To appear in Engineering Analysis with Boundary Elements. When the article is in press on line, you may check for further details directly by entering the following URL into your browser:

http://dx.doi.org/10.1016/j.enganabound.2006.03.005/

Non-steady state heat conduction across an imperfect interface: a dual-reciprocity boundary element approach

Whye-Teong Ang
Division of Engineering Mechanics,
School of Mechanical and Aerospace Engineering,
Nanyang Technological University,
50 Nanyang Avenue, Singapore 639790.
E-mail: mwtang@ntu.edu.sg
http://www.ntu.edu.sg/home/mwtang/

Abstract

The two-dimensional problem of determining the time-dependent temperature in a bimaterial with a homogeneously imperfect interface is considered. A temperature jump which is proportional to the thermal heat flux is assumed across the imperfect interface. Through the use of the corresponding steady-state Green's function for the imperfect interface, a dual-reciprocity boundary element method is derived for the numerical solution of the problem under consideration. To assess the validity and accuracy of the proposed method of solution, some specific problems are solved.

Keywords: Non-steady state heat conduction; Imperfect interface; Bimaterial; Dual-reciprocity boundary element method; Green's function

1 Introduction

Nowadays, multi-layered media comprising two or more dissimilar materials play an important role in modern technology and engineering. In many studies, the layers are often assumed to be perfectly joined or bonded to one another along their common boundaries (see e.g. Ang [1], Berger and Karageorghis [2], Clements [3] and Lee and Kim [4]).

A perfect bond is only an idealisation, however. In reality, microscopic imperfections or gaps are bound to be present along the interfaces of the materials. Thus, in recent years, there is a growing interest among researchers in the investigation of microscopically imperfect interfaces in layered and composite materials (see e.g. Benveniste and Miloh [5], Fan and Sze [6], Benveniste [7], Torquato and Rintoul [8] and other references therein).

For heat conduction in solids, a macroscopic model for a microscopically imperfect interface allows for a temperature jump which is proportional in magnitude to the thermal heat flux at the interface. A special Green's function satisfying the conditions on a straight homogeneously imperfect interface may be obtained in order to derive a boundary element method for solving two-dimensional steady-state heat conduction problems in bimaterials (Ang *et al.* [9]). An alternative method of solution proposed by Ang and Fan [14] employs the corresponding Green's function for the perfect interface and formulate the imperfect interfacial conditions using a hypersingular boundary integral equation.

The present paper considers the two-dimensional problem of determining the time-dependent temperature field in a bimaterial with a straight homogeneously imperfect interface. With the use of the corresponding steady-state Green's function for the straight imperfect interface (as given in Ang *et al.* [9]), an integro-differential formulation is obtained for the problem under consideration. The formulation contains a domain integral in addition to the usual line integrals over the boundary of the bimaterial. To avoid having to discretise the domain into elements, the domain integral is treated by using

the dual-reciprocity method. The problem under consideration is eventually reduced approximately to a system of linear algebraic equations to be solved at consecutive time levels which are separated by small time steps.

With the use of the special Green's function, the imperfect interfacial conditions are automatically satisfied and no integral containing the unknown interfacial temperature and heat flux is present in the integro-differential formulation. Only the conditions on the exterior boundary of the bimaterials have to be taken into consideration in setting up the system of linear algebraic equations at each time level. Thus, the number of unknowns in the system is less than that from the corresponding approach which does not use the special Green's function. Furthermore, with the Green's function, the temperature field in the bimaterial may be calculated with a better accuracy, particularly at points near the interface. The advantages of using special Green's functions in boundary element formulations are well known in the literature (see e.g. Berger [10], Clements [11], Clements and Haselgrove [12] and Cruse *et al.* [13]).

The dual-reciprocity boundary element method was originally introduced by Brebbia and Nardini [15] and Partridge and Brebbia [16] for the numerical solution of dynamic problems in solid mechanics. The method has now been successfully applied to solve a wide range of problems in engineering. For some examples of those problems, one may refer to the papers by Zhu *et al.* [17], Profit *et al.* [18], Ang [19], Ang *et al.* [20] and other relevant references therein.

2 The imperfect interface problem

Referring to an $0xyz$ Cartesian co-ordinate system, consider a body comprising two homogeneous materials with possibly different physical and thermal properties. The geometry of the body is independent of the z co-ordinate. On the $0xy$ plane, the interface separating the two materials is the straight line segment Γ which lies on part of the x-axis, while the exterior boundary

of the body is the simple closed curve C. The curve C consists of two parts: C^+ which lies above the x-axis, and C^- below the axis. A sketch of the geometry is given in Figure 1. The regions enclosed by $C^+ \cup \Gamma$ and $C^- \cup \Gamma$ are denoted by R^+ and R^- respectively.

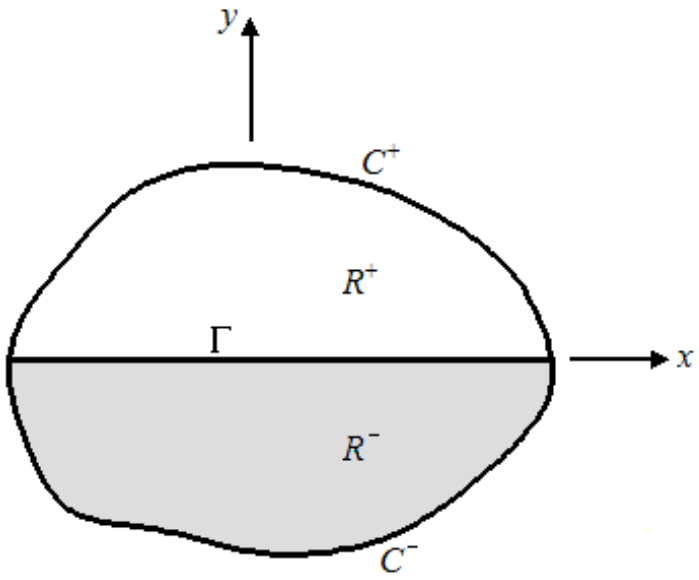

Figure 1. A sketch of the geometry of the problem.

