A numerical method for the wave equation subject to a non-local conservation condition

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Abstract

A numerical method based on an integro-differential equation and local interpolating functions is proposed for solving the one-dimensional wave equation subject to a non-local conservation condition and suitably prescribed initial-boundary conditions. To assess its validity and accuracy, the method is applied to solve several test problems.

This is a pre-print of the article in Applied Numerical Mathematics 56 (2006) 1054-1060.

Keywords: Wave equation; Non-local condition; Integro-differential formulation; Local interpolating functions; Numerical solution

1 Introduction

The development of numerical techniques for solving partial differential equations in physics subject to non-local conservation conditions is a subject of considerable interest. There are many papers that deal with the numerical solution of the diffusion (parabolic) equation with integral conditions giving the specification of mass, e.g. Ang [1], Dehghan [6], Cannon, Lin and Wang [3], Cannon and Matheson [4], Cannon and van der Hoek [5], Gumel, Ang and Twizell [8] and Noye and Dehghan [10]. Although theoretical studies on the existence and uniqueness and the behaviours of solutions for problems governed by the wave (hyperbolic) equation with non-local conditions have received considerable attention in the literature (e.g. Beilin [2] and Kavalloris and Tzanetis [9]), relatively few papers give the numerical solutions of such problems.

In a recent paper, Dehghan [7] describes several finite-difference schemes for the numerical solution of the one-dimensional wave equation (in non-dimensionalized form)

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2} - q(x,t) \text{ (for } 0 \leq x \leq 1 \text{ and } t \geq 0) \tag{1}$$

subject to the non-local conservation condition

$$\int_0^1 \phi(x,t)dx = 0 \text{ for } t \geq 0. \tag{2}$$

In addition to (2), the unknown function ϕ is required to satisfy the initial conditions

$$\phi(x,0) = f(x) \text{ and } \left.\frac{\partial \phi}{\partial t}\right|_{t=0} = g(x) \text{ for } 0 \leq x \leq 1 \tag{3}$$

and the boundary condition

$$\phi(0,t) = 0 \text{ for } t > 0. \tag{4}$$

Here $q(x,t)$, $f(x)$ and $g(x)$ are suitably given functions.

For a more general problem, the non-local condition (2) and the boundary condition (4) may be respectively superceded by

$$\int_0^1 \phi(x,t)dx = E(t) \text{ for } t \geq 0 \tag{5}$$

and

$$\alpha\phi(0,t) + \beta \left.\frac{\partial}{\partial x}[\phi(x,t)]\right|_{x=0} = r(t) \text{ for } t > 0 \tag{6}$$

where α and β are given constants and $r(t)$ and $E(t)$ are suitably prescribed functions.

In the present paper, an alternative numerical method based on an integro-differential formulation of the wave equation and on approximating $\phi(x,t)$ through the use of local interpolating spatial functions is proposed for solving (1) subject to the initial conditions (3), the non-local condition (5) and the boundary condition given by (6). The proposed method is applied to solve several test problems in order to assess its validity and accuracy.

2 Integro-differential formulation

Through partial integrations of the wave equation (1) with respect to x, one may derive the integro-differential equation

$$\begin{aligned} 2\phi(\xi,t) &= \phi(0,t) + \phi(1,t) + \xi\theta(t) + (\xi-1)\omega(t) \\ &\quad + \int_0^1 |x-\xi|(\frac{\partial^2}{\partial t^2}[\phi(x,t)] - q(x,t))dx, \end{aligned} \tag{7}$$

where θ and ω are the boundary flux functions defined by

$$\theta(t) = \left.\frac{\partial}{\partial x}[\phi(x,t)]\right|_{x=0} \text{ and } \omega(t) = \left.\frac{\partial}{\partial x}[\phi(x,t)]\right|_{x=1}. \tag{8}$$

The problem under consideration may now be reformulated as one that requires finding $\phi(x,t)$ from (7) together with (3), (5) and (6).

