A PENNY-SHAPED CRACK IN AN INHOMOGENEOUS ELASTIC MATERIAL UNDER AXISYMMETRIC TORSION

W. T. Ang
Department of Applied Mathematics
University of Adelaide, South Australia

Abstract

The problem of a penny-shaped crack in an inhomogeneous elastic material under axisymmetric torsion is considered here. The shear modulus of the material is assumed to exhibit a slight variation in the direction perpendicular to the crack. A solution to the problem in series form is proposed, and the first two terms of the series are obtained analytically by using a Hankel transform technique.

Note. This has been a draft of the published paper (by WT Ang) in the journal *Solid Mechanics Archives* Vol. 12 (No. 2) (1987) pp. 391-422. (*Solid Mechanics Archives* was edited by Professor John Roorda of the Solid Mechanics Division at the University of Waterloo, Canada, before its last issue was published around 1990.) The analysis in this draft is essentially the one given in the published paper, although there may be differences in the numbering of the equations as well as other descriptive details. The published paper also contains slightly more analytical details including an appendix containing used results taken from the table of Hankel transforms in Erdélyi *et al* [2].

1 Introduction

The present paper is concerned with an extension of the recent work in Ang and Clements [1] to the case of a penny-shaped crack in an inhomogeneous elastic material under axisymmetric torsion. The shear modulus of the material is assumed to exhbit a slight variation in the direction perpendicular to the crack. A solution to the problem in series form is proposed, and the first two terms of the series are derived analytically by using a Hankel transform technique. The truncated series obtained by retaining only the first two terms of the series is then used to derive an approximate expression for the relevant stress intensity factor. Specific cases of the problem, such as the case in which the shear modulus varies linearly, are considered.

2 Basic equations for axisymmetric torsion

With reference to a cylindrical coordinate system (r, θ, z), consider an elastic material under axisymmetric torsion. The only non-zero component of the displacement is u_θ which is a function of r and z. The only non-zero stresses are

$$\sigma_{\theta z} = \mu \frac{\partial u_\theta}{\partial z} \text{ and } \sigma_{\theta r} = \mu [\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}], \tag{1}$$

where μ is the shear modulus of the material.

In the absence of body forces, the equilibrium equation is given by

$$\frac{\partial}{\partial r}[\sigma_{\theta r}] + \frac{\partial}{\partial z}[\sigma_{\theta z}] + \frac{2}{r}\sigma_{\theta r} = 0. \tag{2}$$

Following Ang and Clements [1], we take the shear modulus to be given by

$$\mu = \mu_0 + \epsilon \mu_1(z), \tag{3}$$

where μ_0 is a constant, ϵ is some positive real constant such that $|\epsilon| << 1$ and μ_1 is a continuous and differentiable function of z in the solution domain of interest.

Substutition of (1) and (3) into (2) yields

$$[\mu_0 + \epsilon \mu_1(z)][\nabla^2 u_\theta - \frac{1}{r^2} u_\theta] + \epsilon \mu_1'(z) \frac{\partial u_\theta}{\partial z} = 0. \tag{4}$$

Note that the operator ∇^2 is defined by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{5}$$

Assume that the displacement u_θ may be written in the form

$$u_\theta = \sum_{n=0}^{\infty} \epsilon^n \Phi_n(r, z). \tag{6}$$

From (1), the stress $\sigma_{\theta z}$ may be written as

$$\sigma_{\theta z} = \sigma_{\theta z}^{(0)} + \epsilon \sigma_{\theta z}^{(1)} + O(\epsilon^2), \tag{7}$$

where

$$\sigma_{\theta z}^{(0)} = \mu_0 \frac{\partial \Phi_0}{\partial z} \text{ and } \sigma_{\theta z}^{(1)} = \mu_0 \frac{\partial \Phi_1}{\partial z} + \mu_1 \frac{\partial \Phi_0}{\partial z}. \tag{8}$$

Substitution of (6) into (4) gives

$$\begin{aligned} \nabla^2 \Phi_0 - \frac{1}{r^2} \Phi_0 &= 0 \\ \nabla^2 \Phi_n - \frac{1}{r^2} \Phi_n &= f_{n-1}(r, z) \text{ for } n \geq 1, \end{aligned} \tag{9}$$

where

$$f_n(r, z) = -\frac{1}{\mu_0} [\mu_1(z)(\nabla^2 \Phi_n - \frac{1}{r^2} \Phi_n) + \mu_1'(z) \frac{\partial \Phi_n}{\partial z}]. \tag{10}$$

