A dual-reciprocity boundary element method for a class of elliptic boundary value problems for nonhomogeneous anisotropic media

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Abstract

A dual-reciprocity boundary element method is proposed for the numerical solution of a two-dimensional boundary value problem governed by an elliptic partial differential equation with variable coefficients. The boundary value problem under consideration has applications in a wide range of engineering problems of practical interest, such as in the calculation of antiplane stresses or temperature in nonhomogeneous anisotropic media. The proposed numerical method is applied to solve specific test problems.

Keywords Elliptic partial differential equation, anisotropic nonhomogeneous media, boundary element method, dual-reciprocity method.

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1 Introduction

Of particular interest here is the numerical solution of the elliptic partial differential equation (PDE)

$$\frac{\partial}{\partial x_i}(\lambda_{ij}\frac{\partial u}{\partial x_j}) = 0 \text{ in } R, \tag{1}$$

subject to

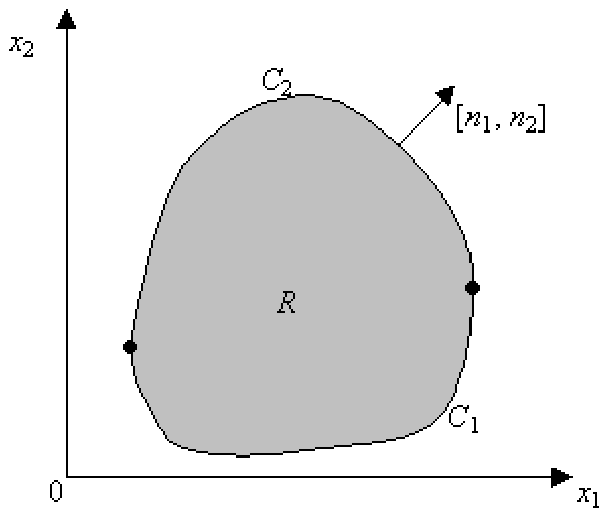
$$\begin{aligned} u(x_1, x_2) &= t(x_1, x_2) \text{ for } (x_1, x_2) \in C_1, \\ v(x_1, x_2) &= h(x_1, x_2) \text{ for } (x_1, x_2) \in C_2, \end{aligned} \tag{2}$$

where the Einsteinian convention of summing over a repeated index is adopted for latin subscripts running from 1 to 2, R is a two-dimensional region bounded by a simple closed curve C on the $0x_1x_2$ plane, $u(x_1, x_2)$ is the unknown function to be determined, λ_{ij} are non-negative coefficients satisfying the symmetry property $\lambda_{ij} = \lambda_{ji}$ and the strict inequality $\lambda_{12}^2 - \lambda_{11}\lambda_{22} < 0$ at all points in the region $R \cup C$, C_1 and C_2 are non-intersecting curves such that $C_1 \cup C_2 = C$, $v(x_1, x_2) = \lambda_{ij}(x_1, x_2)n_i(x_1, x_2)\partial u/\partial x_j$, $n_i(x_1, x_2)$ are the components of the unit normal outward vector to R at the point (x_1, x_2) on C, and t and h are suitably prescribed functions. (Refer to Figure 1.) If v is specified at all points on C (i.e. if $C_1 = \emptyset$) then, to ensure compatibility with (1), the function h in (2) is required to satisfy

$$\oint_C h(x_1, x_2) ds(x_1, x_2) = 0. \tag{3}$$

The boundary value problem (BVP) defined by (1) and (2) has extensive applications in engineering problems involving both nonhomogeneous isotropic and anisotropic media (such as functionally graded materials). For example, if u denotes the steady-state temperature in a two-dimensional solid then under certain assumptions (1) is a manifestation of the law of conservation of energy, λ_{ij} are the thermal conductivity coefficients of the solid and $-v$ represents the heat flux. In general, the thermal conductivity coefficients

Figure 1: A sketch of the geometry of the problem. The solution domain R is bounded by a simple closed curve which consists of two parts C_1 and C_2 on which u and $v = \lambda_{ij} n_i \partial u / \partial x_j$ are respectively specified. The unit normal outward vector to R is given by $[n_1, n_2]$.



may vary from one point to another in the solid. For a thermally isotropic solid, λ_{ij} take the form $\lambda_{ij} = k\delta_{ij}$, where δ_{ij} is the Kronecker-delta. The coefficients λ_{ij} are constant if the solid is thermally homogeneous. Thus, for a solid which is both thermally homogeneous and isotropic, (1) reduces to the two-dimensional Laplace equation.

