Hypersingular integral equations for a thermoelastic problem of multiple planar cracks in an anisotropic medium

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Abstract

The problem of calculating the thermoelastic stress around an arbitrary number of arbitrarily-located planar cracks in an infinite anisotropic medium is considered. The cracks open up under the action of suitably prescribed heat flux and traction. With the aid of suitable integral solutions, we reduce the problem to solving a system of Hadamard finite-part (hypersingular) integral equations. The hypersingular integral equations are solved for specific cases of the problem.

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1 INTRODUCTION

Anisotropic structures can now be found in an increasingly wider range of applications in modern technology. For example, synthetic materials, such as plywoods and carbon-carbon composites, which exhibit anisotropic behaviours are widely used in the design and construction of modern vehicles (e.g. aircrafts and high speed trains). The need to assess the reliability and integrity of these structures has given rise to a vast body of literature on cracks in anisotropic media. The majority of the papers in the literature, e.g. Sollero and Aliabadi [9], Ang [1], Sweeney [12] and Stroh [10], ignored the effect of heat flow on the stress distribution around the cracks.

There are relatively fewer studies which analyse the thermoelastic stress around cracks in anisotropic solids. Using integral transform techniques, Atkinson and Clements [2] calculated the thermoelastic stress around a planar crack in an infinite anisotropic medium. Similar problems involving a single crack in an infinite anisotropic material were also solved by Hwu [6], Tsai [13] and Wu [14]. Sturla and Barber [11] extended the work in Atkinson and Clements [2] to allow for the possibility that the crack may be partially closed. Clements [3] examined the thermoelastic problem of a planar crack between bonded dissimilar anisotropic materials.

In the present paper, the problem of calculating the thermoelastic stress around an arbitrary number of arbitrarily-located planar cracks in an infinite anisotropic medium is considered. The problem is reduced to the task of solving a system of Hadamard finite-part (hypersingular) integral equations. The unknown functions in the integral equations are the jumps in the temperature across opposite crack faces and the crack-opening displacements. For convenience, we assume that the cracks open up under the action of suitably prescribed heat flux and traction. It may be worth mentioning that the analysis presented is valid for the most general anisotropic material, i.e. it does not require the material to possess any particular symmetries in its anisotropy.

2 TEMPERATURE AROUND CRACKS

2.1 Mathematical formulation

By referring to a Cartesian coordinate system $0x_1x_2x_3$, consider an infinite anisotropic elastic medium in which there are N arbitrarily-located planar cracks having geometries that do not vary along the x_3-axis. The cracks denoted by $C^{(1)}$, $C^{(2)}, \cdots$, and $C^{(N)}$ are assumed not to intersect with one another. On the $0x_1x_2$ plane, the tips of a typical crack $C^{(k)}$ are given by $(a^{(k)}, b^{(k)})$ and $(c^{(k)}, d^{(k)})$. Refer to Figure 1.

The cracks are acted upon by suitably prescribed heat flux. The heat flux generated by the cracks are required to vanish at infinity. The problem is to determine the temperature throughout the elastic medium, particularly around the cracks.

For plane problems, the steady-state temperature in a homogeneous anisotropic medium is given by the function $T(\mathbf{x})$ ($\mathbf{x} = (x_1, x_2)$) which satisfies the heat conduction equation (Nowacki [8])

$$\lambda_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} = 0, \tag{1}$$

where $\lambda_{ij} = \lambda_{ji}$ $(i, j = 1, 2, 3)$ are the constant heat conduction coefficients satisfying the strict inequality $\lambda_{11}\lambda_{22} - \lambda_{12}^2 > 0$. The convention of summing over a repeated index is adopted only for latin subscripts which run from 1 to 3.

The mathematical task is to solve (1) subject to the conditions on the cracks given by

$$\begin{aligned} \lambda_{ij} n_i^{(m)} \frac{\partial}{\partial x_j} (T(\mathbf{x})) \quad &\rightarrow \quad -S_i(\mathbf{y}) n_i^{(m)} \\ \text{as } \mathbf{x} \quad &\rightarrow \quad \mathbf{y} \in C^{(m)} \ (m = 1, 2, \cdots, N), \end{aligned} \tag{2}$$

where $\mathbf{n}^{(m)} = [n_1^{(m)}, n_2^{(m)}] = [\{d^{(m)} - b^{(m)}\}/\ell^{(m)}, \{a^{(m)} - c^{(m)}\}/\ell^{(m)}]$ is a unit normal vector to the crack $C^{(m)}$, $\ell^{(m)} = \sqrt{\{d^{(m)} - b^{(m)}\}^2 + \{a^{(m)} - c^{(m)}\}^2}$

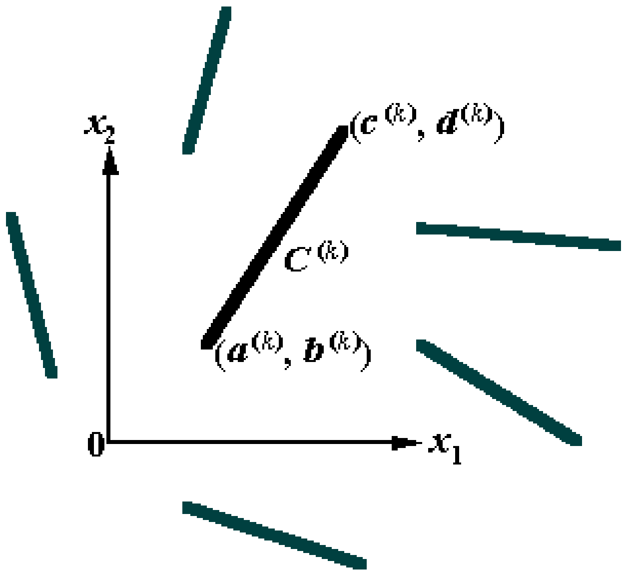

Figure 1: Planar cracks in an anisotropic medium.

and $S_i(\xi_1, \xi_2)$ is a suitably prescribed function. In addition to (2), it is also required that the heat flux generated by the cracks vanishes at infinity.