3 Approximation of $\phi(x,t)$

To find $\phi(x,t)$ from (3), (5) and (6) and (7), we make the approximation

$$\phi(x,t) \simeq \sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}\sigma_n(x)\phi_m(t), \tag{9}$$

where $\phi_m(t) = \phi(\xi_m, t)$, ξ_1, ξ_2, $\cdots$, ξ_{N-1} and ξ_N are N distinct well-spaced nodes selected from the interval $[0,1]$ with $\xi_1 = 0$ and $\xi_N = 1$, $\sigma_n(x) = 1 + |x - \xi_n|^{3/2}$ is the local interpolating function centred about ξ_n and c_{nm} are constant coefficients defined by

$$\sum_{k=1}^{N} \sigma_n(\xi_k)c_{pk} = \begin{cases} 1 & \text{if} \quad n = p, \\ 0 & \text{if} \quad n \neq p. \end{cases} \tag{10}$$

The nodal functions $\phi_1(t)$, $\phi_2(t)$, $\cdots$, $\phi_{N-1}(t)$ and $\phi_N(t)$ may be regarded as unknown. The integro-differential equation (7) and the approximation (9) can be used together with (3), (5) and (6) to reduce the problem under consideration into an initial-value problem governed by a system of second order linear ordinary differential equations.

4 Initial-value problem

If we substitute (9) into (7) and let $\xi = \xi_r$ for $r = 1, 2, \cdots, N$, we obtain the system of ordinary differential equations

$$\begin{aligned} & 2\phi_r(t) + S_r(t) \\ = \quad & \phi_1(t) + \phi_N(t) + \xi_r\theta(t) + (\xi_r - 1)\omega(t) + \sum_{m=1}^{N} F_{rm}\phi''_m(t) \\ & \qquad\qquad\qquad\qquad \text{for } r = 1, 2, \cdots, N, \end{aligned} \tag{11}$$

where the prime denotes differentiation with respect to t and

$$S_r(t) = \int_0^1 |x - \xi_r| q(x,t)dx,$$

$$
\begin{aligned}
F_{rm} \quad = \quad & \sum_{n=1}^{N} c_{nm}(\frac{1}{2}[(1-\xi_r)^2+\xi_r^2] \\
& +\frac{2}{5}[(1-\xi_r)(1-\xi_n)^{5/2}+\xi_r\xi_n^{5/2}] \\
& -\frac{4}{35}[(1-\xi_n)^{7/2}+\xi_n^{7/2}]+\frac{8}{35}\left|\xi_r-\xi_n\right|^{7/2}). \qquad (12)
\end{aligned}
$$

The initial conditions (3) and the boundary condition (6) respectively give rise to

$$\phi_r(0)=f(\xi_r) \text{ and } \phi_r'(0)=g(\xi_r) \text{ for } r=1,2,\cdots,N, \qquad (13)$$

and

$$\alpha\phi_1(t)+\beta\theta(t)=r(t) \text{ for } t>0. \qquad (14)$$

With (9), the non-local conservation condition (5) can be approximately written as

$$\sum_{m=1}^{N} G_m\phi_m(t)=E(t), \qquad (15)$$

where

$$G_m=\sum_{n=1}^{N} c_{nm}(1+\frac{2}{5}\left|1-\xi_n\right|^{5/2}+\frac{2}{5}\xi_n^{5/2}). \qquad (16)$$

Thus, the initial-value problem is to solve (11), (14) and (15) subject to (13).

5 Numerical procedure

A numerical procedure for solving (1) subject to (3), (5) and (6) can be obtained by reducing (11) into a system of linear algebraic equations.

