e_n\,, \tag{16}$$

where $\mathcal{S}_k = \{S \subset \mathbb{Z}_N : |S| = k\}$ and $|S|$ denotes the cardinality of the set S.[4]

[4]The notation $\mathbb{Z}_N$ is used to denote $\mathbb{Z}_N \triangleq \{1, 2, \ldots, N\}$.

Since $\mathbf{e} \in \mathbb{R}_+^N$, each of the ESFs satisfy $\mathfrak{E}_k(\mathbf{e}) \geq 0$. Noting the fact that $\mathfrak{E}_1(\mathbf{e}) = 1$, the equation (15) can be lower bounded as

$$\prod_{k=1}^{N} \left[1 + \frac{c_{\text{MPSK}}\Gamma_{\text{tot}}}{m \sin^2\theta} e_k\right] \geq 1 + \left[\frac{c_{\text{MPSK}}\Gamma_{\text{tot}}}{m \sin^2\theta}\right], \quad (17)$$

where the equality in (17) can be achieved if and only if

$$\sum_{k=2}^{N} \left[\frac{c_{\text{MPSK}}\Gamma_{\text{tot}}}{m \sin^2\theta}\right]^k \mathfrak{E}_k(\mathbf{e}) = 0 \,. \quad (18)$$

The condition in (18) is satisfied for each $\theta \in (0, \Theta]$ if and only if the ESFs satisfy

$$\mathfrak{E}_k(\mathbf{e}) = 0 \qquad k = 2, \ldots, N. \quad (19)$$

Without loss of generality, considering the ordered PDP $e_1 \geq e_2 \geq \cdots \geq e_N \geq 0$, the condition in (19) together with feasibility condition of the optimization problem in (14) imply that

$$e_1 = 1 \,, \quad (20)$$

$$e_k = 0 \qquad k = 2, 3, \ldots, N. \quad (21)$$

Therefore, the worst PDP is the one in which all the SNRs are concentrated at only one of the branches, and the rest of the branches have no energy, i.e., the worst PDP is given by $\mathbf{e}_{\text{w}} = (1, 0, \ldots, 0)$.

□

This worst PDP gives rise to the maximum SEP which is given by

$$P_{e,\max}(\Gamma_{\text{tot}}) = \frac{1}{\pi}\int_0^{\Theta} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\text{MPSK}}\Gamma_{\text{tot}}/m}\right]^m d\theta. \quad (22)$$

Note that (22) is the SEP of single-branch reception in Nakagami-m fading channel (i.e., without diversity).

C. Extensions

Our analysis thus far considers the coherent detection of MPSK modulation in independent Nakagami-m channels with non-identical means. We deliberately restrict our analysis to this case as it captures all of the essential ideas. We now describe the two immediate extensions of our methodology. First, Definitions 2 and 3 and Theorems 1 and 2 are valid for any two dimensional modulation schemes with polygonal decision regions, whose SEP can be written as a weighted sum of canonical integrals in the form of (2) with positive weights [14], [18]. Second, our analysis can be extended to characterize the effect of channel correlations by using eigenvalues of the correlation matrix. It was shown in [14] that SEP in correlated Nakagami-m channels can be written in terms of the eigenvalues of the correlation matrix. Similar analysis shows that the best correlation matrix has equal eigenvalues implying i.i.d. channels as given by Theorem 1. Similarly, the worst correlation matrix has only one non-zero eigenvalue, which is positive, implying no diversity as given by Theorem 2.

IV. Conclusions

In this paper, we derived the best and worst power dispersion profiles for the MRC receiver in Nakagami-m channels. The results are valid for a large class of two dimensional modulation schemes. We proved that uniform PDPs give the minimum SEP in a broader class of Nakagami fading channels. As expected, our results for the special case of $m = 1$ agree with the results of [3] for Rayleigh fading.

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