Another practical example in which the BVP is applicable involves the antiplane static deformation of an infinitely-long anisotropic elastic cylinder with a uniform cross-section given by R. The antiplane deformation is such that the only non-zero components of the displacement and the traction are perpendicular to the $0x_1x_2$ plane and are given by $u(x_1, x_2)$ and $v(x_1, x_2)$ respectively. For this example, λ_{ij} are the elastic shear moduli of the anisotropic solid, the only non-zero stresses are given by $\sigma_{i3} = \lambda_{ij} \partial u / \partial x_j$, and (1) is the equilibrium equation.

The boundary element method (BEM) for the numerical solution of the BVP for the special case in which λ_{ij} are constants (i.e. the case of homogeneous media) is well established, see e.g. Clements [8]. In general, for spatially varying λ_{ij}, it is mathematically difficult to derive a suitable fundamental solution of (1) which can be employed to obtain a boundary integral formulation for the BVP. If the fundamental solution for the homogeneous media is used instead, the resulting integral formulation includes not only a boundary integral but also a domain integral. To deal with the domain integral in an effective manner or to obtain alternative formulations that do not require the solution domain to be discretized, various approaches were proposed for particular nonhomogeneous isotropic media, e.g. Clements [7] and Ang, Kusuma and Clements [2] [special fundamental solutions for the case $\lambda_{ij} = \delta_{ij} X(x) Y(y)$], Rangogni [12] [BEM and perturbation techniques], Kassab and Divo [10] [the idea of a generalized fundamental solution], Park and Ang [11] and Ang, Park and Kang [1] [a complex variable BEM for $\lambda_{ij} = \delta_{ij} X(x) Y(y)$], and Tanaka, Matsumoto and Suda [13] [a dual-reciprocity method for treating the domain integral]. Other relevant references on the BEM for nonhomogeneous isotropic media include Cheng [5]-[6] and Gipson, Ortiz and Shaw [9].

In the present paper, we consider the case in which coefficients of the nonhomogeneous anisotropic media take the form

$$\lambda_{ij}(x_1, x_2) = \lambda_{ij}^{(0)} g(x_1, x_2), \tag{4}$$

where g is a given positive function that can be partially differentiated at least twice with respect to x_i and $\lambda_{ij}^{(0)}$ are non-negative constants satisfying $\lambda_{ij}^{(0)} = \lambda_{ji}^{(0)}$ and $[\lambda_{12}^{(0)}]^2 - \lambda_{11}^{(0)} \lambda_{22}^{(0)} < 0$. After using a substitution to re-write (1) in a suitable form, we employ the fundamental solution for the corresponding homogeneous anisotropic media, which takes the form of a simple logarithmic function, to derive an integral formulation for the BVP under consideration. With such a fundamental solution, the integral formulation inevitably contains a domain integral over the region R. To use the formulation for deriving a BEM for the numerical solution of the BVP, we apply the dual-reciprocity method (DRM) proposed by Brebbia and Nardini [4] to convert the domain integral into a line integral approximately. The DRM requires us to collocate at points in $R \cup C$, but the discretization of the region R into tiny elements is not needed. Thus, in the proposed approach for solving numerically (1) and (2) with (4), only the curve boundary C has to be discretized. In the literature, the term 'dual-reciprocity boundary element method' (DRBEM) is used to describe such a BEM approach. The DRBEM outlined in the present paper is applicable for physically suitable g given by any general function that varies spatially in a sufficiently smooth manner. To assess the applicability of the method, it is used to solve some specific problems.

2 Integral equation

With the substitution

$$u(x_1, x_2) = \frac{1}{\sqrt{g(x_1, x_2)}} w(x_1, x_2), \tag{5}$$

we find that (1) with (4) can be re-written as

$$\lambda_{ij}^{(0)} \frac{\partial^2 w}{\partial x_i \partial x_j} = \kappa(x_1, x_2) w, \tag{6}$$

where κ is given by

$$\kappa(x_1, x_2) = \frac{1}{\sqrt{g(x_1, x_2)}} \lambda_{ij}^{(0)} \frac{\partial^2}{\partial x_i \partial x_j} [\sqrt{g(x_1, x_2)}]. \tag{7}$$