We approximate $\phi_n(t)$ $(n = 1, 2, \cdots, N)$ as a quartic function of time t over the interval $\tau \leq t \leq \tau + 3\Delta t$, that is,

$$
\begin{aligned}
& \phi_n(t) \\
\simeq\ & \frac{1}{(\Delta t)^4}[\frac{1}{24}(t-\tau-\Delta t)(t-\tau-2\Delta t)(t-\tau-3\Delta t)(t-\tau-4\Delta t)\phi_n(\tau) \\
& -\frac{1}{6}(t-\tau)(t-\tau-2\Delta t)(t-\tau-3\Delta t)(t-\tau-4\Delta t)\phi_n(\tau+\Delta t) \\
& +\frac{1}{4}(t-\tau)(t-\tau-\Delta t)(t-\tau-3\Delta t)(t-\tau-4\Delta t)\phi_n(\tau+2\Delta t) \\
& -\frac{1}{6}(t-\tau)(t-\tau-\Delta t)(t-\tau-2\Delta t)(t-\tau-4\Delta t)\phi_n(\tau+3\Delta t) \\
& +\frac{1}{24}(t-\tau)(t-\tau-\Delta t)(t-\tau-2\Delta t)(t-\tau-3\Delta t)\phi_n(\tau+4\Delta t)] \\
& \text{for } \tau \leq t \leq \tau + 4\Delta t. \qquad (17)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \phi_n'(t) \\
\simeq\ & \frac{1}{(\Delta t)^4}[(\frac{1}{6}[t-\tau]^3 - \frac{5}{4}[t-\tau]^2\Delta t + \frac{35}{12}[t-\tau][\Delta t]^2 - \frac{25}{12}[\Delta t]^3)\phi_n(\tau) \\
& +(-\frac{2}{3}[t-\tau]^3 + \frac{9}{2}[t-\tau]^2\Delta t - \frac{26}{3}[t-\tau][\Delta t]^2 + 4[\Delta t]^3)\phi_n(\tau+\Delta t) \\
& +([t-\tau]^3 - 6[t-\tau]^2\Delta t + \frac{19}{2}[t-\tau][\Delta t]^2 - 3[\Delta t]^3)\phi_n(\tau+2\Delta t) \\
& +(-\frac{2}{3}[t-\tau]^3 + \frac{7}{2}[t-\tau]^2\Delta t - \frac{14}{3}[t-\tau][\Delta t]^2 + \frac{4}{3}[\Delta t]^3)\phi_n(\tau+3\Delta t) \\
& +(\frac{1}{6}[t-\tau]^3 - \frac{3}{4}[t-\tau]^2\Delta t + \frac{11}{12}[t-\tau][\Delta t]^2 - \frac{1}{4}[\Delta t]^3)\phi_n(\tau+4\Delta t)] \\
& \text{for } \tau \leq t \leq \tau + 4\Delta t, \qquad (18)
\end{aligned}
$$

and

$$
\begin{aligned}
& \phi_n''(t) \\
\simeq \; & \frac{1}{(\Delta t)^4}[(\frac{1}{2}[t-\tau]^2 - \frac{5}{2}[t-\tau]\Delta t + \frac{35}{12}[\Delta t]^2)\phi_n(\tau) \\
& +(-2[t-\tau]^2 + 9[t-\tau]\Delta t - \frac{26}{3}[\Delta t]^2)\phi_n(\tau+\Delta t) \\
& +(3[t-\tau]^2 - 12[t-\tau]\Delta t + \frac{19}{2}[\Delta t]^2)\phi_n(\tau+2\Delta t) \\
& +(-2[t-\tau]^2 + 7[t-\tau]\Delta t - \frac{14}{3}[\Delta t]^2)\phi_n(\tau+3\Delta t) \\
& +(\frac{1}{2}[t-\tau]^2 - \frac{3}{2}[t-\tau]\Delta t + \frac{11}{12}[\Delta t]^2)\phi_n(\tau+4\Delta t)] \\
& \qquad \text{for } \tau \le t \le \tau + 4\Delta t. \qquad (19)
\end{aligned}
$$