If the temperature in the body is independent of z and given by $T(x, y, t)$, then together with the classical Fourier's law of heat conduction in thermally isotropic solids, the energy equation gives rise to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\rho^{\pm} c^{\pm}}{\kappa^{\pm}} \frac{\partial T}{\partial t} \text{ in } R^{\pm}, \tag{1}$$

where κ^+ and κ^- are the constant thermal conductivities of the materials in R^+ and R^- respectively, ρ^+ and ρ^- are the constant densities of the materials in R^+ and R^- respectively, and c^+ and c^- are the constant specific heat capacities of the materials in R^+ and R^- respectively.

The bond between the materials in R^+ and R^- at the interface Γ is microscopically damaged. A macroscopic model for the heat conduction across

the imperfect interface is given by

$$\kappa^+ \left.\frac{\partial T}{\partial y}\right|_{y=0^+} = \kappa^- \left.\frac{\partial T}{\partial y}\right|_{y=0^-} = \lambda \Delta T(x,t) \quad \text{for } a < x < b \text{ and } t > 0, \tag{2}$$

where λ is a positive coefficient and $\Delta T(x,t) = T(x,0^+,t) - T(x,0^-,t)$ is the temperature jump across the interface Γ, if Γ is given by the line segment $a < x < b$, $y = 0$.

In the present paper, the interface is assumed to be homogeneously imperfect. Thus, λ is taken to be a given positive constant.

The problem then is to determine the temperature in the body by solving (1) subject to the interfacial condition (2), the initial-boundary conditions

$$\begin{aligned} T(x,y,0) = g_0(x,y) \quad & \text{for } (x,y) \in R^+ \cup R^-, \\ \left.\begin{aligned} T(x,y,t) = g_1(x,y) \quad & \text{if } (x,y) \in D_1 \\ P(x,y,t) = g_2(x,y) \quad & \text{if } (x,y) \in D_2 \\ \alpha(x,y)T(x,y,t) + \beta(x,y)P(x,y,t) & \\ = g_3(x,y) \quad & \text{if } (x,y) \in D_3 \end{aligned}\right\} & \text{for } t \geq 0, \end{aligned} \tag{3}$$

where D_1, D_2 and D_3 are non-intersecting curves such that $D_1 \cup D_2 \cup D_3 = C$, g_0, g_1, g_2 and g_3 are given functions of x and y, α and β are given functions such that α and β are not zero at all points on D_3 and $P(x,y) = -\kappa(x,y)[n_x \partial T/\partial x + n_y \partial T/\partial y])$ is the heat flux in the direction of the unit outward normal vector $[n_x(x,y), n_y(x,y)]$ to C at (x,y). Note that $\kappa(x,y) = \kappa^\pm$ for $(x,y) \in R^\pm$.

3 Integro-differential formulation

Applying the reciprocal theorem (in Clements [21]) for the two-dimensional Laplace's equation to (1) and (2), one may derive the integro-differential

equation

$$\gamma(\xi,\eta)T(\xi,\eta,t)$$
$$= \iint\limits_{R^+\cup R^-} \Phi(x,y,\xi,\eta)\rho(x,y)c(x,y)\frac{\partial}{\partial t}T(x,y,t)dxdy$$
$$+ \int\limits_C [T(x,y,t)\kappa(x,y)\frac{\partial}{\partial n}\Phi(x,y,\xi,\eta) + \Phi(x,y,\xi,\eta)P(x,y,t)]ds(x,y), \quad (4)$$

where $\gamma(\xi,\eta) = 1$ if (x,y) lies in the interior of the region enclosed by $C = C^+ \cup C^-$, $\gamma(\xi,\eta) = 1/2$ if (x,y) lies on a smooth part of C, $\rho(x,y)c(x,y) = \rho^\pm c^\pm$ for $(x,y) \in R^\pm$ and $\Phi(x,y,\xi,\eta)$ is the special Green's function in Ang *et al.* [9] as given by

$$\Phi(x,y,\xi,\eta) = \frac{1}{2\pi\kappa^\pm}\,\mathrm{Re}\{\ln([x-\xi]+i[y-\eta])\} + \Phi^\pm \quad \text{for } (x,y) \in R^\pm, \quad (5)$$

where $i = \sqrt{-1}$ and

$$\Phi^+(x,y,\xi,\eta) = -\frac{\kappa^- - \kappa^+}{2\pi\kappa^+(\kappa^- + \kappa^+)}\,\mathrm{Re}\{H(-\eta)\ln([x-\xi]+i[y-\eta]) + H(\eta)\ln([x-\xi]+i[y+\eta])\}$$
$$+ \frac{1}{2\pi}\,\mathrm{Re}\{\int\limits_0^\infty G^+(\tau,\xi,\eta)\exp(i\tau[x+iy])d\tau\}, \quad (6)$$

$$\Phi^-(x,y,\xi,\eta) = \frac{\kappa^- - \kappa^+}{2\pi\kappa^-(\kappa^- + \kappa^+)}\,\mathrm{Re}\{H(-\eta)\ln([x-\xi]+i[-y-\eta]) + H(\eta)\ln([x-\xi]-i[y-\eta])\}$$
$$+ \frac{1}{2\pi}\,\mathrm{Re}\{\int\limits_0^\infty G^-(\tau,\xi,\eta)\exp(-i\tau[x+iy])d\tau\}, \quad (7)$$

$$(\lambda[1+\frac{\kappa^+}{\kappa^-}]+\kappa^+\tau)G^+(\tau,\xi,\eta) = -\frac{\kappa^-}{\kappa^+}(\lambda[1+\frac{\kappa^+}{\kappa^-}]+\kappa^+\tau)\overline{G}^-(\tau,\xi,\eta)$$
$$= H(-\eta)[1-\frac{\kappa^- - \kappa^+}{\kappa^- + \kappa^+}]\exp(-i\tau[\xi+i\eta])$$
$$- H(\eta)[1+\frac{\kappa^- - \kappa^+}{\kappa^- + \kappa^+}]\exp(-i\tau[\xi-i\eta]), \quad (8)$$

where H is the unit-step Heaviside function and the overhead bar denotes the complex conjugate of a complex number.