If we pretend that the right hand side of (6) is known, i.e. if we regard (6) as a Poisson's equation, we can apply the analysis in Clements [8] to derive the integral equation

$$
\begin{aligned}
&\gamma(\xi_1,\xi_2)w(\xi_1,\xi_2) \\
&= \iint_R \kappa(x_1,x_2)w(x_1,x_2)\Phi(x_1,x_2,\xi_1,\xi_2)dx_1dx_2 \\
&+ \oint_C [\Gamma(x_1,x_2,\xi_1,\xi_2)w(x_1,x_2) \\
&\quad - \Phi(x_1,x_2,\xi_1,\xi_2)\lambda_{ij}^{(0)}n_i(x_1,x_2)\frac{\partial}{\partial x_j}\{w(x_1,x_2)\}]ds(x_1,x_2), \qquad (8)
\end{aligned}
$$

where $\gamma(\xi_1,\xi_2) = 0$ if $(\xi_1,\xi_2) \notin R \cup C$, $\gamma(\xi_1,\xi_2) = 1$ if $(\xi_1,\xi_2) \in R$, $0 < \gamma(\xi_1,\xi_2) < 1$ if $(\xi_1,\xi_2) \in C$ [$\gamma(\xi_1,\xi_2) = 1/2$ if (ξ_1,ξ_2) lies on a smooth part of C] and

$$
\Phi(x_1,x_2,\xi_1,\xi_2) = \frac{1}{2\pi\sqrt{\lambda_{11}^{(0)}\lambda_{22}^{(0)} - [\lambda_{12}^{(0)}]^2}} \mathrm{Re}\{\ln(x_1-\xi_1+\tau[x_2-\xi_2])\},
$$

$$
\Gamma(x_1,x_2,\xi_1,\xi_2) = \frac{1}{2\pi\sqrt{\lambda_{11}^{(0)}\lambda_{22}^{(0)} - [\lambda_{12}^{(0)}]^2}} \mathrm{Re}\left\{\frac{L(x_1,x_2)}{(x_1-\xi_1+\tau[x_2-\xi_2])}\right\},
$$

$$
L(x_1,x_2) = (\lambda_{11}^{(0)} + \tau\lambda_{12}^{(0)})n_1(x_1,x_2) + (\lambda_{21}^{(0)} + \tau\lambda_{22}^{(0)})n_2(x_1,x_2),
$$

$$
\tau = \frac{-\lambda_{12}^{(0)} + i\sqrt{\lambda_{11}^{(0)}\lambda_{22}^{(0)} - [\lambda_{12}^{(0)}]^2}}{\lambda_{22}^{(0)}} \quad (i = \sqrt{-1}). \qquad (9)
$$

With (6), we can re-write the integral equation (10) as:

$$
\begin{aligned}
&\gamma(\xi_1,\xi_2)\sqrt{g(\xi_1,\xi_2)}u(\xi_1,\xi_2) \\
&= \iint_R \kappa(x_1,x_2)\sqrt{g(x_1,x_2)}u(x_1,x_2) \\
&\quad \times \Phi(x_1,x_2,\xi_1,\xi_2)dx_1dx_2
\end{aligned}
$$

$$
\begin{aligned}
&+ \oint_C [\Gamma(x_1,x_2,\xi_1,\xi_2)\sqrt{g(x_1,x_2)}u(x_1,x_2) \\
&- [u(x_1,x_2)\lambda_{ij}^{(0)} n_i(x_1,x_2)\frac{\partial}{\partial x_j}\{\sqrt{g(x_1,x_2)}\} \\
&\quad + \frac{v(x_1,x_2)}{\sqrt{g(x_1,x_2)}}]\Phi(x_1,x_2,\xi_1,\xi_2)]ds(x_1,x_2).
\end{aligned}
\tag{10}
$$

Notice that $v(x_1, x_2) = \lambda_{ij}(x_1, x_2)n_i(x_1, x_2)\partial u/\partial x_j$ (as defined earlier on).

In the following section, the integral equation (10) is used to derive a DRBEM for the numerical solution of the boundary value problem defined by (1) and (2) with λ_{ij} as given by (4).

3 DRBEM

For the DRBEM, let us discretize the curve C into N straight line (boundary) elements denoted $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N-1)}$ and $C^{(N)}$, i.e. we make the approximation:

$$
C \simeq C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(N-1)} \cup C^{(N)} \tag{11}
$$

As we shall see later on, the DRBEM requires us to collocate equations at points on the boundary C and in the interior of R. For this purpose, we select N points on the boundary C given by $(\xi_1^{(1)}, \xi_2^{(1)})$, $(\xi_1^{(2)}, \xi_2^{(2)})$, $\cdots$, $(\xi_1^{(N-1)}, \xi_2^{(N-1)})$ and $(\xi_1^{(N)}, \xi_2^{(N)})$, and L well-spaced out points in the interior of the region R as denoted by $(\xi_1^{(N+1)}, \xi_2^{(N+1)})$, $(\xi_1^{(N+2)}, \xi_2^{(N+2)})$, $\cdots$, $(\xi_1^{(N+L-1)}, \xi_2^{(N+L-1)})$ and $(\xi_1^{(N+L)}, \xi_2^{(N+L)})$. For convenience, for $p = 1, 2, \cdots, N$, we take $(\xi_1^{(p)}, \xi_2^{(p)})$ to be the midpoint of the line element $C^{(p)}$.