If we let $t = \tau + j\Delta t$ (for $j = 2,\ 3, 4$) in (11), after using (18) and (19), we obtain

$$
\begin{aligned}
& 2\phi_r(\tau + j\Delta t) + S_r(\tau + j\Delta t) \\
= \; & \phi_1(\tau + j\Delta t) + \phi_N(\tau + j\Delta t) + \xi_r\theta(\tau + j\Delta t) + (\xi_r - 1)\omega(\tau + j\Delta t) \\
& +\frac{1}{(\Delta t)^2}\sum_{m=1}^{N} F_{rm}[(\frac{1}{2}j^2 - \frac{5}{2}j + \frac{35}{12} + \frac{25}{48}[-2j^2 + 9j - \frac{26}{3}])\phi_m(\tau) \\
& \qquad +\frac{\Delta t}{4}(-2j^2 + 9j - \frac{26}{3})\phi_m'(\tau) \\
& \qquad +(3j^2 - 12j + \frac{19}{2} + \frac{3}{4}[-2j^2 + 9j - \frac{26}{3}])\phi_m(\tau + 2\Delta t) \\
& \qquad +(-2j^2 + 7j - \frac{14}{3} - \frac{1}{3}[-2j^2 + 9j - \frac{26}{3}])\phi_m(\tau + 3\Delta t) \\
& \qquad +(\frac{1}{2}j^2 - \frac{3}{2}j + \frac{11}{12} + \frac{1}{16}[-2j^2 + 9j - \frac{26}{3}])\phi_m(\tau + 4\Delta t)] \\
& \qquad \text{for } r = 1, 2, \cdots, N \text{ and } j = 2, 3, 4. \qquad (20)
\end{aligned}
$$

Letting $t = \tau + j\Delta t$ (for $j = 2,\ 3, 4$) in (14) yields

$$
\alpha\phi_1(\tau + j\Delta t) + \beta\theta(\tau + j\Delta t) = r(\tau + j\Delta t) \text{ for } j = 2, 3, 4. \qquad (21)
$$

Similarly, (15) gives

$$\sum_{m=1}^{N} G_m \phi_m(\tau + j\Delta t) = E(t + j\Delta t) \text{ for } j = 2, 3, 4. \tag{22}$$

If $\phi_m(\tau)$ and $\phi'_m(\tau)$ ($m = 1, 2, \cdots, N$) are known, we may regard (20), (21) and (22) as a system of $3N + 6$ linear algebraic equations in $3N + 6$ unknowns given by $\phi_m(\tau + j\Delta t)$, $\theta(\tau + j\Delta t)$ and $\omega(\tau + j\Delta t)$ for $m = 1, 2, \cdots, N$ and $j = 2, 3, 4$. We may solve for the unknowns as follows.

Work out $\phi_m(0)$ and $\phi'_m(0)$ using the initial conditions (13). Let $\tau = 0$ in (20), (21) and (22) and solve for $\phi_m(2\Delta t)$, $\theta(2\Delta t)$, $\omega(2\Delta t)$, $\phi_m(3\Delta t)$, $\theta(3\Delta t)$, $\omega(3\Delta t)$, $\phi_m(4\Delta t)$, $\theta(4\Delta t)$ and $\omega(4\Delta t)$. With $\tau = 0$, apply (18) with $t = 0$ to calculate $\phi_m(\Delta t)$ using the known values of $\phi'_m(0)$, $\phi_m(2\Delta t)$, $\phi_m(3\Delta t)$ and $\phi_m(4\Delta t)$. Still with $\tau = 0$, calculate $\phi'_m(4\Delta t)$ using (18) with $t = 4\Delta t$. With $\phi_m(4\Delta t)$ and $\phi'_m(4\Delta t)$ now known, we can let $\tau = 4\Delta t$ in (20), (21) and (22) to solve for $\phi_m(6\Delta t)$, $\theta(6\Delta t)$, $\omega(6\Delta t)$, $\phi_m(7\Delta t)$, $\theta(7\Delta t)$, $\omega(7\Delta t)$, $\phi_m(8\Delta t)$, $\theta(8\Delta t)$ and $\omega(8\Delta t)$ and then calculate $\phi_m(5\Delta t)$ and $\phi'_m(8\Delta t)$ using (18) with $t = 4\Delta t$ and $t = 8\Delta t$ respectively. Repeating the process, we can let $\tau = 8\Delta t$, $12\Delta t$, $16\Delta t$, $\cdots$ to solve for the unknowns at higher and higher time levels.