Notice that the conditions in (2) describe the applied heat flux on the crack faces.

For our purpose here, we shall regard $C^{(m)}$ as the straight line segment from $(a^{(m)}, b^{(m)})$ to $(c^{(m)}, d^{(m)})$.

2.2 Hypersingular integral equations

To solve (1) subject to (2), for any point $\mathbf{x}$ in the interior of the cracked anisotropic medium, we take

$$T(\mathbf{x}) = \sum_{m=1}^{N} \int_{C^{(m)}} r(\mathbf{y}) \Lambda(\mathbf{x}; \mathbf{y}; \mathbf{n}^{(m)}) dS(\mathbf{y}), \tag{3}$$

where $\mathbf{y} = (y_1, y_2)$, $r(\mathbf{y})$ is an arbitrary function yet to be determined and

$$\Lambda(\mathbf{x};\mathbf{y};\mathbf{n}^{(m)}) = \frac{1}{2\pi}\,\mathrm{Re}\left\{\frac{\rho\Omega_k n_k^{(m)}}{(y_1 - x_1) + \tau(y_2 - x_2)}\right\}, \tag{4}$$

with $\Omega_k = \lambda_{k1} + \tau\lambda_{k2}$, $\tau = \left(-\lambda_{12} + i\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}\right)/\lambda_{22}$, $i = \sqrt{-1}$ and $\rho = 1/\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}$.

It is an easy matter to verify via direct substitution that (3) together with (4) satisfies the heat conduction equation (1) in the domain of interest. Equation (3) may be obtained from the boundary integral equation of (1). For further details of the boundary integral equation, refer to Clements [5].

Using (3) together with (4), we find that conditions (2) give rise to:

$$\begin{aligned}&\frac{1}{\pi}\chi^{(q)}\mathcal{H}\int_{-1}^{1}\frac{R^{(q)}(t)dt}{(t-u)^2} + \frac{1}{4\pi}\sum_{m=1}^{N}\int_{-1}^{1} R^{(m)}(t)K^{(mq)}(t,u)dt\\ = \;& -S_i(X_1^{(q)}(u), X_2^{(q)}(u))n_i^{(q)} \text{ for } -1 < u < 1 \; (q = 1, 2, \cdots, N),\end{aligned} \tag{5}$$

where $R^{(q)}(t) = r(X_1^{(q)}(t), X_2^{(q)}(t))$, $2X_1^{(q)}(u) = c^{(q)} + a^{(q)} + (c^{(q)} - a^{(q)})u$, $2X_2^{(q)}(u) = d^{(q)} + b^{(q)} + (d^{(q)} - b^{(q)})u$, $\chi^{(q)} = \mathrm{Re}\{\rho\ell^{(q)}[\Omega_k n_k^{(q)}]^2[c^{(q)} - a^{(q)} + \tau(d^{(q)} - b^{(q)})]^{-2}\}$,

$$\begin{aligned}&K^{(mq)}(t,u)\\ = \;& \mathrm{Re}\left\{\frac{\rho\ell^{(m)}\Omega_k n_k^{(m)}\Omega_i n_i^{(q)}(1-\delta_{mq})}{\left[X_1^{(m)}(t) - X_1^{(q)}(u) + \tau\left(X_2^{(m)}(t) - X_2^{(q)}(u)\right)\right]^2}\right\},\end{aligned} \tag{6}$$

δ_{mq} is the kronecker-delta, and $\mathcal{H}$ indicates that the integral is to be interpreted in the Hadamard finite-part sense or more specifically

$$\begin{aligned}\mathcal{H}\int_{-1}^{1}\frac{R(t)dt}{(t-u)^2} \overset{\text{def}}{=} \lim_{\varepsilon\to 0^+}&\left[\int_{-1}^{1}\frac{(t-u)^2R(t)dt}{[(t-u)^2+\varepsilon^2]^2} - \frac{\pi}{2\varepsilon}R(u)\right]\\ &-\frac{R(u)}{2}[\frac{1}{1-u} + \frac{1}{1+u}] \text{ for } -1 < u < 1.\end{aligned} \tag{7}$$

[Note. In the version published in *Engineeing Analysis with Boundary Elements*, the term on the second line of (7) is missing.]

Equations (5) constitute a system of Hadamard finite-part singular (hypersingular) integral equations containing $R^{(q)}(t)$ as unknown functions. Numerical methods for solving the integral equations are available (e.g. Kaya and Erdogan [7]). Once $R^{(q)}(t)$ for $-1 < t < 1$, or $r(\mathbf{y})$ for $\mathbf{y} \in C^{(q)}$, is determined from (5), the temperature throughout the whole anisotropic medium can be evaluated using (3) together with (4).

If a physical interpretation of the function $r(\mathbf{y})$ in (3) is required, we can use the limit

$$\lim_{\varepsilon \to 0^{\pm}} \varepsilon \int_{-1}^{1} \frac{w(s)ds}{\left([s-t]^2 + \varepsilon^2\right)} = \pm\pi w(t) \quad \text{for } -1 < t < 1 \tag{8}$$

to show that $r(\mathbf{y})$ is directly proportional to the jump in the temperature across opposite faces of the cracks. [Notice that in (8) it is assumed that $w(s)$ is differentiable infinitely many times over the open interval $(-1, 1)$.] More precisely, it can be shown that

$$\lim_{\varepsilon \to 0^{+}} \left[T(\mathbf{y} - \varepsilon\mathbf{n}^{(m)}) - T(\mathbf{y} + \varepsilon\mathbf{n}^{(m)})\right] = r(\mathbf{y}) \quad \text{for } \mathbf{y} \in C^{(m)}. \tag{9}$$

3 STRESS AROUND CRACKS

3.1 Mathematical formulation

Having determined the temperature field, we are now interested in finding the stress distribution around the cracks. The cracks are assumed to open up under the action of applied heat flux (as described in Section 2) and suitably prescribed traction. The stress generated by the cracks are required to vanish at infinity.