To apply the dual-reciprocity method (DRM) of Brebbia and Nardini [4] to transform the domain integral in (6) into a line integral, we first make the approximation

$$
\kappa(x_1,x_2)\sqrt{g(x_1,x_2)}u(x_1,x_2) \simeq \sum_{p=1}^{N+L} a^{(p)}\sigma^{(p)}(x_1,x_2), \tag{12}
$$

where $a^{(p)}$ are constants to be determined and

$$\sigma^{(p)}(x_1,x_2) = 1 + \left([x_1 - \xi_1^{(p)} + \mathrm{Re}\{\tau\}\{x_2 - \xi_2^{(p)}\}]^2 + [\mathrm{Im}\{\tau\}\{x_2 - \xi_2^{(p)}\}]^2\right)$$
$$+ \left([x_1 - \xi_1^{(p)} + \mathrm{Re}\{\tau\}\{x_2 - \xi_2^{(p)}\}]^2 + [\mathrm{Im}\{\tau\}\{x_2 - \xi_2^{(p)}\}]^2\right)^{3/2}. \tag{13}$$

It should be noted that for $\lambda_{ij}^{(0)} = \delta_{ij}$ (Kronecker-delta) we find that $\tau = i$ and (13) reduces to give the local interpolating functions suggested by Zhang and Zhu [14]. Of course, the choice of the interpolating functions is not unique, and for reasons why (13) is preferred to some other forms, refer to [14].

We can let (x_1, x_2) in (12) be given by $(\xi_1^{(m)}, \xi_2^{(m)})$ for $m = 1, 2, \cdots, N+L$, to set up a system of linear algebraic equations in $a^{(p)}$ which can be inverted to obtain

$$a^{(p)} = \sum_{m=1}^{N+L} \sqrt{g(\xi_1^{(m)}, \xi_2^{(m)})}\, u^{(m)} \kappa(\xi_1^{(m)}, \xi_2^{(m)}) \chi^{(mp)}, \tag{14}$$

where $u^{(m)} = u(\xi_1^{(m)}, \xi_2^{(m)})$ $(m = 1, 2, \cdots, N+L)$ and $\chi^{(mp)}$ are constants defined by

$$\sum_{m=1}^{N+L} \sigma^{(p)}(\xi_1^{(m)}, \xi_2^{(m)}) \chi^{(mr)} = \begin{cases} 1 & \text{if} \quad p = r, \\ 0 & \text{if} \quad p \neq r. \end{cases} \tag{15}$$

Using (12) and (14) and applying the DRM, we find that the double integral in (10) can be approximately re-written as

$$\iint\limits_R \kappa(x_1,x_2)\sqrt{g(x_1,x_2)}\,u(x_1,x_2)\Phi(x_1,x_2,\xi_1,\xi_2)ds(x_1,x_2)$$
$$\simeq \sum_{m=1}^{N+L} \sqrt{g(\xi_1^{(m)}, \xi_2^{(m)})}\, u^{(m)} \kappa(\xi_1^{(m)}, \xi_2^{(m)}) \sum_{p=1}^{N+L} \chi^{(mp)} \Psi^{(p)}(\xi_1, \xi_2), \tag{16}$$

where

$$\Psi^{(p)}(\xi_1,\xi_2)=\gamma(\xi_1,\xi_2)\theta^{(p)}(\xi_1,\xi_2)+\oint\limits_{C}\Phi(x_1,x_2,\xi_1,\xi_2)\beta^{(p)}(x_1,x_2)ds(x_1,x_2)$$
$$-\oint\limits_{C}\theta^{(p)}(x_1,x_2)\Gamma(x_1,x_2,\xi_1,\xi_2)ds(x_1,x_2) \tag{17}$$

with

$$\begin{aligned}
&\left(\frac{\lambda_{11}^{(0)}\lambda_{22}^{(0)}-[\lambda_{12}^{(0)}]^2}{\lambda_{22}^{(0)}}\right)\theta^{(p)}(x_1,x_2)\\
&=\frac{1}{4}\left([x_1-\xi_1^{(p)}+\mathrm{Re}\{\tau\}\{x_2-\xi_2^{(p)}\}]^2+[\mathrm{Im}\{\tau\}\{x_2-\xi_2^{(p)}\}]^2\right)\\
&+\frac{1}{16}\left([x_1-\xi_1^{(p)}+\mathrm{Re}\{\tau\}\{x_2-\xi_2^{(p)}\}]^2+[\mathrm{Im}\{\tau\}\{x_2-\xi_2^{(p)}\}]^2\right)^2\\
&+\frac{1}{25}\left([x_1-\xi_1^{(p)}+\mathrm{Re}\{\tau\}\{x_2-\xi_2^{(p)}\}]^2+[\mathrm{Im}\{\tau\}\{x_2-\xi_2^{(p)}\}]^2\right)^{5/2}.
\end{aligned} \tag{18}$$

