2D Potential Problems with Periodic Boundary Conditions

We will explain here how the codes in Chapter 1 of the book "A Beginner's Course in Boundary Element Methods" can be modified to solve a particular 2D potential problem with periodic boundary conditions. The particular problem requires solving

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \text{ for } -\infty<x<\infty,\ 0<y<b, \tag{1}$$

subject to the boundary conditions

$$\begin{aligned}\phi(x,0) &= f(x) \text{ for } -\infty<x<\infty,\\ \left.\frac{\partial\phi}{\partial n}\right|_{y=b} &= g(x) \text{ for } -\infty<x<\infty,\end{aligned} \tag{2}$$

where f and g are given functions which are periodic with period $a>0$, that is, they satisfy the periodic conditions

$$f(x+a)=f(x) \text{ and } g(x+a)=g(x). \tag{3}$$

The problem defined by (1)-(3) may be reformulated as one which requires solving

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \text{ for } 0<x<a,\ 0<y<b, \tag{4}$$

subject to

$$\begin{aligned}\phi(x,0) &= f(x) \text{ for } 0<x<a,\\ \left.\frac{\partial\phi}{\partial n}\right|_{y=b} &= g(x) \text{ for } 0<x<a,\\ \phi(0,y) &= \phi(a,y) \text{ for } 0<y<b,\\ \left.\frac{\partial\phi}{\partial n}\right|_{x=0} &= -\left.\frac{\partial\phi}{\partial n}\right|_{x=a} \text{ for } 0<y<b.\end{aligned} \tag{5}$$

There are several different ways to modify the codes in Chapter 1 of the book to solve the problem defined by (4)-(5). One possible way is as explained below.

Discretize each of the vertical sides of the rectangular solution domain into N_1 elements and each of the horizontal sides into N_2 elements by placing

$2N_1 + 2N_2$ points in counterclockwise fashion on the boundary as explained in Chapter 1. We have a total of $2N_1+2N_2$ elements, that is, $N = 2N_1+2N_2$. The first N_1 elements denoted by $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N_1-1)}$ and $C^{(N_1)}$ elements lie on the left vertical side of the rectangular domain, that is, the side, $x = 0$, $0 < y < b$. The elements $C^{(N_1+1)}$, $C^{(N_1+2)}$, $\cdots$, $C^{(N_1+N_2-1)}$ and $C^{(N_1+N_2)}$ lie on the bottom horizontal side $y = 0$, $0 < x < a$; $C^{(N_1+N_2+1)}$, $C^{(N_1+N_2+2)}$, $\cdots$, $C^{(2N_1+N_2-1)}$ and $C^{(2N_1+N_2)}$ lie on the right vertical side $x = a$, $0 < y < b$; $C^{(2N_1+N_2+1)}$, $C^{(2N_1+N_2+2)}$, $\cdots$, $C^{(2N_1+2N_2-1)}$ and $C^{(2N_1+2N_2)}$ on the top horizontal side $y = b$, $0 < x < a$. With such a discretization, we see that the last two lines of (5) give

$$\left.\begin{array}{lcl}\overline{\phi}^{(k)} - \overline{\phi}^{(2N_1+N_2+1-k)} & = & 0 \\ \overline{p}^{(k)} + \overline{p}^{(2N_1+N_2+1-k)} & = & 0\end{array}\right\} \text{ for } k = 1, 2, \cdots , N_1. \tag{6}$$

If $C^{(k)}$ is a horizontal element and ϕ is specified on $C^{(k)}$, we give `BCT(k)` the value 0 and `BCV(k)` the specified value of ϕ. If $C^{(k)}$ is a horizontal element and $\partial\phi/\partial n$ is specified on $C^{(k)}$, we give `BCT(k)` the value 1 and `BCV(k)` the specified value of $\partial\phi/\partial n$. If $C^{(k)}$ is a vertical elements then both $\overline{\phi}^{(k)}$ and $\overline{p}^{(k)}$ are unknowns. If $C^{(k)}$ is a vertical element, we give `BCT(k)` the value of 2 (the value of `BCV(k)` is not important here, as it is not needed in the code).

We have $4N_1+2N_2$ unknowns in our formulation. The boundary integral equation for the 2D Laplace's equation together with the boundary conditions on the horizontal sides and (6) can be used to generate $4N_1 + 2N_2$ linear algebraic equations. If the unknowns are $Z^{(1)}$, $Z^{(2)}$, $\cdots$, $Z^{(4N_1+2N_2-1)}$ and $Z^{(4N_1+2N_2)}$ then we choose $Z^{(k)} = \overline{\phi}^{(k)}$ for $k = 1, 2, \cdots , N_1$, $Z^{(N_1+k)} = \overline{p}^{(k)}$ for $k = 1, 2, \cdots , N_2$, $Z^{(N_1+N_2+k)} = \overline{\phi}^{(k)}$ for $k = 1, 2, \cdots , N_1$, $Z^{(2N_1+N_2+k)} = \overline{\phi}^{(k)}$ for $k = 1, 2, \cdots , N_2$, $Z^{(2N_1+2N_2+k)} = \overline{p}^{(k)}$ for $k = 1, 2, \cdots , N_1$, and $Z^{(2N_1+2N_2+2N_1+1-k)} = \overline{p}^{(2N_1+N_2+1-k)}$ for $k = 1, 2, \cdots , N_1$. The subroutine `CELAP1` has to modified accordingly. It is modifed and renamed `CELAPPER` as listed in the file CELAPPER.for.

For a particular example to test the code, take $b = 0.50$, $f(x) = \sin(x)$ and $g(x) = \exp(0.50)\sin(x)$. The functions $f(x)$ and $g(x)$ are periodic with period $a = 2\pi$. The main program for setting up this problem is in the file EXPRD.for. In the main program, the subroutine `CELAPPER` is called. Other subprograms required to run the program are `CELAP2`, `CPF` and `SOLVER` (together with `DAXPY`, `DSCAL` and `IDAMAX`) (all listed in the book). `CELAP2` is called to compute the solution at any selected point in the domain $0 < x <$

2π, $0 < y < 0.50$. The exact value given by $\phi(x,y) = \exp(y)\sin(x)$ is also printed out by the program. At selected points, numerical results obtained using $N_1 = 10$ and $N_2 = 30$ are compared with the exact solution in the table below.

(x, y)	Numerical	Exact
(0,1, 0.2)	0.1202	0.1219
(3.0,0.4)	0.2077	0.2105
(5.0,0.01)	-0.9748	-0.9686
(6.0,0.45)	-0.4382	-0.4382

Whye-Teong Ang, 10 January 2009