If the displacement and the stress are denoted by $u_k(\mathbf{x})$ and $\sigma_{ij}(\mathbf{x})$ respectively then

$$\sigma_{ij}(\mathbf{x}) = c_{ijkr}\frac{\partial u_k}{\partial x_r} - \beta_{ij}T(\mathbf{x}), \tag{10}$$

where c_{ijkp} and $\beta_{ij} = \beta_{ji}$ are, respectively, the constant elastic moduli and stress-temperature coefficients of the anisotropic medium.

Substituting (10) into the equilibrium equations, we find that

$$c_{ijkr}\frac{\partial^2 u_k}{\partial x_j \partial x_r} = \beta_{ij}\frac{\partial T}{\partial x_j}. \tag{11}$$

Notice that the temperature $T(\mathbf{x})$ is regarded as known from Section 2. Hence, (11) is an inhomogeneous system of partial differential equations. For further details, refer to Clements [4].

The mathematical task is to solve (11) subject to the conditions

$$\begin{aligned} \sigma_{kj}(\mathbf{x})n_j^{(m)} &\rightarrow -s_{kj}(\mathbf{y})n_j^{(m)} \\ \text{as } \mathbf{x} &\rightarrow \mathbf{y} \in C^{(m)} \ (m = 1, 2, \cdots, N), \end{aligned} \tag{12}$$

where $-s_{kj}n_j^{(m)}$ denotes the k-th component of the prescribed traction on the crack $C^{(m)}$. In addition, it is also required that $\sigma_{kj} \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$.

3.2 A particular solution

To find a particular solution of (11), for any point $\mathbf{x}$ in the interior of the cracked anisotropic medium, let

$$v_k(\mathbf{x}) = \sum_{m=1}^{N} \int_{C^{(m)}} r(\mathbf{y}) V_k(\mathbf{x}; \mathbf{y}; \mathbf{n}^{(m)}) dS(\mathbf{y}), \tag{13}$$

where $r(\mathbf{y})$ is as determined in Section 2 and $V_k(\mathbf{x}; \mathbf{y}; \mathbf{n}^{(m)})$ are functions yet to be determined.

The system (11) holds if $V_k(\mathbf{x}; \mathbf{y}; \mathbf{n}^{(m)})$ satisfy

$$c_{ijkr}\frac{\partial^2 V_k}{\partial x_j \partial x_r} = \beta_{ij}\frac{\partial \Lambda}{\partial x_j} \text{ for } \mathbf{x} \neq \mathbf{y}, \tag{14}$$

where $\Lambda(\mathbf{x}; \mathbf{y}; \mathbf{n}^{(m)})$ is as given by (4).

From Clements [3], solutions of (14) can be written as

$$\begin{aligned} V_k(\mathbf{x}; \mathbf{y}; \mathbf{n}^{(m)}) &= -\frac{1}{2\pi}\,\mathrm{Re}\left\{\rho D_k \Omega_p n_p^{(m)} \ln(y_1 - x_1 + \tau[y_2 - x_2])\right\} \\ &\quad +\frac{1}{2\pi}\,\mathrm{Re}\sum_{\alpha}\left\{A_{k\alpha} f_\alpha(z_\alpha)\right\}, \end{aligned} \tag{15}$$

where the summation over the greek subscript is from 1 to 3, $f_\alpha(z_\alpha)$ is a holomorphic function of z_α in the interior of the cracked anisotropic medium, $z_\alpha = x_1 + p_\alpha x_2$, $A_{k\alpha}$ and p_α are constants as defined in reference [3] and D_k $(k = 1, 2, 3)$ are constants satisfying the system

$$\left(c_{i1k1} + \tau c_{i1k2} + \tau c_{i2k1} + \tau^2 c_{i2k2}\right) D_k = \beta_{i1} + \tau\beta_{i2}. \tag{16}$$

For a given $\mathbf{y}$, to ensure that $V_k(\mathbf{x};\mathbf{y};\mathbf{n}^{(m)})$ is single-valued at any point $\mathbf{x} \neq \mathbf{y}$, i.e. in order that

$$\oint_{\mathbf{x}\in C} \frac{d}{dS}\left[V_k(\mathbf{x};\mathbf{y};\mathbf{n}^{(m)})\right] dS(\mathbf{x}) = 0$$

holds for any closed curve C in the $0x_1x_2$ plane ($\mathbf{y} \notin C$), we choose

$$f_\alpha(z_\alpha) = \rho N_{\alpha j} D_j \Omega_p n_p^{(m)} \ln(y_1 + p_\alpha y_2 - z_\alpha), \tag{17}$$

where $[N_{\alpha j}]$ is the inverse of $[A_{k\alpha}]$.

From (10), the stress which corresponds to (13) together with (15) and (17) is given by

$$t_{kj}(\mathbf{x}) = \sum_{m=1}^{N} \int_{C^{(m)}} r(\mathbf{y}) W_{kj}(\mathbf{x};\mathbf{y};\mathbf{n}^{(m)}) dS(\mathbf{y}) - \beta_{kj} T(\mathbf{x}), \tag{18}$$

where

$$\begin{aligned} W_{kj}(\mathbf{x};\mathbf{y};\mathbf{n}^{(m)}) &= \frac{1}{2\pi}\operatorname{Re}\left\{\frac{\rho B_{kj}\Omega_p n_p^{(m)}}{(y_1 - x_1 + \tau[y_2 - x_2])}\right\} \\ &\quad -\frac{1}{2\pi}\operatorname{Re}\sum_{\alpha}\left\{\frac{\rho L_{kj\alpha} N_{\alpha s} D_s \Omega_p n_p^{(m)}}{(y_1 + p_\alpha y_2 - z_\alpha)}\right\}, \end{aligned} \tag{19}$$

where $B_{kj} = [c_{kjp1} + \tau c_{kjp2}] D_p$ and $L_{kj\alpha} = [c_{kjp1} + p_\alpha c_{kjp2}] A_{p\alpha}$.