and

$$\beta^{(p)}(x_1,x_2)=\lambda_{ik}^{(0)}n_i(x_1,x_2)\frac{\partial\theta^{(p)}}{\partial x_k}. \tag{19}$$

The integral equation (10) together with (11) and (16) may be used to derive

$$\begin{aligned}
&\gamma(\xi_1^{(n)},\xi_2^{(n)})\sqrt{g(\xi_1^{(n)},\xi_2^{(n)})}u^{(n)}\\
&=\sum_{m=1}^{N+L}\sqrt{g(\xi_1^{(m)},\xi_2^{(m)})}u^{(m)}\kappa(\xi_1^{(m)},\xi_2^{(m)})\sum_{p=1}^{N+L}\chi^{(mp)}\Psi^{(p)}(\xi_1^{(n)},\xi_2^{(n)})\\
&\quad+\sum_{m=1}^{N}\sqrt{g(\xi_1^{(m)},\xi_2^{(m)})}u^{(m)}\int\limits_{C^{(m)}}\Gamma(x_1,x_2,\xi_1^{(n)},\xi_2^{(n)})ds(x_1,x_2)\\
&\quad-\sum_{m=1}^{N}[u^{(m)}\lambda_{ij}^{(0)}n_i^{(m)}\ \frac{\partial}{\partial x_j}\{\sqrt{g(x_1,x_2)}\}\bigg|_{(x_1,x_2)=(\xi_1^{(m)},\xi_2^{(m)})}\\
&\qquad+\frac{v^{(m)}}{\sqrt{g(\xi_1^{(m)},\xi_2^{(m)})}}]\int\limits_{C^{(m)}}\Phi(x_1,x_2,\xi_1^{(n)},\xi_2^{(n)})ds(x_1,x_2),\\
&\qquad\qquad\text{for } n=1,2,\cdots,N+L,
\end{aligned} \tag{20}$$

where $v^{(m)} = v(\xi_1^{(m)}, \xi_2^{(m)})$ $(m = 1, 2, \cdots, N)$ and $[n_1^{(m)}, n_2^{(m)}]$ is the outward unit normal vector to $C^{(m)}$. Notice that, in deriving (20), we let (x_1, x_2) in (10) be given by $(\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1, 2, \cdots, N + L$, and in the integrands of the line integrals over $C^{(m)}$, we approximate the functions multiplied to $\Phi(x_1, x_2, \xi_1^{(n)}, \xi_2^{(n)})$ and $\Gamma(x_1, x_2, \xi_1^{(n)}, \xi_2^{(n)})$ as constants given by the values (of the functions) at the midpoint of $C^{(m)}$.

In view of the boundary conditions (2), either $u^{(m)}$ or $v^{(m)}$ (not both) is known for $m = 1, 2, \cdots, N$. Being the values of u at the interior collocation points $(\xi_1^{(N+1)}, \xi_2^{(N+1)})$, $(\xi_1^{(N+2)}, \xi_2^{(N+2)})$, $\cdots$, $(\xi_1^{(N+L-1)}, \xi_2^{(N+L-1)})$ and $(\xi_1^{(N+L)}, \xi_2^{(N+L)})$, the constants $u^{(N+1)}$, $u^{(N+2)}$, $\cdots$, $u^{(N+L-1)}$ and $u^{(N+L)}$ are not known. Thus, the system (20) consists of $N + L$ linear algebraic equations which can be solved for $N + L$ unknowns given by either $u^{(m)}$ or $v^{(m)}$ for $m = 1, 2, \cdots, N$ and $u^{(N+n)}$ for $n = 1, 2, \cdots, L$.

4 Specific problems

We shall now apply the DRBEM proposed above to solve some specific problems.