6 Specific test problems

Problem 1. For the first test problem, in the wave equation (1), the initial conditions (3), the non-local condition (5) and the boundary condition (6), take

$$\begin{aligned} q(x,t) &= 0, \;\; f(x) = 0, \;\; g(x) = \pi\cos(\pi x), \\ E(t) &= 0, \;\; \alpha = 1, \;\; \beta = 0, \;\; r(t) = \sin(\pi t). \end{aligned} \tag{23}$$

With (23), it may be easily verified that the exact solution to the problem is given by

$$\phi(x,t) = \cos(\pi x)\sin(\pi t). \tag{24}$$

To apply the numerical procedure in Section 5 to solve the problem here, we select the nodes $\xi_1, \xi_2, \cdots, \xi_{N-1}$ and ξ_N to be given by $\xi_i = (i-1)/(N-1)$ for $i = 1, 2, \cdots, N$. For $N = 21$ and various values of Δt, the absolute errors of the numerically obtained ϕ at selected points and at time $t = 1/2$ are given in Table 1.

Table 1. Absolute errors of the numerical values of $\phi(x,t)$ at selected points and at $t = 1/2$, as computed using $N = 21$ and various values of Δt.

x	Exact ϕ	$\Delta t = 0.125$ Absolute error	$\Delta t = 0.0625$ Absolute error	$\Delta t = 0.03125$ Absolute error
0.10	0.95106	3.4×10^{-3}	3.0×10^{-4}	1.9×10^{-5}
0.20	0.80902	5.3×10^{-3}	6.1×10^{-4}	2.0×10^{-5}
0.30	0.58779	5.5×10^{-3}	7.0×10^{-4}	5.3×10^{-5}
0.40	0.30902	3.5×10^{-3}	4.5×10^{-4}	3.8×10^{-5}
0.50	0.00000	3.3×10^{-14}	1.5×10^{-14}	3.3×10^{-15}
0.60	-0.30902	3.5×10^{-3}	4.5×10^{-4}	3.8×10^{-5}
0.70	-0.58779	5.5×10^{-3}	7.0×10^{-4}	5.3×10^{-5}
0.80	-0.80902	5.3×10^{-3}	6.1×10^{-4}	2.0×10^{-5}
0.90	-0.95106	3.4×10^{-3}	3.0×10^{-4}	1.9×10^{-5}
1.00	-1.00000	2.3×10^{-13}	2.0×10^{-13}	3.5×10^{-13}

From Table 1, it is obvious that there is a significant reduction in the absolute errors of the numerical values of ϕ when the computation is refined, that is, the numerical values of ϕ converge to the exact solution (24) as the time-step Δt decreases from 0.125 to 0.03125 units. It may also be of interest to compare our results here with those obtained by Dehghan [7] for the same test problem. In [7], numerical values of ϕ at $t = 1/2$ and at the points in

Table 1 are computed using various finite-difference schemes with 101 evenly spaced grid points and a very small time-step of about 0.008 units. The accuracy of the numerical values in Table 1, obtained using only 21 nodal points and relatively larger time-steps, is as good compared with that of those given in [7]. When we repeat our calculations using 41 nodal points and a time-step of 0.015625 units, our numerical values are definitely more accurate than those in [7].