The stress $t_{kj}(\mathbf{x})$ is discontinuous across each of the cracks. Specifically, from (8), (9), (18) and (19), we find that

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\left[t_{kj}(\mathbf{x} - \varepsilon\mathbf{n}^{(m)}) - t_{kj}(\mathbf{x} + \varepsilon\mathbf{n}^{(m)})\right] \\ &= \mu_{kj}^{(m)} r(\mathbf{x}) \ \text{ for } \mathbf{x} \in C^{(m)} \ (m = 1, 2, \cdots, N). \end{aligned} \tag{20}$$

where

$$\mu_{kj}^{(m)} = -\beta_{kj} + \mathrm{Re}\left\{\frac{2i\rho B_{kj}\Omega_p n_p^{(m)}\left(n_1^{(m)} + \overline{\tau} n_2^{(m)}\right)}{\tau - \overline{\tau}}\right\}$$
$$- \mathrm{Re}\sum_{\alpha}\left\{\frac{2i\rho L_{kj\alpha}N_{\alpha s}D_s\Omega_p n_p^{(m)}\left(n_1^{(m)} + \overline{p}_\alpha n_2^{(m)}\right)}{p_\alpha - \overline{p}_\alpha}\right\}, \tag{21}$$

where $\overline{z}$ denotes the complex conjugate of z.

3.3 Hypersingular integral equations

For the solution of (11) subject to (12), we let the displacement be given by

$$u_k(\mathbf{x}) = v_k(\mathbf{x}) + \phi_k(\mathbf{x}), \tag{22}$$

where $v_k(\mathbf{x})$ are given by (13) together with (15)-(17) and $\phi_k(\mathbf{x})$ are functions yet to be determined.

From (11) and (22), for $\mathbf{x}$ in the interior of the cracked anisotropic medium, the functions $\phi_k(\mathbf{x})$ must satisfy the homogeneous system of partial differentiable equations

$$c_{ijkr}\frac{\partial^2\phi_k}{\partial x_j\partial x_r} = 0. \tag{23}$$

Using (10), we find that the stress which corresponds to $u_k(\mathbf{x})$ in (22) is given by

$$\sigma_{kj}(\mathbf{x}) = t_{kj}(\mathbf{x}) + \varphi_{kj}(\mathbf{x}), \tag{24}$$

where $t_{ij}(\mathbf{x})$ are given by (18) together with (19) and $\varphi_{kj}(\mathbf{x}) = c_{kjpr}\partial\phi_p/\partial x_r$.

For the crack problem under consideration, to ensure that the functions $\sigma_{kj}n_j^{(m)}$ are continuous across the opposite faces of the crack $C^{(m)}$, from (20) and (24), we require that

$$\lim_{\varepsilon\to 0^+}\left[\varphi_{kj}(\mathbf{x} - \varepsilon\mathbf{n}^{(m)}) - \varphi_{kj}(\mathbf{x} + \varepsilon\mathbf{n}^{(m)})\right]n_j^{(m)}$$
$$= -\mu_{kj}^{(m)}n_j^{(m)}r(\mathbf{x}) \text{ for } \mathbf{x} \in C^{(m)}\ (m = 1, 2, \cdots, N). \tag{25}$$

Conditions (12) now become:

$$\begin{aligned}\varphi_{kj}(\mathbf{x})n_j^{(m)} &\to -\left[s_{kj}(\mathbf{y})+t_{kj}(\mathbf{y})\right]n_j^{(m)}\\ \text{as } \mathbf{x} &\to \mathbf{y}\in C^{(m)}\ (m=1,2,\cdots,N).\end{aligned} \tag{26}$$

Our task now is to find the solution of (23) which satisfies the conditions (25) and (26).

If the system (23) holds in a region R (on the $0x_1x_2$ plane) bounded by a closed curve ∂R, it can be shown that (see Clements [5])

$$\phi_k(\mathbf{x}) = \int_{\partial R}\left[\phi_r(\mathbf{y})\Gamma_{rk}(\mathbf{x};\mathbf{y};\mathbf{n}) - \varphi_{rp}(\mathbf{y})n_p(\mathbf{y})\Phi_{rk}(\mathbf{x};\mathbf{y})\right]dS(\mathbf{y}) \text{ for } \mathbf{x}\in R, \tag{27}$$

where $\mathbf{n}=[n_1(\mathbf{y}),n_2(\mathbf{y})]$ is the unit normal outward vector to R at the point $\mathbf{y}=(y_1,y_2)\in\partial R$ and

$$\begin{aligned}\Phi_{rk}(\mathbf{x};\mathbf{y}) &= \frac{1}{2\pi}\,\mathrm{Re}\sum_\alpha\left\{A_{r\alpha}N_{\alpha j}\ln(y_1-x_1+p_\alpha[y_2-x_2])\right\}d_{jk},\\ \Gamma_{rk}(\mathbf{x};\mathbf{y};\mathbf{n}) &= \frac{1}{2\pi}\,\mathrm{Re}\sum_\alpha\left\{L_{rj\alpha}N_{\alpha p}(y_1-x_1+p_\alpha[y_2-x_2])^{-1}\right\}n_j(\mathbf{y})d_{pk},\end{aligned} \tag{28}$$

where d_{jk} are constants defined by the relation

$$-\frac{i}{2}\sum_\alpha\left\{L_{j2\alpha}N_{\alpha p}-\overline{L}_{j2\alpha}\overline{N}_{\alpha p}\right\}d_{pk}=\delta_{jk}. \tag{29}$$

Equations (27) are used in the indirect boundary element formulation of boundary value problems involving (23).