Problem 1

Solve the elliptic PDE

$$\begin{aligned} &\frac{\partial}{\partial x_1}\left([x_1^2 - 2x_1x_2 + 2]^2[2\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}]\right) \\ &+\frac{\partial}{\partial x_2}\left([x_1^2 - 2x_1x_2 + 2]^2[\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}]\right) \\ &= 0 \text{ in the region } 0 < x_1 < 1,\ 0 < x_2 < 1, \end{aligned} \tag{21}$$

subject to the boundary conditions

$$\begin{aligned} v(x_1, 0) &= 0 \text{ for } 0 < x_1 < 1, \\ v(1, x_2) &= -1 + 8x_2 - 4x_2^2 \text{ for } 0 < x_2 < 1, \\ v(x_1, 1) &= 2x_1 \text{ for } 0 < x_1 < 1, \\ u(0, x_2) &= \frac{1}{2}(1 - x_2) \text{ for } 0 < x_2 < 1. \end{aligned} \tag{22}$$

Notice that for this particular BVP we may take

$$\frac{1}{2}\lambda_{11} = \lambda_{22} = \lambda_{12} = \lambda_{21} = [x_1^2 - 2x_1x_2 + 2]^2. \tag{23}$$

and the function $v(x_1, x_2)$ [for (x_1, x_2) lying on the boundary of the square region $0 < x_1 < 1$, $0 < x_2 < 1$] is defined by

$$v = [x_1^2 - 2x_1x_2 + 2]^2[(2n_1 + n_2)\frac{\partial u}{\partial x_1} + (n_1 + n_2)\frac{\partial u}{\partial x_2}]. \tag{24}$$

It is easy to check that the BVP defined by (21)-(22) has the exact solution

$$u(x_1, x_2) = \frac{1 + x_1 - x_2}{x_1^2 - 2x_1x_2 + 2}. \tag{25}$$

Table 1: A comparison of the numerical values of v with the exact ones at various points on the boundary $x_1 = 0$, $0 < x_2 < 1$.

(x_1, x_2)	$N_0 = 10$ $N_1 = 4$	$N_0 = 30$ $N_1 = 9$	Exact
$(0.000, 0.950)$	-2.2882	-2.1509	-2.1900
$(0.000, 0.750)$	-2.7379	-2.7475	-2.7500
$(0.000, 0.550)$	-2.9822	-2.9883	-2.9900
$(0.000, 0.350)$	-2.8974	-2.9073	-2.9100
$(0.000, 0.150)$	-2.4769	-2.5035	-2.5100

We shall attempt to solve the BVP approximately by using the DRBEM and compare the numerical solution obtained with the exact one in (25). For the DRBEM, each side of the square is divided into N_0 boundary elements, each of length $1/N_0$. The interior collocation points are taken to be $(k/(N_1 + 1), p/(N_1+1))$ for $k, p = 1, 2, \cdots, N_1$. (Thus, in the notation of the preceding section, $N = 4N_0$ and $L = N_1^2$.) For this particular BVP, v is not known on the boundary $x_1 = 0$, $0 < x_2 < 1$. Thus, to assess the accuracy of the DRBEM, we make a comparison of the numerical values of v on that part of the boundary with the exact ones given by $v(0, x_2) = -2 - 4x_2 + 4x_2^2$ for

$0 < x_2 < 1$. The numerical values of v at selected points on the boundary $x_1 = 0$, $0 < x_2 < 1$ as obtained using $(N_0, N_1) = (10, 4)$ and $(N_0, N_1) = (30, 9)$ are given in the second and third columns of Table 1 respectively, and compared with the exact values in the last column. Similarly, in Table 2, we compare the numerical values of u at various points in the interior of the square region with the exact ones. In both tables, it is clear that the accuracy of the numerical values improve significantly when N_0 increases from 10 to 30 and N_1 from 4 to 9.

Table 2: A comparison of the numerical values of u with the exact ones at various points in the interior of the square.

(x_1, x_2)	$N_0 = 10$ $N_1 = 4$	$N_0 = 30$ $N_1 = 9$	Exact
$(0.400, 0.200)$	0.5979	0.5996	0.6000
$(0.600, 0.200)$	0.6577	0.6598	0.6604
$(0.400, 0.400)$	0.5424	0.5433	0.5435
$(0.600, 0.400)$	0.6365	0.6379	0.6383
$(0.400, 0.600)$	0.4757	0.4761	0.4762
$(0.600, 0.600)$	0.6087	0.6095	0.6098
$(0.400, 0.800)$	0.3945	0.3947	0.3947
$(0.600, 0.800)$	0.5709	0.5713	0.5714