Note that the imperfect interfacial condition (2) is automatically satisfied by (4) together with (5), (6), (7) and (8). The corresponding integro-differential formulation for the perfect interface may be obtained by replacing (8) with $G^+(\tau,\xi,\eta)=0$ and $G^-(\tau,\xi,\eta)=0$ that is, by setting the integrals over the interval $[0,\infty)$ in (6) and (7) to be zero.

4 Dual-reciprocity boundary element method

A dual-reciprocity boundary element method for the numerical solution of the imperfect interface problem can be obtained from (4) as follows.

The curves C^+ and C^- are discretised into N^+ and N^- straight line elements respectively. Denote the elements from C^+ by C_1^+, C_2^+, $\cdots$, $C_{N^+-1}^+$ and $C_{N^+}^+$ and those from C^- by C_1^-, C_2^-, $\cdots$, $C_{N^--1}^-$ and $C_{N^-}^-$.

The midpoint of the element $C_k^\pm$ ($k=1,\,2,\,\cdots,\,N^\pm$) is $(\xi_k^\pm,\eta_k^\pm)$. Select M^+ and M^- well-spaced out collocation points in the interior of R^+ and R^- respectively. The $M^\pm$ points in $R^\pm$ are denoted by $(\xi_m^\pm,\eta_m^\pm)$ for $m=N^\pm+1$, $N^\pm+2$, $\cdots$, $N^\pm+M^\pm$.

To treat the domain integral in (4) using the dual-reciprocity method, the partial derivative $\partial T/\partial t$ is approximated as

$$\begin{aligned}&\frac{\partial}{\partial t}T(x,y,t)\\ \simeq\ &\sum_{k=1}^{N^\pm+M^\pm}\frac{d}{dt}[T_k^\pm(t)]\sum_{j=1}^{N^\pm+M^\pm}\chi_{kj}^\pm\sigma_j^\pm(x,y)\ \text{for}\ (x,y)\in R^\pm,\end{aligned}\tag{9}$$

where $T_k^\pm(t)=T(\xi_k^\pm,\eta_k^\pm,t)$, the local interpolating functions $\sigma_j^\pm(x,y)$ are given by

$$\sigma_j^\pm(x,y)=1+([x-\xi_j^\pm]^2+[y-\eta_j^\pm]^2)+([x-\xi_j^\pm]^2+[y-\eta_j^\pm]^2)^{3/2},\tag{10}$$

and the coefficients $\chi_{kj}^{\pm}$ are defined by

$$\sum_{k=1}^{N^{\pm}+M^{\pm}} \sigma_j^{\pm}(\xi_k^{\pm}, \eta_k^{\pm})\chi_{kp}^{\pm} = \delta_{jp}, \tag{11}$$

where δ_{jp} is the Kronecker-delta. The local interpolating functions in (10) are those introduced by Zhang and Zhu [22].

With (9), the domain integral in (4) can now be approximated as

$$\begin{aligned} &\iint\limits_{R^+\cup R^-} \Phi(x,y,\xi,\eta)\rho(x,y)c(x,y)\frac{\partial}{\partial t}T(x,y,t)dxdy \\ \simeq &\sum_{k=1}^{N^+ + M^+} \frac{\rho^+ c^+}{\kappa^+}\Psi_k^+(\xi,\eta)\frac{d}{dt}[T_k^+(t)] + \sum_{k=1}^{N^- + M^-} \frac{\rho^- c^-}{\kappa^-}\Psi_k^-(\xi,\eta)\frac{d}{dt}[T_k^-(t)], \end{aligned} \tag{12}$$

where

$$\begin{aligned} \Psi_k^{\pm}(\xi,\eta) = &\sum_{j=1}^{N^{\pm}+M^{\pm}} \chi_{kj}^{\pm}[\gamma^{\pm}(\xi,\eta)\theta_j^{\pm}(\xi,\eta) \\ &+ \int\limits_{C^{\pm}\cup\Gamma} \kappa^{\pm}\Phi(x,y,\xi,\eta)\frac{\partial}{\partial n}\theta_j^{\pm}(x,y)ds(x,y) \\ &- \int\limits_{C^{\pm}\cup\Gamma} \kappa^{\pm}\theta_j^{\pm}(x,y)\frac{\partial}{\partial n}\Phi(x,y,\xi,\eta)ds(x,y)], \end{aligned}$$

$$\gamma^+(\xi,\eta) = \begin{cases} 1 & \text{if } (\xi,\eta) \in R^+ \\ 1/2 & \text{if } (\xi,\eta) \in C^+ \text{ (on smooth part)} \\ 0 & \text{if } (\xi,\eta) \in R^- \cup C^- \end{cases}$$

$$\gamma^-(\xi,\eta) = \begin{cases} 1 & \text{if } (\xi,\eta) \in R^- \\ 1/2 & \text{if } (\xi,\eta) \in C^- \text{ (on smooth part)} \\ 0 & \text{if } (\xi,\eta) \in R^+ \cup C^+ \end{cases}$$

$$\theta_j^{\pm}(x,y) = \frac{1}{4}([x-\xi_j^{\pm}]^2 + [y-\eta_j^{\pm}]^2) + \frac{1}{16}([x-\xi_j^{\pm}]^2 + [y-\eta_j^{\pm}]^2)^2 + \frac{1}{25}([x-\xi_j^{\pm}]^2 + [y-\eta_j^{\pm}]^2)^{5/2}. \tag{13}$$

Note that $C^{\pm} \cup \Gamma$ denotes the closed curve enclosing $R^{\pm}$ and the unit normal vector $[n_x, n_y]$ on $C^{\pm} \cup \Gamma$ points out of $R^{\pm}$.