3 A penny-shaped crack problem

3.1 Statement of the problem

Consider an infinite elastic material with a penny-shaped crack in its interior. The crack lies in the region $0 \leq r < a$, $0 \leq \theta < 2\pi$, $z = 0$, where a is a positive constant. The shear modulus is given by (3) with μ_1 being an even function of z. The material is subject to a small axisymmetric torsion so that an internal stress $\sigma_{\theta z} = s_0(r)$ acts on the crack. The elastic field generated by the crack vanishes at infinity. The problem is to find the displacement and stress fields throughout the material. Specifically, we are interested in finding out how the inhomogeneity of the material affects the stress intensity factor K defined by

$$K = \lim_{r \to a^+} (r - a)^{1/2} \sigma_{\theta z}(r, 0). \tag{11}$$

From (7), K may be rewritten as

$$K = K^{(0)} + \epsilon K^{(1)} + O(\epsilon^2), \tag{12}$$

where

$$K^{(i)} = \lim_{r\to a^+} (r-a)^{1/2} \sigma_{\theta z}^{(i)}(r,0) \text{ for } i = 0, 1. \tag{13}$$

Mathematically, the problem is to solve (4) in the half-space $z > 0$ subject to

$$u_\theta(r,0) = 0 \text{ for } r > a, \tag{14}$$

and

$$\sigma_{\theta z}(r,0) = s_0(r) \text{ for } 0 \leq r < a. \tag{15}$$

Assume that the problem admits a series solution of the form (6) and that we are interested in finding only the first two terms of the series. The task of solving (4) in the half-space $z > 0$ subject to (14) and (15) can then be reduced to following two problems.

Problem 1. Solve (in the half-space $z > 0$)

$$\nabla^2\Phi_0 - \frac{1}{r^2}\Phi_0 = 0, \tag{16}$$

subject to

$$\Phi_0(r,0) = 0 \text{ for } r > a, \tag{17}$$

and

$$\sigma_{\theta z}^{(0)}(r,0) = s_0(r) \text{ for } 0 \leq r < a. \tag{18}$$

Problem 2. Solve (in the half-space $z > 0$)

$$\nabla^2\Phi_1 - \frac{1}{r^2}\Phi_1 = -\frac{\mu_1'(z)}{\mu_0}\frac{\partial\Phi_0}{\partial z}, \tag{19}$$

subject to

$$\Phi_1(r,0) = 0 \text{ for } r > a, \tag{20}$$

and

$$\sigma_{\theta z}^{(1)}(r,0) = 0 \text{ for } 0 \leq r < a. \tag{21}$$

3.2 Solution of problem 1

A solution of (16) is given by

$$\Phi_0(r,z) = \int_0^\infty \psi(\xi)\exp(-\xi z)J_1(\xi r)d\xi, \tag{22}$$

where J_1 is a Bessel function of order 1 and ψ is an arbitrary function yet to be determined. Note Φ_0 as given above tends to 0 as $r^2 + z^2 \to \infty$.

Now, let

$$\psi(\xi) = \xi^m \int_0^a g(t)J_\nu(\xi t)dt, \tag{23}$$

where J_ν is a Bessel function of order ν and g is an arbitrary function to be determined. From (22) and (23), we find that

$$\Phi_0(r,z) = \int_0^a r^{-1/2}g(t)\int_0^\infty \xi^{m-1/2}J_\nu(\xi t)J_1(\xi r)(\xi r)^{1/2}d\xi dt. \tag{24}$$

From the table of Hankel transforms in Erdélyi *et al* [2], we find that $\Phi_0(r,0) = 0$ for $r > t$, if we select $m = 1/2$ and $\nu = 3/2$. The inequality $r > t$ is always true if $r > a$ (since $0 < t < a$). This implies that the condition (17) is satisfied if we take ψ to be

$$\psi(\xi) = \xi^{1/2}\int_0^a g(t)J_{3/2}(\xi t)dt. \tag{25}$$

It follows that

$$\sigma_{\theta z}^{(0)}(r,0) = -\frac{\mu_0\tau}{r^2}\frac{d}{dr}\int_0^{\min(r,a)}\frac{r^{3/2}g(t)dt}{(r^2-t^2)^{1/2}} \quad (\tau = \sqrt{\frac{2}{\pi}}), \tag{26}$$

if we make use of some results from the table of Hankel transforms in Erdélyi *et al* [2].

Condition (18) now becomes

$$\frac{d}{dr}\int_0^r \frac{r^{3/2}g(t)dt}{(r^2-t^2)^{1/2}} = -\frac{r^2}{\mu_0}\tau s_0(r) \text{ for } 0 \le r < a, \tag{27}$$

which can be inverted to give

$$g(t) = -\frac{2}{\mu_0\sqrt{2\pi t}}\int_0^t \frac{u^2 s_0(u)du}{(t^2-u^2)^{1/2}} \text{ for } 0 < t < a. \tag{28}$$

From (13) and (26), the stress intensity factor $K^{(0)}$ may be given by (after integration by parts)

$$K^{(0)} = \frac{\mu_0 g(a)}{a\sqrt{\pi}}. \tag{29}$$

Note that $g(a)$ can be evaluated either analytically or numerically from (28).