Problem 2

Find the antiplane displacement $u(x_1, x_2)$ in an elastic slab occupying the region $0 < x_1 < \ell$, $0 < x_2 < \ell$, $-\infty < x_3 < \infty$, where ℓ is a given positive real number. The side $x_2 = 0$ of the slab is perfectly bonded to a rigid wall, the side $x_2 = \ell$ is acted upon by a constant shear stress and the remaining two sides $x_1 = 0$ and $x_1 = \ell$ are stress-free. Thus, the relevant boundary conditions for the problem are given by

$$\begin{aligned} &u(x_1, 0) = 0 \text{ for } 0 < x_1 < \ell, \\ &v(x_1, \ell) = v_0 \text{ for } 0 < x_1 < \ell, \\ &v(0, x_2) = v(\ell, x_2) = 0 \text{ for } 0 < x_2 < \ell, \end{aligned} \quad (26)$$

where v_0 is a given constant and v is the antiplane traction. The elastic slab is nonhomogeneous with its shear moduli given by

$$\lambda_{11} = \frac{1}{2}\lambda_{22} = \lambda[(1+\frac{2x_2}{\ell})^2 + \frac{1}{10}\sin(\frac{\pi x_2}{\ell})] \text{ and } \lambda_{12} = \lambda_{21} = 0, \tag{27}$$

where λ is a given positive constant.

Azis, Clements and Budhi [3] had computed the non-dimensionalized displacement $\lambda u/(v_0\ell)$ at various points in the slab using a BEM together with a perturbation scheme. Here we apply the DRBEM to calculate $\lambda u/(v_0\ell)$ approximately. The boundary of the square region is discretized into $4N_0$ equal length elements and the N_1^2 interior collocation points are selected as in Problem 1 above. In Table 3, the numerical results obtained using $(N_0, N_1) = (20, 9)$ and $(N_0, N_1) = (40, 19)$ are compared with those given by Azis, Clements and Budhi [3] and also with those obtained using the ANSYS finite element analysis software. For the finite element method (FEM), the slab is simply modeled as consisting of 10 homogeneous layers denoted by $L^{(1)}$, $L^{(2)}, \cdots$, $L^{(9)}$ and $L^{(10)}$, where $L^{(k)} = \{(x_1, x_2, x_3) : 0 < x_1 < \ell,$ $(k-1)\ell/10 < x_2 < k\ell/10,\ -5\ell < x_3 < 5\ell\}$. The layers are perfectly bonded to one another. The shear moduli of the homogeneous layer $L^{(k)}$ are given by

$$\lambda_{11} = \frac{1}{2}\lambda_{22} = \lambda[(1+\frac{[2k-1]}{10})^2 + \frac{1}{10}\sin(\frac{\pi[2k-1]}{20})] \text{ and } \lambda_{12} = \lambda_{21} = 0. \tag{28}$$

Several thousand elements are employed in the FEM model. The FEM solution for $\lambda u/(v_0\ell)$ given in the last column of Table 3 are the values of the non-dimensionalized displacement on the plane $x_3 = 0$ of the FEM model of the slab.

The numerical values of $\lambda u/(v_0\ell)$ obtained using the DRBEM show convergence when N_0 is increased from 20 to 40 and N_1 from 9 to 19. The DRBEM solution also shows a reasonably good agreement with the numerical values given by Azis, Clements and Budhi [3]and the FEM. As the numerical values given in [3] were obtained by taking only a few terms in a series solution and each of the terms was calculated using a BEM scheme, we expect the more direct DRBEM solution as given in the second and the third columns of Table 3 to be more accurate than the numerical values in the fourth column. At each of the points, the numerical value of the displacement given by the DRBEM is greater than that of Azis, Clements and Budhi [3] but less than that given by the FEM.

Table 3: A comparison of the numerical values of $\lambda u/(v_0\ell)$ with those given by Azis, Clements and Budhi [3] at various points in the interior of the square.

(x_1, x_2)	$N_0 = 20$ $N_1 = 9$	$N_0 = 40$ $N_1 = 19$	Azis, Clements and Budhi [3]	FEM
$(0.500, 0.100)$	0.0411	0.0412	0.0395	0.0413
$(0.500, 0.200)$	0.0700	0.0701	–	0.0704
$(0.500, 0.300)$	0.0916	0.0917	0.0901	0.0921
$(0.500, 0.400)$	0.1085	0.1086	–	0.1091
$(0.500, 0.500)$	0.1219	0.1221	0.1206	0.1227
$(0.500, 0.600)$	0.1330	0.1332	–	0.1339
$(0.500, 0.700)$	0.1422	0.1425	0.1410	0.1432
$(0.500, 0.800)$	0.1502	0.1504	–	0.1511
$(0.500, 0.900)$	0.1570	0.1572	0.1556	0.1580