If one takes (ξ, η) in (4) to be given by $(\xi_m^{\pm}, \eta_m^{\pm})$ for $m = 1, 2, \cdots, N^{\pm} + M^{\pm}$, and if one assumes that $T(x,y,t) \simeq T_k^{\pm}(t)$ and $P(x,y,t) \simeq P_k^{\pm}$ for $(x,y) \in C_k^{\pm}$, after using (12), one obtains

$$\begin{aligned}
&\gamma(\xi_m^+, \eta_m^+) T_m^+(t) \\
&= \sum_{k=1}^{N^+ + M^+} \frac{\rho^+ c^+}{\kappa^+} \Psi_k^+(\xi_m^+, \eta_m^+) \frac{d}{dt}[T_k^+(t)] + \sum_{k=1}^{N^- + M^-} \frac{\rho^- c^-}{\kappa^-} \Psi_k^-(\xi_m^+, \eta_m^+) \frac{d}{dt}[T_k^-(t)] \\
&+ \sum_{k=1}^{N^+} \int_{C_k^+} [\kappa^+ \frac{\partial}{\partial n} \Phi(x,y,\xi_m^+, \eta_m^+) T_k^+(t) + \Phi(x,y,\xi_m^+, \eta_m^+) P_k^+(t)] ds(x,y) \\
&+ \sum_{k=1}^{N^-} \int_{C_k^-} [\kappa^- \frac{\partial}{\partial n} \Phi(x,y,\xi_m^+, \eta_m^+) T_k^-(t) + \Phi(x,y,\xi_m^+, \eta_m^+) P_k^-(t)] ds(x,y) \\
&\qquad\qquad \text{for } m = 1, 2, \cdots, N^+ + M^+,
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\gamma(\xi_m^-, \eta_m^-) T_m^-(t) \\
&= \sum_{k=1}^{N^+ + M^+} \frac{\rho^+ c^+}{\kappa^+} \Psi_k^+(\xi_m^-, \eta_m^-) \frac{d}{dt}[T_k^+(t)] + \sum_{k=1}^{N^- + M^-} \frac{\rho^- c^-}{\kappa^-} \Psi_k^-(\xi_m^-, \eta_m^-) \frac{d}{dt}[T_k^-(t)] \\
&+ \sum_{k=1}^{N^+} \int_{C_k^+} [\kappa^+ \frac{\partial}{\partial n} \Phi(x,y,\xi_m^-, \eta_m^-) T_k^+(t) + \Phi(x,y,\xi_m^-, \eta_m^-) P_k^+(t)] ds(x,y) \\
&+ \sum_{k=1}^{N^-} \int_{C_k^-} [\kappa^- \frac{\partial}{\partial n} \Phi(x,y,\xi_m^-, \eta_m^-) T_k^-(t) + \Phi(x,y,\xi_m^-, \eta_m^-) P_k^-(t)] ds(x,y) \\
&\qquad\qquad \text{for } m = 1, 2, \cdots, N^- + M^-.
\end{aligned} \tag{15}$$

If $T_k^{\pm}(t)$ and its first order derivative with respect to t are approximated using

$$\begin{aligned} T_k^{\pm}(t) &\simeq \frac{1}{2}[T_k^{\pm}(t-\frac{1}{2}\Delta t)+T_k^{\pm}(t+\frac{1}{2}\Delta t)], \\ \frac{d}{dt}[T_k^{\pm}(t)] &\simeq \frac{T_k^{\pm}(t+\frac{1}{2}\Delta t)-T_k^{\pm}(t-\frac{1}{2}\Delta t)}{\Delta t}, \end{aligned} \tag{16}$$

where Δt is a small positive number, then (14) and (15) can be approximately rewritten as

$$\begin{aligned} &\frac{1}{2}\gamma(\xi_m^+,\eta_m^+)[T_m^+(t-\frac{1}{2}\Delta t)+T_m^+(t+\frac{1}{2}\Delta t)] \\ &= \sum_{k=1}^{N^++M^+}\frac{\rho^+c^+}{\kappa^+\Delta t}\Psi_k^+(\xi_m^+,\eta_m^+)[T_k^+(t+\frac{1}{2}\Delta t)-T_k^+(t-\frac{1}{2}\Delta t)] \\ &+ \sum_{k=1}^{N^-+M^-}\frac{\rho^-c^-}{\kappa^-\Delta t}\Psi_k^-(\xi_m^+,\eta_m^+)[T_k^-(t+\frac{1}{2}\Delta t)-T_k^-(t-\frac{1}{2}\Delta t)] \\ &+ \sum_{k=1}^{N^+}\{\frac{1}{2}\kappa^+[T_k^+(t-\frac{1}{2}\Delta t)+T_k^+(t+\frac{1}{2}\Delta t)]\int\limits_{C_k^+}\frac{\partial}{\partial n}\Phi(x,y,\xi_m^+,\eta_m^+)ds(x,y) \\ &\qquad + P_k^+(t)\int\limits_{C_k^+}\Phi(x,y,\xi_m^+,\eta_m^+)ds(x,y)\} \\ &+ \sum_{k=1}^{N^-}\{\frac{1}{2}\kappa^-[T_k^-(t-\frac{1}{2}\Delta t)+T_k^-(t+\frac{1}{2}\Delta t)]\int\limits_{C_k^-}\frac{\partial}{\partial n}\Phi(x,y,\xi_m^+,\eta_m^+)ds(x,y) \\ &\qquad + P_k^-(t)\int\limits_{C_k^-}\Phi(x,y,\xi_m^+,\eta_m^+)ds(x,y)\} \\ &\qquad\qquad \text{for } m=1,2,\cdots,N^++M^+, \end{aligned} \tag{17}$$