3.3 Solution of problem 2

Let

$$\Phi_1(r,z) = \int_0^\infty \omega(\xi,z) J_1(\xi r) d\xi. \tag{30}$$

From (22), we find that (30) is a solution of (19) if

$$(\xi r)^2 J_1''(\xi r) + \xi r J_1'(\xi r) + [\frac{r^2}{\omega}\frac{\partial^2 \omega}{\partial z^2} - \frac{r^2 \xi}{\omega \mu_0}\mu_1'(z)\psi(\xi)\exp(-\xi z) - 1] J_1(\xi r) = 0. \tag{31}$$

Equation (31) holds if ω satisfies

$$\frac{\partial^2 \omega}{\partial z^2} - \xi^2 \omega = \frac{\xi}{\mu_0}\mu_1'(z)\psi(\xi)\exp(-\xi z). \tag{32}$$

The general solution of (32) is

$$\omega(\xi,z) = \alpha(\xi)\exp(\xi z) + [\beta(\xi) + \phi(\xi,z)]\exp(-\xi z), \tag{33}$$

where α and β are arbitrary functions to be determined and

$$\phi(\xi,z) = \frac{\xi}{\mu_0}\psi(\xi)\exp(2\xi z)\int^z \mu_1(t)\exp(-2\xi t) dt. \tag{34}$$

We set $\alpha(\xi) = 0$ since we require $\Phi_1 \to 0$ as $z \to \infty$. Thus, from (30) and (33), we find that

$$\Phi_1(r,z) = \int_0^\infty [\beta(\xi) + \phi(\xi,z)]\exp(-\xi z) J_1(\xi r) d\xi. \tag{35}$$

Condition (20) is satisfied if β is chosen to be

$$\beta(\xi) = \xi^{1/2} \int_0^a h(t) J_{3/2}(\xi t) dt - \phi(\xi, 0), \tag{36}$$

where h is yet to be determined.

From (8), (35) and (36), we find that

$$\sigma_{\theta z}^{(1)}(r, 0) = \mu_0 \int_0^\infty \phi_z(\xi, 0) J_1(\xi r) d\xi - \frac{\tau}{r^2} \int_0^{\min(r,a)} \frac{u^{3/2}[\mu_0 h(u) + \mu_1(0) g(u)] du}{(r^2 - u^2)^{1/2}}. \tag{37}$$

Thus, by using (27), condition (21) may be rewritten as

$$\frac{d}{dr} \int_0^r \frac{t^{3/2} h(t) dt}{(r^2 - t^2)^{1/2}} = \tau r^2 [\int_0^\infty \phi_z(\xi, 0) J_1(\xi r) d\xi + \frac{\mu_1(0)}{\mu_0^2} s_0(r)] \text{ for } 0 \leq r < a. \tag{38}$$

Equation (38) can be inverted to obtain

$$h(t) + \frac{\mu_1(0)}{\mu_0} g(t) = \frac{2}{\sqrt{2\pi t}} \int_0^t \int_0^\infty \frac{u^2 J_1(\xi u)}{(t^2 - u^2)^{1/2}} \phi_z(\xi, 0) d\xi du \text{ for } 0 \leq t < a. \tag{39}$$

It follows that the stress intensity factor $K^{(1)}$ defined in (13) is given by

$$\begin{aligned} K^{(1)} &= \mu_0 \lim_{r \to a^+} (r - a)^{1/2} \int_0^\infty \phi_z(\xi, 0) J_1(\xi r) d\xi \\ &\quad + \frac{2\mu_0}{\pi a \sqrt{2a}} \int_0^a \int_0^\infty \frac{u^2 J_1(\xi u)}{(a^2 - u^2)^{1/2}} \phi_z(\xi, 0) d\xi du. \end{aligned} \tag{40}$$

4 Uniform torsion

Take $s_0(r) = -\sigma_0$ (constant), that is, the crack is acted upon by a constant torsion. From (28), we obtain

$$g(t) = \frac{t^2 \sigma_0}{2\mu_0 \tau \sqrt{t}}. \tag{41}$$

From (29), $K^{(0)}$ is then given by

$$K^{(0)} = \frac{a\sigma_0}{2\sqrt{2a}}. \tag{42}$$

Through (25) and (41) together with the result (in Abramowitz and Stegun [3])

$$J_{3/2}(z) = \frac{\tau}{\sqrt{z}}[\frac{\sin(z)}{z^2} - \frac{\cos(z)}{z}], \tag{43}$$

we obtain

$$\psi(\xi) = \frac{\sigma_0}{2\mu_0\xi^2}[2(1-\cos(a\xi)) - a\xi\sin(a\xi)]. \tag{44}$$

Consider the following cases.