Problem 2. For the second test problem, take

$$\begin{aligned} q(x,t) &= (\pi^2 + \frac{1}{4}) \exp(-\frac{1}{2}t) \sin(\pi x), \\ f(x) &= \sin(\pi x), \;\; g(x) = -\frac{1}{2} \sin(\pi x), \\ E(t) &= \frac{2}{\pi} \exp(-\frac{1}{2}t), \\ \alpha = 1, \;\; \beta &= 1, \; r(t) = \pi \exp(-\frac{1}{2}t). \end{aligned} \tag{25}$$

The exact solution for this particular test problem is given by

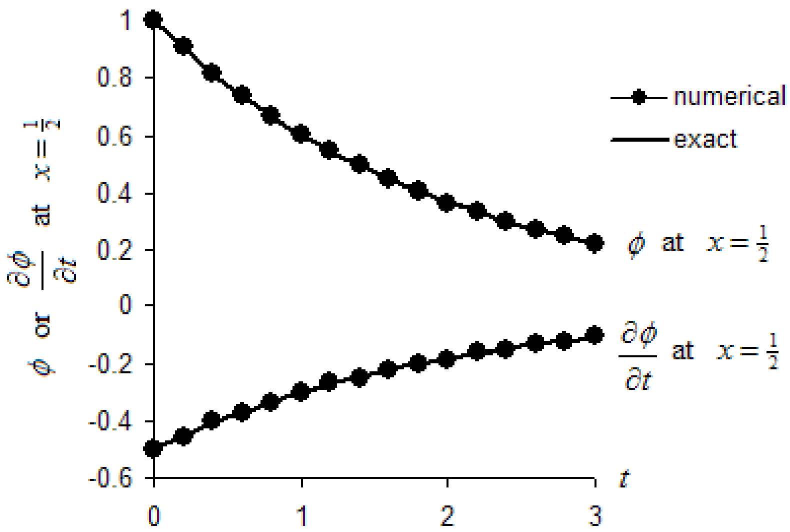
$$\phi(x,t) = \exp(-\frac{1}{2}t) \sin(\pi x). \tag{26}$$

With $q(x,t)$ as given in (25), the function $S_r(t)$ in (11) may be written as

$$S_r(t) = (1 + \frac{1}{4\pi^2}) \exp(-\frac{1}{2}t)(-2\sin(\pi\xi_r) + \pi). \tag{27}$$

As in the first test problem, the nodes ξ_1, ξ_2, $\cdots$, ξ_{N-1} and ξ_N are taken to be given by $\xi_i = (i-1)/(N-1)$ for $i = 1, 2, \cdots, N$. Using $N = 41$ and $\Delta t = 0.05$, we compute ϕ and $\partial\phi/\partial t$ numerically at the point $x = 1/2$ for $0 \leq t \leq 3$. The numerical values obtained are compared graphically with the exact ones in Figure 1. As the numerical and exact values agree to at least 2 significant figures, their graphs are almost visually indistinguishable.

Figure 1



7 Discussion and conclusion

A numerical method has been successfully developed and implemented on the computer for solving the one-dimensional wave equation subject to a non-local conservation condition and suitably prescribed initial-boundary conditions. It (the method) reduces the problem under consideration into a system of linear algebraic equations which may be written in matrix form as $\mathbf{AX} = \mathbf{B}$ over a certain time interval consisting of several consecutive time levels. The system can be used to obtain a time-stepping scheme for finding an approximate solution to the problem at higher and higher time levels. In implementing the time-stepping scheme, the square matrix $\mathbf{A}$ has to be evaluated and processed only once in order to solve linear algebraic equations, if the absolute difference between two consecutive time levels (that is, Δt)

remains constant. Thus, for example, if the LU decomposition technique together with backward substitutions is applied to solve the linear algebraic equations, then the square matrix has to be decomposed only once.

Numerical results obtained for specific test problems (with known exact solutions) indicate that accurate numerical solutions can be obtained by using the numerical method presented here. Convergence of the numerical solutions to the exact ones is observed when the calculations are refined by reducing Δt or by increasing the number of nodal (collocation) points in the spatial domain of the problem.

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