and

$$
\begin{aligned}
&\frac{1}{2}\gamma(\xi_m^-,\eta_m^-)[T_m^-(t-\frac{1}{2}\Delta t)+T_m^-(t+\frac{1}{2}\Delta t)] \\
&\quad=\sum_{k=1}^{N^+ +M^+}\frac{\rho^+c^+}{\kappa^+\Delta t}\Psi_k^+(\xi_m^-,\eta_m^-)[T_k^+(t+\frac{1}{2}\Delta t)-T_k^+(t-\frac{1}{2}\Delta t)] \\
&\quad+\sum_{k=1}^{N^- +M^-}\frac{\rho^-c^-}{\kappa^-\Delta t}\Psi_k^-(\xi_m^-,\eta_m^-)[T_k^-(t+\frac{1}{2}\Delta t)-T_k^-(t-\frac{1}{2}\Delta t)] \\
&\quad+\sum_{k=1}^{N^+}\{\frac{1}{2}\kappa^+[T_k^+(t-\frac{1}{2}\Delta t)+T_k^+(t+\frac{1}{2}\Delta t)]\int_{C_k^+}\frac{\partial}{\partial n}\Phi(x,y,\xi_m^-,\eta_m^-)ds(x,y) \\
&\qquad\qquad+P_k^+(t)\int_{C_k^+}\Phi(x,y,\xi_m^-,\eta_m^-)ds(x,y)\} \\
&\quad+\sum_{k=1}^{N^-}\{\frac{1}{2}\kappa^-[T_k^-(t-\frac{1}{2}\Delta t)+T_k^-(t+\frac{1}{2}\Delta t)]\int_{C_k^-}\frac{\partial}{\partial n}\Phi(x,y,\xi_m^-,\eta_m^-)ds(x,y) \\
&\qquad\qquad+P_k^-(t)\int_{C_k^-}\Phi(x,y,\xi_m^-,\eta_m^-)ds(x,y)\} \\
&\qquad\qquad\qquad\text{for } m=1,2,\cdots,N^-+M^-. \qquad (18)
\end{aligned}
$$

If $T_p^{\pm}(t-\frac{1}{2}\Delta t)$ is assumed known for $p=1,\ 2,\ \cdots,\ N^{\pm}+M^{\pm}$, then (17) and (18) constitute a system of $N^+ + M^+ + N^- + M^-$ linear algebraic equations containing $N^+ + M^+ + N^- + M^-$ unknown functions. The unknown functions are given by:

(a) $T_m^{\pm}(t+\frac{1}{2}\Delta t)$ for $m = N^{\pm}+1,\ N^{\pm}+2,\ \cdots,\ N^{\pm}+M^{\pm}$, that is, the temperature at selected points in R^+ and R^-,

(b) $T_k^{\pm}(t+\frac{1}{2}\Delta t)$ if the heat flux $P(x,y,t)$ is known on $C_k^{\pm}$,

(c) $T_k^{\pm}(t+\frac{1}{2}\Delta t)$ if $\alpha(x,y)T(x,y,t)+\beta(x,y)P(x,y,t)$ is specified on $C_k^{\pm}$ and $P_k^{\pm}(t)$ is replaced by $[g_3(\xi_k^{\pm},\eta_k^{\pm})-\frac{1}{2}\alpha(\xi_k^{\pm},\eta_k^{\pm})(T_k^+(t-\frac{1}{2}\Delta t)+T_k^+(t+\frac{1}{2}\Delta t))]/\beta(\xi_k^{\pm},\eta_k^{\pm})$, where α, β and g_3 are given functions in (3),

(d) $P_k^{\pm}(t)$ if the temperature $T(x, y, t)$ is known on $C_k^{\pm}$.

A time-stepping scheme may be used to solve for the unknowns in (17) and (18) at different time levels as follows. In (17) and (18), let $t = \frac{1}{2}\Delta t$. With $T_p^{\pm}(0)$ known (from the initial condition) for $p = N^{\pm} + 1,\ N^{\pm} + 2,\ \cdots,\ N^{\pm} + M^{\pm}$, the unknowns given by $T_m^{\pm}(\Delta t)$ for $m = N^{\pm} + 1,\ N^{\pm} + 2,\ \cdots,\ N^{\pm} + M^{\pm}$ and by either $T_k^{\pm}(\Delta t)$ or $P_k^{\pm}(\frac{1}{2}\Delta t)$ for $k = 1,\ 2,\ \cdots,\ N^{\pm}$ can then be determined. With $T_p^{\pm}(\Delta t)$ now known for $p = N^{\pm} + 1,\ N^{\pm} + 2,\ \cdots,\ N^{\pm} + M^{\pm}$, one can proceed on to determine the unknowns given by $T_m^{\pm}(2\Delta t)$ for $m = N^{\pm} + 1,\ N^{\pm} + 2,\ \cdots,\ N^{\pm} + M^{\pm}$ and by either $T_k^{\pm}(2\Delta t)$ or $P_k^{\pm}(\frac{3}{2}\Delta t)$ for $k = 1,\ 2,\ \cdots,\ N^{\pm}$ by letting $t = \frac{3}{2}\Delta t$ in (17) and (18). The process can be repeated with $t = \frac{5}{2}\Delta t,\ \frac{7}{2}\Delta t,\ \cdots$, until one reaches the desired time level.

5 Specific problems

Problem 1. We take R^+ to be the region $0 < x < 1,\ 0 < y < 1/2$, and R^- to be $0 < x < 1,\ -1/2 < y < 0$. (Refer to Figure 2.) The materials occupying R^+ and R^- are taken to be such that $k^+ = 4/3$, $k^- = 9/8$, $\rho^+ c^+/\kappa^+ = 1$ and $\rho^- c^-/\kappa^- = 64/81$.