From (27), we find that $\varphi_{qs}(\mathbf{x}) = c_{qskr}\partial\phi_k/\partial x_r$ is given by

$$\begin{aligned}&\varphi_{qs}(\mathbf{x})\\ =&\int_{\partial R}\left[\phi_r(\mathbf{y})\Xi_{rqs}(\mathbf{x};\mathbf{y};\mathbf{n}^{(m)}) + \varphi_{rp}(\mathbf{y})n_p(\mathbf{y})\Theta_{rqs}(\mathbf{x};\mathbf{y})\right]dS(\mathbf{y}) \text{ for } \mathbf{x}\in R,\end{aligned} \tag{30}$$

where

$$\begin{aligned}\Theta_{rqs}(\mathbf{x};\mathbf{y}) &= \frac{1}{2\pi}\,\mathrm{Re}\sum_\alpha\left\{\frac{A_{r\alpha}N_{\alpha p}G_{qs\alpha p}}{y_1-x_1+p_\alpha[y_2-x_2]}\right\},\\ \Xi_{rqs}(\mathbf{x};\mathbf{y};\mathbf{n}) &= \frac{1}{2\pi}\,\mathrm{Re}\sum_\alpha\left\{\frac{L_{rj\alpha}N_{\alpha p}G_{qs\alpha p}}{(y_1-x_1+p_\alpha[y_2-x_2])^2}\right\}n_j(\mathbf{y}),\end{aligned} \tag{31}$$

where $G_{qs\alpha p} = (c_{qsk1} + p_\alpha c_{qsk2})d_{pk}$.

Now if the region R covers the entire $0x_1x_2$ plane and contains cracks $C^{(1)}$, $C^{(2)}, \cdots$, and $C^{(N)}$ in its interior, as in the case of the crack problem presently under consideration, and if the functions $\varphi_{rp}(\mathbf{x})$ behave as $O(|\mathbf{x}|^{-s})$ $(s > 0)$ for very large $|\mathbf{x}|$, then, applying (25) and (30), we obtain, for $\mathbf{x}$ in the interior of the cracked anisotropic medium,

$$\varphi_{qs}(\mathbf{x}) = \sum_{m=1}^{N} \int_{C^{(m)}} \left[\Delta\phi_p(\mathbf{y})\Xi_{pqs}(\mathbf{x};\mathbf{y};\mathbf{n}^{(m)}) - \mu_{pj}^{(m)} n_j^{(m)} r(\mathbf{y})\Theta_{pqs}(\mathbf{x};\mathbf{y})\right] dS(\mathbf{y}), \tag{32}$$

where

$$\lim_{\varepsilon\to 0^+} \left[\phi_p(\mathbf{x} - \varepsilon\mathbf{n}^{(m)}) - \phi_p(\mathbf{x} + \varepsilon\mathbf{n}^{(m)})\right] = \Delta\phi_p(\mathbf{x}) \ \text{ for } \mathbf{x} \in C^{(m)}. \tag{33}$$

Notice that, from (18), (24) and (32), it is clear that $\sigma_{qs}(\mathbf{x})$ behave as $O(|\mathbf{x}|^{-1})$ for very large $|\mathbf{x}|$. Such an asymptotic behaviour was also reported by Sturla and Barber [11].

Substituting (18) and (32) into conditions (26), we obtain

$$\begin{aligned}
&\mathcal{H}\int_{-1}^{1} \frac{\gamma_{rq}^{(i)}\Delta\phi_r^{(i)}(t)dt}{(t-u)^2} + \sum_{m=1}^{N} \frac{1}{2}\ell^{(m)}(1-\delta_{im}) \int_{-1}^{1} \Delta\phi_r^{(m)}(t)\Upsilon_{rq}^{(mi)}(t,u)dt \\
= \;& -s_{qs}(X_1^{(i)}(u), X_2^{(i)}(u))n_s^{(i)} + \mathcal{P}\int_{-1}^{1} \frac{\kappa_q^{(i)} R^{(i)}(t)dt}{(t-u)} \\
&+ \sum_{m=1}^{N} \frac{1}{2}\ell^{(m)}(1-\delta_{im}) \int_{-1}^{1} R^{(m)}(t)\Psi_q^{(mi)}(t,u)dt \\
&\text{for } -1 < u < 1 \ (q = 1,2,3;\ i = 1,2,\cdots,N),
\end{aligned} \tag{34}$$

where $\Delta\phi_r^{(i)}(t) = \Delta\phi_r(X_1^{(i)}(t), X_2^{(i)}(t))$, $\mathcal{P}$ indicates that the integral is to be interpreted in the Cauchy principal sense and

$$\begin{aligned}
\gamma_{rq}^{(i)} &= \frac{1}{\pi}\ell^{(i)} n_p^{(i)} n_s^{(i)} \operatorname{Re}\sum_\alpha \left\{ \frac{L_{rp\alpha}N_{\alpha j}G_{qs\alpha j}}{[c^{(i)} - a^{(i)} + p_\alpha(d^{(i)} - b^{(i)})]^2} \right\}, \\
\Upsilon_{rq}^{(mi)}(t,u) &= \Xi_{rqs}(X_1^{(i)}(u), X_2^{(i)}(u); X_1^{(m)}(t), X_2^{(m)}(t); \mathbf{n}^{(m)})n_s^{(i)}, \\
\kappa_q^{(i)} &= \frac{1}{2\pi}\ell^{(i)} n_s^{(i)} n_p^{(i)} \operatorname{Re}\left\{ \frac{\rho\Omega_p\left(\beta_{qs} - B_{qs}\right)}{c^{(i)} - a^{(i)} + \tau(d^{(i)} - b^{(i)})} \right.
\end{aligned}$$

$$+\sum_{\alpha}\frac{\mu_{rp}^{(i)}A_{r\alpha}N_{\alpha j}G_{qs\alpha j}+\rho L_{qs\alpha}N_{\alpha k}D_k\Omega_p}{c^{(i)}-a^{(i)}+p_\alpha(d^{(i)}-b^{(i)})}\Bigg\},$$

$$\begin{aligned}\Psi_q^{(mi)}(t,u) &= \mu_{rj}^{(m)}n_j^{(m)}\Theta_{rqs}(X_1^{(i)}(u),X_2^{(i)}(u);X_1^{(m)}(t),X_2^{(m)}(t))n_s^{(i)}\\ &\quad -W_{qs}(X_1^{(i)}(u),X_2^{(i)}(u);X_1^{(m)}(t),X_2^{(m)}(t);\mathbf{n}^{(m)})n_s^{(i)}\\ &\quad +\beta_{qs}\Lambda(X_1^{(i)}(u),X_2^{(i)}(u);X_1^{(m)}(t),X_2^{(m)}(t);\mathbf{n}^{(m)})n_s^{(i)}.\end{aligned} \quad (35)$$

Equations (34) constitute a system of hypersingular integral equations from which we can solve for the unknown functions $\Delta\phi_r^{(i)}(t)$ after we have determined the functions $R^{(i)}(t)$ from (5). Once $R^{(i)}(t)$ and $\Delta\phi_r^{(i)}(t)$ are known, the stress $\sigma_{kj}(\mathbf{x})$ can be evaluated using (18), (24) and (32).