Case 1. $\mu_1 = k|z|$ (k is a positive constant). From (34), we obtain

$$\phi(\xi, z) = -\frac{k\psi(\xi)}{4\mu_0\xi}[2\xi z + 1], \tag{45}$$

and hence

$$\phi(\xi, z) = -\frac{k\psi(\xi)}{2\mu_0}. \tag{46}$$

Using (44) and (46) together with the fact that $J_\nu(z)$ behaves as $O(z^{-1/2})$ for large $|z|$, we find that the integrand of the first integral in (40) behaves as $O(\xi^{-3/2})$ for large ξ. The integrand is also bounded everywhere within the interval $0 \leq \xi < \infty$. Thus, the integral is well defined for all $r \geq 0$ and does not contribute to the stress intensity factor $K^{(1)}$. It follows that

$$\begin{aligned} K^{(1)} &= -\frac{k\sigma_0}{2\mu_0\pi\sqrt{2a}}\int_0^a \frac{u^2du}{(a^2-u^2)^{1/2}} \\ &\times \int_0^\infty \frac{1}{\xi^2}[2(1-\cos(a\xi)) - a\xi\sin(a\xi)]J_1(\xi u)d\xi. \end{aligned} \tag{47}$$

With the help of the table of Hankel transforms, the integral in (47) can be evaluated to obtain

$$K^{(1)} = \frac{k\sigma_0 a^2(1-\lambda)}{4\pi\mu_0\sqrt{2a}}, \tag{48}$$

where $\lambda = 4(\ln(2) + 1/2)/3 > 1$. Thus, for this particular case in which $\mu = \mu_0 + \epsilon|z|$, the stress intensity factor K is lower than the corresponding stress intensity factor for a material with constant shear modulus μ_0. It is interesting to note that as μ_0 decreases the difference between these stress intensity factors becomes more pronounced. This observation is consistent with that in Ang and Clements [1] for the case in which the variation of the elastic modulus is linear.

Case 2. $\mu_1 = kz^2$ (k is a positive constant). From (34), we obtain

$$\phi(\xi, z) = -\frac{k\psi(\xi)}{4\mu_0\xi^2}[2\xi^2 z^2 + 2\xi z + 1]. \tag{49}$$

As in Case 1, the first integral in (40) is finite for $r \geq 0$ and does not contribute to the stress intensity factor $K^{(1)}$. Hence

$$\begin{aligned} K^{(1)} &= -\frac{k\sigma_0}{2\mu_0\pi\sqrt{2a}} \int_0^a \frac{u^2 du}{(a^2 - u^2)^{1/2}} \\ &\quad \times \int_0^\infty [\frac{2}{\xi^3}(1 - \cos(a\xi)) - \frac{a}{\xi^2}\sin(a\xi)] J_1(\xi u) d\xi, \end{aligned} \tag{50}$$

which eventually gives (after some analytical manipulations using results from the table of Hankel transforms)

$$K^{(1)} = -\frac{k\sigma_0 a^3}{48\mu_0\sqrt{2a}}. \tag{51}$$

As in Case 1, the variation of the shear modulus considered here has the effect of reducing the stress intensity factor K. Also, decreasing μ_0 has the effect of increasing the magnitude of $K^{(1)}$.

Case 3. $\mu_1 = k(z - \delta)^2$ if $0 < z \leq \delta$ and $\mu_1 = 0$ if $z > \delta$ (k and δ are positive constants). The results in Cases 1 and 2 may be used to derive

$$K^{(1)} = \frac{k\sigma_0 a^2}{2\mu_0\sqrt{2a}}(-\frac{a}{24} + |1 - \lambda|\frac{\delta}{\pi}). \tag{52}$$

From (52), it is obvious that $K^{(1)} > 0$ if $a < 24|1 - \lambda|\delta/\pi$, that is, the crack is less stable in the particular inhomogeneous material considered here than in a homogeneous material with shear modulus μ_0 if the crack length a is shorter than a critical value which depends on δ. Note that if δ is very large it is more likely for the crack to be less stable.

References

[1] W. T. Ang and D. L. Clements, On some crack problems for inhomogeneous elastic materials, International Journal of Solids Structures 23 (1987) 1089-1104.

[2] A. Erdélyi *et al,* Tables of Integral Transforms, Vol. 2, 1954, McGraw-Hill.

[3] M. Abramowitz and A. Stegun, Handbooks of Mathematical Functions, 1970, Dover.