With $\rho^+ c^+/\kappa^+ = 1$ and $\rho^- c^-/\kappa^- = 64/81$, a solution of (1) is given by

$$T(x,y,t) = \begin{cases} [\frac{1}{2}\cos(y) + \frac{3}{8}\sin(y)] \\ \times\ [\exp(-x) + \exp(-t)] \\ +\ [\frac{27}{64}\sin(\frac{8}{9}y) + \frac{1}{2}\cos(\frac{8}{9}y)]\exp(-\frac{8}{9}x) & \text{for } (x,y) \in R^+ \\ \\ [-\frac{1}{2}\cos(\frac{8}{9}y) + \frac{1}{2}\sin(\frac{8}{9}y)] \\ \times\ [\exp(-\frac{8}{9}x) + \exp(-t)] \\ +\ [\frac{4}{9}\sin(y) - \frac{1}{2}\cos(y)]\exp(-x) & \text{for } (x,y) \in R^-. \end{cases} \tag{19}$$

One may easily verify that (19) satisfies the interface condition (2) with $\lambda = 1/2$.

To devise a test problem, we use (19) to generate boundary values of the temperature T on the sides $y = \pm 1/2,\ 0 < x < 1$, the boundary values of the

flux $-\kappa(x,y)\partial[T(x,y,t)]/\partial n$ on $x = 0$, $-1/2 < y < 1/2$ and also on $x = 1$, $-1/2 < y < 1/2$, and the initial temperature (at $t = 0$) throughout the square region. The proposed numerical procedure is then applied to solve (1) subject to the initial-boundary data thus generated and the interface condition (2). If the proposed numerical method really works, it should recover the solution (19) approximately.

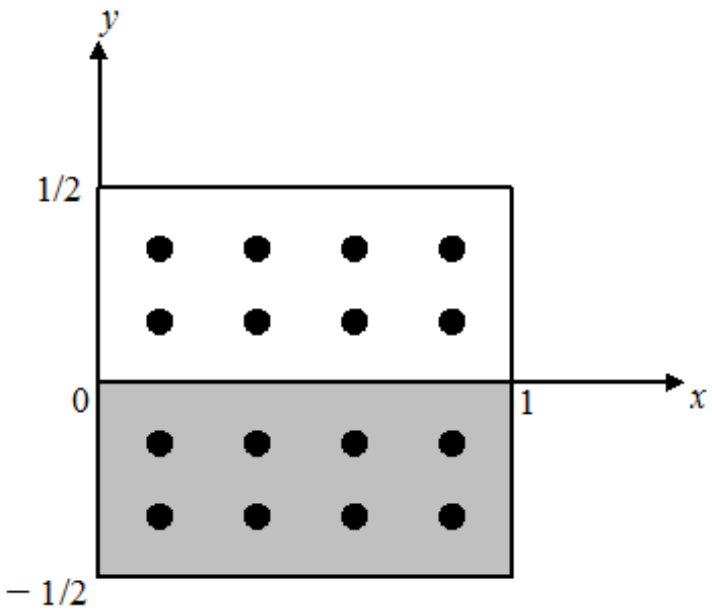

Figure 2. Geometrical sketch of Problem 1 with interior collocation points generated using $P = 2$.

To implement the method, each side of the square region $0 < x < 1$, $-1/2 < y < 1/2$, is divided into J boundary elements of equal length. To avoid ambiguity, we require a boundary element to be in either R^+ or R^- but not partly in both the regions. One of the endpoints of the element is allowed to be on Γ. Thus, J is selected to be an even (positive) integer and $N^+ = N^- = 2J$. Furthermore, the $M^+ + M^-$ points in the interior of R^+ and R^- are chosen to be given by $(i/(2P+1), j/[2(P+1)])$ for $i = 1, 2,$

$\cdots$, $2P$ and $j = \pm 1, \pm 2, \cdots, \pm P$, where P is a positive integer. Note that $M^+ = M^- = 2P^2$. Thus, there are $4P^2$ interior collocation points. For example, the 16 interior collocation points generated using $P = 2$ are shown in Figure 2. For calculating numerically the functions $\Psi_k^{\pm}(\xi, \eta)$ in (13), the interface Γ is divided into J straight line segments of equal length.

For the test problem under consideration here, the heat flux is specified on the vertical sides of the bimaterial (hence the temperature is not known a priori on those sides). The absolute errors of the numerical values of the temperature at selected points on the vertical sides, at time $t = 1.0$, as obtained using the dual-reciprocity boundary element method (DRBEM) (with the special Green's function) as described in Section 4 with $J = 10$, $P = 2$ and $\Delta t = 0.25$, are shown in the second column of Table 1.

The test problem is also solved using the conventional dual-reciprocity boundary element method without the use of the special Green's function (that is, the fundamental solution (5) is used with $\Phi^{\pm} = 0$). Without the special Green's function, a separate dual-reciprocity boundary element formulation has to be obtained for each of the regions R^+ and R^-. On the imperfect interface, both the temperature and the heat flux appear as unknown functions to be determined. The formulations are then coupled through the use of the imperfect interfacial condition (2). To ensure a fair comparison of the numerical results between the two dual-reciprocity boundary element approaches, the conventional dual-reciprocity boundary element method is implemented using as much as possible the same details given in Section 4. For example, the same dual-reciprocity procedure for treating the domain integrals, the same time-stepping scheme and the same approximations over the boundary elements are used.

The absolute errors of the temperature at selected points on the vertical sides of the bimaterial, at time $t = 1.0$, as obtained using the conventional dual-reciprocity boundary element method with $J = 10$ (40 boundary elements on the exterior boundary and 10 boundary elements each on the lower and upper parts of the interface), $P = 2$ and $\Delta t = 0.25$, are given in the

third column of Table 1.

Table 1. Absolute errors of the numerical values of the temperature T at selected points on the vertical sides of the bimaterial at time $t = 1.0$.