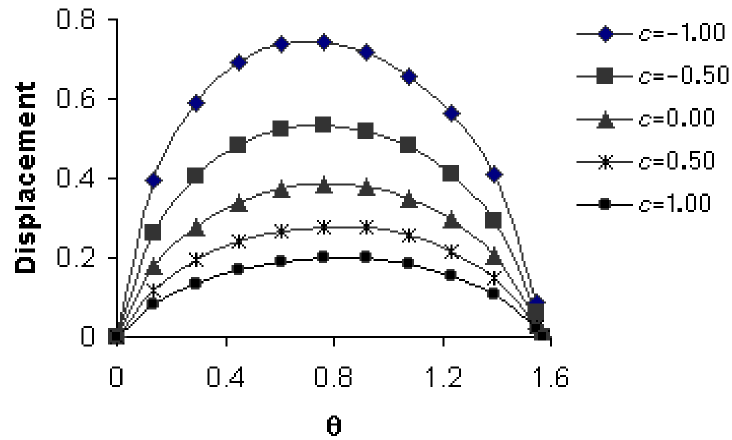
Problem 3

Find the antiplane displacement $u(x_1, x_2)$ in an infinitely-long elastic cylinder with a uniform cross-section that occupies the quarter-circular region $x_1^2 + x_2^2 < \ell^2$, $x_1 > 0$, $x_2 > 0$, where ℓ is a given positive real number. The boundary conditions are:

$$\begin{aligned} u(x_1, 0) &= 0 \text{ for } 0 < x_1 < \ell, \\ v(x_1, x_2) &= v_0 \text{ for } x_1^2 + x_2^2 = \ell^2, x_1 > 0, x_2 > 0, \\ u(0, x_2) &= 0 \text{ for } 0 < x_2 < \ell, \end{aligned} \tag{29}$$

where v_0 is a given constant and v is the antiplane traction.

Figure 2: Plots of the non-dimensionalized displacement $\lambda u/(v_0\ell)$ (on the curved part of the boundary) against the angle θ $(0 < \theta < \pi/2)$ for $c = -1.00$, -0.500, 0.000, 0.500 and 1.00.



The elastic property of the cylinder is functionally graded such that its shear moduli are given by

$$\lambda_{11} = \frac{1}{2}\lambda_{22} = \lambda \exp[\frac{c}{\ell^2}(x_1^2 + x_2^2)] \text{ and } \lambda_{12} = \lambda_{21} = 0, \tag{30}$$

where λ is a given positive constant and c is also a constant.

For the DRBEM, each of the flat parts of the boundary (i.e. $x_1 = 0$, $0 < x_2 < \ell$, and $x_2 = 0$, $0 < x_1 < \ell$) as well as the curved part $(x_1^2 + x_2^2 = \ell^2, x_1 > 0, x_2 > 0)$ is discretized into N_0 elements (so that $N = 3N_0$). The interior collocation points are taken to be given by $(\{k/(N_1+1)\}\cos(n\pi/\{2(N_1+1)\}), \{k/(N_1+1)\}\sin(n\pi/\{2(N_1+1)\}))$ for $k = 1, 2, \cdots, N_1$ and $n = 1, 2, \cdots, N_1$, where N_1 is a positive integer. (Thus, there are N_1^2 interior collocation points.) The displacement is not known on the curved part of the boundary. In Figure 2, we plot the non-dimensionalized displacement $\lambda u/(v_0\ell)$ on (on the curved part of the boundary) (obtained numerically using $(N_0, N_1) = (40, 8)$) against the angle θ $[\theta = \arctan(x_2/x_1)$, $x_1^2 + x_2^2 = \ell^2, x_1 > 0, x_2 > 0]$ for $c = -1.00$, -0.500, 0, 0.500 and 1.00. For a given θ, it is obvious that the displacement decreases in magnitude as the parameter c increases from -1.00 to 1.00. (If c increases, the shear modulus at any given point in the material increases.) The graphs in Figure 2 give qualitatively acceptable results, as the displacement is expected to be of smaller magnitude for a material of greater strength.

5 Summary

The task of solving a class of two-dimensional boundary value problems (BVPs) governed by an elliptic partial differential equation (PDE) that arises frequently in the formulation of engineering problems involving nonhomogeneous anisotropic media is considered. With an appropriate substitution, the PDE is re-cast in a form that allows the BVPs to be formulated in terms of an integral equation suitable for the development of a dual-reciprocity boundary element method (DRBEM). An DRBEM is proposed for the numerical solution of the BVPs. To assess the validity and accuracy of the proposed DRBEM, it is applied to solve several specific problems. Some of the specific problems have known solutions. The numerical results obtained by using the DRBEM agree favorably with the known solution. Convergence in the numerical values obtained is observed when the number of boundary elements and interior collocation points is increased (at least within the range of the number of collocation points used in the numerical calculations for the given specific examples).

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