4 SPECIFIC CASES

4.1 A single planar crack

For the thermoelastic problem under consideration in Sections 2 and 3, take $N=1$, $(a^{(1)},b^{(1)})=(-a,0)$ and $(c^{(1)},d^{(1)})=(a,0)$ where a is a positive real constant. Physically, this corresponds to the case of a single planar cracks of length $2a$ lying on the $x_2=0$ plane.

If the prescribed heat flux on the crack is given by $S_i(x_1,0)=\delta_{i2}S$ for $-a<x_1<a$ (S is a given constant), we find that the system (5) reduces to:

$$-\frac{1}{2\rho a\pi}\mathcal{H}\int_{-1}^{1}\frac{R^{(1)}(t)dt}{(t-u)^2}=S \quad \text{for } -1<u<1. \quad (36)$$

The hypersingular integral equation (36) has the solution (see, e.g. Kaya and Erdogan [7])

$$R^{(1)}(t)=2\rho aS\sqrt{1-t^2} \text{ for } -1<t<1, \quad (37)$$

which gives us the difference in the temperature on top and bottom crack faces at the point $\mathbf{x}=(at,0)$.

The system (34) becomes:

$$\frac{1}{\pi a}\mathcal{H}\int_{-1}^{1}\frac{\Delta\phi_i^{(1)}(t)dt}{(t-u)^2} = \nu_{qi}\left[s_{q2}(u)-2\rho a\kappa_q Su\right] \quad \text{for } -1<u<1, \tag{38}$$

where $\gamma_{rq}\nu_{qi}=\delta_{ri}$ and

$$\begin{aligned}
\gamma_{rq} &= \frac{1}{2}\,\mathrm{Re}\left\{\sum_\alpha L_{r2\alpha}N_{\alpha j}G_{q2\alpha j}\right\}\\
\kappa_q &= \frac{1}{2}\,\mathrm{Re}\left\{\sum_\alpha\left(\mu_r A_{r\alpha}N_{\alpha j}G_{q2\alpha j}+iL_{q2\alpha}N_{\alpha k}D_k\right)\right.\\
&\quad \left.+i\left(\beta_{q2}-B_{q2}\right)\right\},\\
\mu_r &= -\beta_{r2}+2\,\mathrm{Re}\left\{-\frac{B_{r2}\overline{\tau}}{\tau-\overline{\tau}}+\sum_\alpha\frac{L_{r2\alpha}N_{\alpha s}D_s\overline{p}_\alpha}{p_\alpha-\overline{p}_\alpha}\right\}.
\end{aligned} \tag{39}$$

If we assume that $s_{q2}(u)$ are constant functions, i.e. $s_{q2}(u)=P_q$ (P_q are given constants), then (38) gives us the crack-opening displacements

$$\Delta\phi_i^{(1)}(t)=a\nu_{qi}\left[a\rho\kappa_q St-P_q\right]\sqrt{1-t^2} \quad \text{for } -1<t<1. \tag{40}$$

The solution given by (40) is physically acceptable only if $\Delta\phi_2^{(1)}(t)\geq 0$ for $-1<t<1$, i.e. only if the constant tractions P_q satisfy the inequality

$$\min_{t\in(-1,1)}\nu_{q2}\left[a\rho\kappa_q St-P_q\right]\geq 0. \tag{41}$$

If the inequality in (41) does not hold, the crack may be partially closed. For such a case, (36), (37), (38) and (40) will have to be modified appropriately to take into consideration the region of contact between opposite crack faces. The region of contact may be determined in the manner of Sturla and Barber [11].

4.2 Two coplanar cracks

We shall now solve the hypersingular integral equations (5) and (34) numerically for particular cases of the problem involving a particular transversely-isotropic medium containing a pair of equal length planar cracks.

Specifically, we take the transverse planes of the material to be parallel to the $0x_2x_3$ plane so that equation (1) is given by

$$\lambda_{11}\frac{\partial^2 T}{\partial x_1^2}+\lambda_{22}\frac{\partial^2 T}{\partial x_2^2}=0, \tag{42}$$

and the first two equations of (11) are

$$\begin{aligned} C\frac{\partial^2 u_1}{\partial x_1^2}+L\frac{\partial^2 u_1}{\partial x_2^2}+(F+L)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} &= \beta_{11}\frac{\partial T}{\partial x_1}, \\ A\frac{\partial^2 u_2}{\partial x_2^2}+L\frac{\partial^2 u_2}{\partial x_1^2}+(F+L)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} &= \beta_{22}\frac{\partial T}{\partial x_2}, \end{aligned} \tag{43}$$

where A, L, F and C are independent elastic constants characterising the elastic behaviour of the material. (Assuming that we are not interested in the antiplane deformation of the material, we ignore the third equation which does not contain u_1 and u_2.)

For the systems of partial differential equations above, details involved in the calculation of constants such as p_α, $A_{k\alpha}$, $L_{kj\alpha}$ and $N_{\alpha k}$ are described in, for example, Clements [5]. We may take the latin and greek subscripts in these constants as running from 1 to 2 (instead of 3 as in the general analysis).

For the cracks, we take $N=2$, with $(a^{(1)},b^{(1)})=(a,0)$, $(c^{(1)},d^{(1)})=(a+2\ell,0)$, $(a^{(2)},b^{(2)})=(-a-2\ell,0)$ and $(c^{(2)},d^{(2)})=(-a,0)$, where $a>0$ and $\ell>0$ are given constants. This corresponds to a pair of coplanar cracks, each of length 2ℓ. The inner tips of the cracks are separated by a distance $2a$. The applied heat flux and traction at points $\mathbf{y}$ on both cracks are given by $S_i(\mathbf{y})=\delta_{i2}\lambda_{22}S$ (S is a given constant) and $s_{kj}(\mathbf{y})=0$ [in (2) and (12)] respectively.