Point (x, y)	DRBEM with Green's function Absolute error	DRBEM without Green's function Absolute error	Exact
$(1, 0.45)$	7.1×10^{-3}	1.5×10^{-2}	0.708137
$(1, 0.35)$	1.7×10^{-3}	4.2×10^{-3}	0.688965
$(1, 0.25)$	3.5×10^{-4}	2.5×10^{-3}	0.663431
$(1, 0.15)$	1.8×10^{-4}	3.3×10^{-3}	0.631768
$(1, 0.05)$	1.2×10^{-3}	1.3×10^{-2}	0.594268
$(1, -0.05)$	9.2×10^{-4}	1.1×10^{-2}	-0.598298
$(1, -0.15)$	3.2×10^{-4}	2.5×10^{-3}	-0.644126
$(1, -0.25)$	2.5×10^{-4}	2.0×10^{-3}	-0.684435
$(1, -0.35)$	1.5×10^{-3}	3.6×10^{-3}	-0.718881
$(1, -0.45)$	6.0×10^{-3}	1.3×10^{-2}	-0.747172
$(0, 0.45)$	4.2×10^{-3}	1.2×10^{-2}	1.463786
$(0, 0.35)$	1.1×10^{-3}	3.4×10^{-3}	1.423505
$(0, 0.25)$	6.4×10^{-4}	2.2×10^{-3}	1.370270
$(0, 0.15)$	8.6×10^{-4}	2.3×10^{-3}	1.304560
$(0, 0.05)$	1.6×10^{-3}	8.8×10^{-3}	1.226972
$(0, -0.05)$	1.3×10^{-3}	6.6×10^{-3}	-1.235240
$(0, -0.15)$	7.5×10^{-4}	1.9×10^{-3}	-1.329594
$(0, -0.25)$	6.2×10^{-4}	1.8×10^{-3}	-1.412274
$(0, -0.35)$	1.1×10^{-3}	2.8×10^{-3}	-1.482557
$(0, -0.45)$	3.6×10^{-3}	9.6×10^{-3}	-1.539830

In Table 1, the numerical values of the temperature obtained using the dual-reciprocity boundary element method with the special Green's function appear to be more accurate than those calculated using the conventional dual-reciprocity boundary element method. On the side $-1/2 < y < 1/2$, $x = 0$, at the points $(0, 0.05)$ and $(0, -0.05)$ which are very close to the

interface of the material, the absolute errors of the numerical values of the temperature calculated using the conventional method are more than 5 times greater than those computed using the Green's function approach. On the other side $-1/2 < y < 1/2$, $x = 1$, at $(1, 0.05)$ and $(1, -0.05)$ which are also near the interface, the absolute errors of the conventional method are more than 10 times larger than those of the Green's function approach.

The absolute errors of the numerical values of the temperature at the interior collocation points in the upper region of the bimaterial, at $t = 1.0$, as obtained using the dual-reciprocity boundary element methods with and without the Green's function, with $J = 10$, $P = 2$ and $\Delta t = 0.25$, are given in Table 2. The Green's function approach appears to give more accurate values than the conventional boundary element method at the interior collocation points, except at $(0.80, 0.1667)$ (which is likely a 'fluke'). The average of the absolute errors in the second column of Table 2 (DRBEM with Green's function) is 3.5×10^{-4}, while the average of the absolute errors in the third column (DRBEM without Green's function) is 9.5×10^{-4} (close to thrice the average from the second column).

Table 2. Absolute errors of the numerical values of the temperature T at interior collocation points in the upper region of the bimaterial at time $t = 1.0$.

	DRBEM with Green's function	DRBEM without Green's function	
Point (x, y)	Absolute error	Absolute error	Exact
$(0.20, 0.1667)$	8.9×10^{-5}	4.6×10^{-4}	1.125013
$(0.20, 0.3333)$	8.4×10^{-5}	4.7×10^{-4}	1.209684
$(0.40, 0.1667)$	6.6×10^{-4}	2.0×10^{-3}	0.966687
$(0.40, 0.3333)$	4.0×10^{-4}	9.7×10^{-4}	1.039573
$(0.60, 0.1667)$	7.1×10^{-4}	1.9×10^{-3}	0.835664
$(0.60, 0.3333)$	3.4×10^{-4}	8.0×10^{-4}	0.898397
$(0.80, 0.1667)$	2.9×10^{-4}	6.2×10^{-5}	0.727225
$(0.80, 0.3333)$	2.4×10^{-4}	9.4×10^{-4}	0.781725

Problem 2. For a problem with a more complicated geometry, take $R^{\pm}$ to be given by $1/4 < x^2 + y^2 < 1$, $\pm y > 0$. (Refer to Figure 3.) The materials occupying R^+ and R^- are taken to be such that $k^+ = 1$, $k^- = 1/4$ and $\rho^+ c^+/\kappa^+ = \rho^- c^-/\kappa^- = 1$.

The interfacial condition is given by (2) with $\lambda = 1/4$. The initial and boundary conditions are respectively

$$T(x,y,0) = \begin{cases} \frac{1}{2}\cos(y) + \frac{1}{4}\sin(y) + \cos(\frac{1}{2}x) & \text{for } (x,y) \in R^+ \\ \\ -\frac{1}{2}\cos(y) + \sin(y) + \cos(\frac{1}{2}x) & \text{for } (x,y) \in R^- \end{cases} \tag{20}$$

and

$$T(x,y,t) + P(x,y,t) = \begin{cases} R_{\text{inner}}(x,y,t) & \text{for } (x,y) \in C_{\text{inner}} \\ R_{\text{outer}}(x,y,t) & \text{for } (x,y) \in C_{\text{outer}} \end{cases} \tag{21}$$

where C_{inner} and C_{outer} are given by $x^2+y^2 = 1/4$ and $x^2+y^2 = 1$ respectively and