For this particular case, $R^{(1)}(t)=R^{(2)}(-t)$ and the system (5) reduces to

$$\mathcal{H}\int_{-1}^{1}\frac{R(t)dt}{(t-u)^2}+\int_{-1}^{1}R(t)K(t,u)dt=-2\pi \text{ for } -1<u<1, \tag{44}$$

where $\chi=\sqrt{\lambda_{11}/\lambda_{22}}$, $R(t)=\chi R^{(1)}(t)/(S\ell)$ and

$$K(t,u)=\left[t+u+2\left(\frac{a}{\ell}+1\right)\right]^{-2}. \tag{45}$$

Considering the symmetry of the problem about the x_2-axis, we find that $\Delta\phi_1^{(1)}(t) = -\Delta\phi_1^{(2)}(-t)$ and $\Delta\phi_2^{(1)}(t) = \Delta\phi_2^{(2)}(t) = 0$. Consequently, the system (34) reduces to a single equation, namely:

$$\begin{aligned} &\mathcal{H}\int_{-1}^{1}\frac{\Delta\phi(t)dt}{(t-u)^2} - \int_{-1}^{1}\Delta\phi(t)K(t,u)dt \\ = \;& \mathcal{P}\int_{-1}^{1}\frac{R(t)dt}{(t-u)} - \int_{-1}^{1}R(t)M(t,u)dt \text{ for } -1<u<1, \end{aligned} \tag{46}$$

where $\Delta\phi(t) = \gamma\chi\Delta\phi_1^{(1)}(t)/(\kappa S\ell^2)$, $K(t,u)$ is as given in (45),

$$M(t,u) = \left[t+u+2\left(\frac{a}{\ell}+1\right)\right]^{-1} \tag{47}$$

and

$$\begin{aligned} \gamma &= \operatorname{Re}\sum_{\alpha} L_{12\alpha}N_{\alpha j}G_{12\alpha j}, \\ \kappa &= \operatorname{Re}\left\{\sum_{\alpha}\left[\mu_{22}^{(1)}A_{2\alpha}N_{\alpha j}G_{12\alpha j} + iL_{12\alpha}N_{\alpha k}D_k\right] - iB_{12}\right\}. \end{aligned} \tag{48}$$

The system of singular integral equations (44) and (46) can be solved numerically. There are several different methods of solution. We adopt the collocation technique of Kaya and Erdogan [7], i.e. we make the approximation:

$$\begin{aligned} R(t) &\simeq \sqrt{1-t^2}\sum_{j=1}^{J} r_j U_{j-1}(t), \\ \Delta\phi(t) &\simeq \sqrt{1-t^2}\sum_{j=1}^{J} \phi_j U_{j-1}(t), \end{aligned} \tag{49}$$

where U_j is the j-th order Chebyshev polynomial of the second kind and r_j and ϕ_j are constants yet to be determined.

Substitution of (49) into (44) and (46) yields (for $-1 < u < 1$)

$$\begin{aligned} \sum_{j=1}^{J} r_j\left[-\pi j U_{j-1}(u) + Y_j(u)\right] &= -2\pi, \\ \sum_{j=1}^{J} \phi_j\left[\pi j U_{j-1}(u) + Y_j(u)\right] &= \sum_{j=1}^{J} r_j\left[\pi T_j(u) + Z_j(u)\right], \end{aligned} \tag{50}$$

where $T_j(x) = U_j(x) - xU_{j-1}(x)$ is the j-th order Chebyshev polynomial of the first kind and

$$\begin{aligned} Y_j(u) &= \int_{-1}^{1} K(t,u)\sqrt{1-t^2}U_{j-1}(t)dt, \\ Z_j(u) &= \int_{-1}^{1} M(t,u)\sqrt{1-t^2}U_{j-1}(t)dt. \end{aligned} \quad (51)$$

In deriving (50), we make use of some results in Kaya and Erdogan [7] to compute the Hadamard finite-part and Cauchy principal integrals.

To set up a system of $2J$ linear algebraic equations in $2J$ unknowns r_j and ϕ_j $(j = 1, 2, \cdots, J)$, we take the value of u in (50) to be given (in turn) by

$$u = \cos\left(\frac{[2p-1]\pi}{2J}\right) \quad \text{for } p = 1, 2, \cdots, J. \quad (52)$$

At the crack tips $(a, 0)$ and $(a+2\ell, 0)$, we define the mode II thermoelastic stress intensity factors

$$K_{II}^{-} = \lim_{x\to a^-} \sqrt{2(a-x)}\sigma_{12}(x,0) \text{ and } K_{II}^{+} = \lim_{x\to(a+2\ell)^+} \sqrt{2(x-a-2\ell)}\sigma_{12}(x,0). \quad (53)$$

Once ϕ_j are determined, these stress intensity factors may be computed approximately using

$$K_{II}^{\pm} \simeq \frac{\kappa S \ell^{3/2}}{2\chi} \sum_{j=1}^{J} \phi_j U_{j-1}(\pm 1). \quad (54)$$

Using $J = 20$, we solve the system of linear algebraic equations and compute the non-dimensionalized temperature jump $R(t)$ and the non-dimensionalized crack-opening displacement (COD) $\Delta\phi(t)$ in Figures 2 and 3 respectively for $a/\ell = 1/100$, $1/10$, 1 and 10. From Figure 2, on a given point on a crack, it is apparent that the temperature jump increases as a/ℓ decreases. It is also interesting to note that on the crack $a < x_1 < a + 2\ell$, $x_2 = 0$, the point where the temperature jump is maximum shifts nearer to the tip $(a, 0)$ as a/ℓ decreases. In Figure 3, we find that the COD may be negative in some regions on the cracks near the inner tips $(-a, 0)$ and $(a, 0)$.

In Figure 4, we plot the non-dimensionalized stress intensity factors $K(1) = 2\chi K_{II}^{-}/\left(\kappa S\ell^{3/2}\right)$ and $K(2) = 2\chi K_{II}^{+}/\left(\kappa S\ell^{3/2}\right)$ against $1/100 \leq a/\ell \leq 5$. For

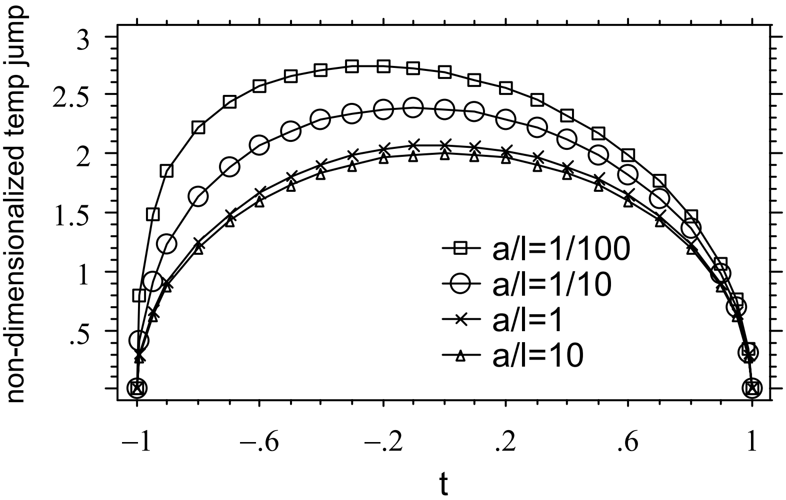

Figure 2: Plots of the non-dimensionalized temperature jump for various values of a/ℓ.

$1/100 \leq a/\ell \leq 5$, $K(1) < 0$, $K(2) > 0$ and $|K(2)| > |K(1)|$. Also, for the same range of a/ℓ, $|K(1)|$ increases while $|K(2)|$ decreases as a/ℓ increases, i.e. the state of stress becomes more severe at the inner tips and less severe at the outer tips if the distance separating the cracks is increased. It appears that $K(1) \to -1$ and $K(2) \to 1$ as $a/\ell \to \infty$.

5 CONCLUSION

Hypersingular integral equations are derived for a two-dimensional thermoelastic crack problem. The integral equations can be used to study the problem for any configuration of planar cracks in any anisotropic full-space. The unknown functions are the temperature jumps across the cracks and the crack-opening displacements. Once they are determined, crack parameters of inter-

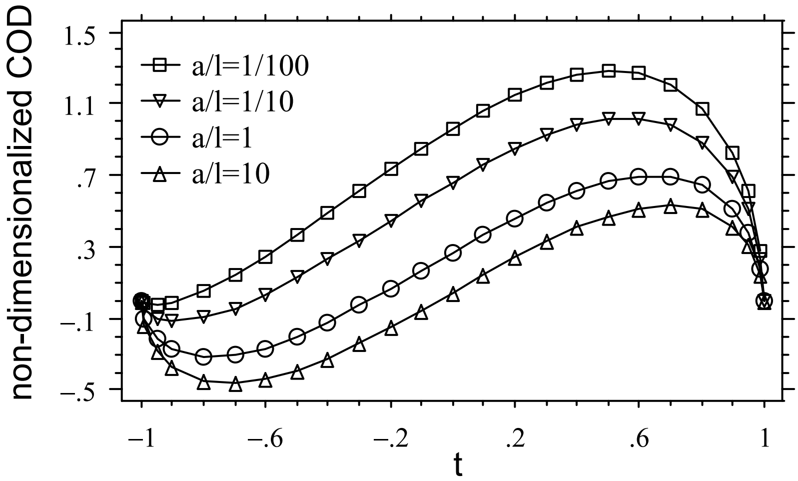

Figure 3: Plots of the non-dimensionalized crack-opening displacement (COD) for various values of a/ℓ.

est, such as the crack tip stress intensity factors, can be readily computed. In general, the hypersingular integral equations have to be solved numerically. For a specific case involving a pair of coplanar cracks in a transversely-isotropic medium, we solve the hypersingular integral equations numerically and compute the relevant crack tip stress intensity factors. The analysis presented can be extended to curved cracks without too much difficulty. With the crack-opening displacements appearing directly in the formulation, the hypersingular approach may prove to be particularly advantageous when the cracks are partially closed under certain thermo-mechanical loading. For partially closed cracks, the hypersingular integral equations may be applied to obtain an iterative procedure for solving the thermoelastic problem. The analysis in the present paper may also be used as a basis for the future development of a hypersingular-boundary integral method for thermoelastic crack

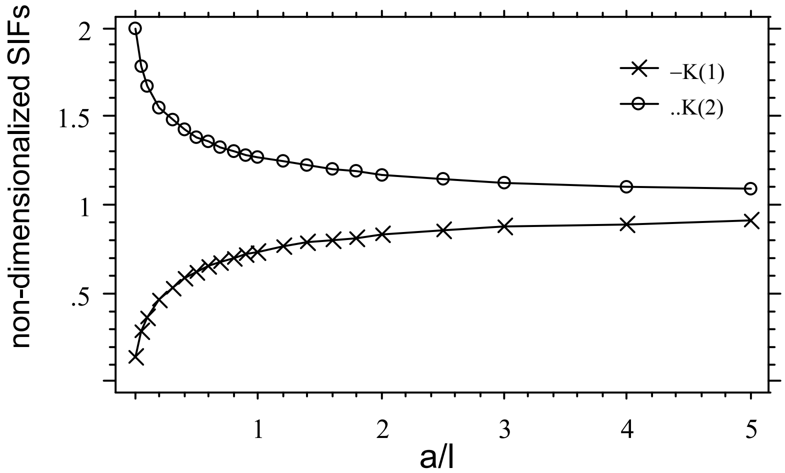

Figure 4: Plots of $-K(1)$ and $K(2)$ against a/ℓ.

problems involving anisotropic bodies of finite extent.

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