$$R_{\text{inner}}(x,y,t) = \begin{cases} [(\frac{1}{2} + \frac{1}{2}y)\cos(y) + (\frac{1}{4} - y)\sin(y)]\exp(-t) & \\ \quad + [\cos(\frac{1}{2}x) - x\sin(\frac{1}{2}x)]\exp(-\frac{1}{4}t) & \text{for } y > 0 \\ \\ [(-\frac{1}{2} + \frac{1}{2}y)\cos(y) + (1 + \frac{1}{4}y)\sin(y)]\exp(-t) & \\ \quad + [\cos(\frac{1}{2}x) - \frac{1}{4}x\sin(\frac{1}{2}x)]\exp(-\frac{1}{4}t) & \text{for } y < 0 \end{cases} \tag{22}$$

$$R_{\text{outer}}(x,y,t) = \begin{cases} [(\frac{1}{2} - \frac{1}{4}y)\cos(y) + (\frac{1}{4} + \frac{1}{2}y)\sin(y)]\exp(-t) & \\ \quad + [\cos(\frac{1}{2}x) + \frac{1}{2}x\sin(\frac{1}{2}x)]\exp(-\frac{1}{4}t) & \text{for } y > 0 \\ \\ [(-\frac{1}{2} - \frac{1}{4}y)\cos(y) + (1 - \frac{1}{8}y)\sin(y)]\exp(-t) & \\ \quad + [\cos(\frac{1}{2}x) + \frac{1}{8}x\sin(\frac{1}{2}x)]\exp(-\frac{1}{4}t) & \text{for } y < 0. \end{cases} \tag{23}$$

Note that on C_{inner} the unit normal vector $[n_x, n_y]$ is taken to point towards the origin of the Cartesian frame $0xy$.

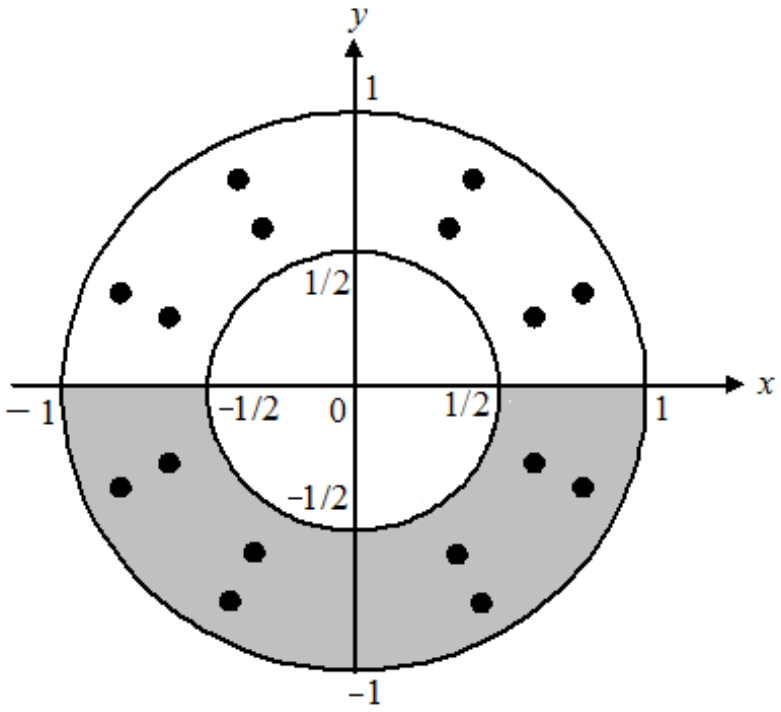

Figure 3. Geometrical sketch of Problem 2 with interior collocation points generated using $P_1 = 4$ and $P_2 = 2$.

It may be verified by direct substitution that the solution of the test problem here is given by

$$T(x,y,t) = \begin{cases} [\frac{1}{2}\cos(y) + \frac{1}{4}\sin(y)]\exp(-t) \\ +\ \cos(\frac{1}{2}x)\exp(-\frac{1}{4}t) & \text{for } (x,y) \in R^+ \\ [-\frac{1}{2}\cos(y) + \sin(y)]\exp(-t) \\ +\ \cos(\frac{1}{2}x)\exp(-\frac{1}{4}t) & \text{for } (x,y) \in R^-. \end{cases} \tag{24}$$

For the present test problem, we may take $C = C_{\text{inner}} \cup C_{\text{outer}}$. Here C is not a simple closed curve as assumed (for convenience) in the general description of the problem in Section 2. However, the method of solution presented

is still applicable by merely discretising C_{inner} and C_{outer} into boundary elements, noting whether the boundary elements lie in the upper or lower regions of the bimaterial and following exactly the procedure described in Section 4.

The circles C_{inner} and C_{outer} are discretised into $6J$ and $12J$ equal length boundary elements such that $N^+ = N^- = 9J$. As in the previous example, for the numerical computation of the functions $\Psi_k^{\pm}(\xi, \eta)$ in (13), the interface Γ is divided into $2J$ straight line segments of equal length. The x and y coordinates of the interior collocation points are chosen to be given respectively by $x = \frac{1}{2}[1 + k/(P_2 + 1)] \cos(\pi[m + \frac{1}{2}]/P_1)$ and $y = \frac{1}{2}[1 + k/(P_2 + 1)] \sin(\pi[m + \frac{1}{2}]/P_1)$ for $m = 0, 1, 2, \cdots, 2P_1 - 1$ and $k = 1, 2, \cdots, P_2$. Thus, there are $2P_1P_2$ interior collocation points.

Table 3. Numerical and exact values of the temperature at selected interior collocation points at time $t = 1.0$.

Point (x, y)	$J = 2$ $\Delta t = 0.50$	$J = 4$ $\Delta t = 0.25$	Exact
$(0.6159, 0.2551)$	0.934421	0.941542	0.943357
$(0.7699, 0.3189)$	0.917405	0.923434	0.925306
$(0.2551, 0.6159)$	0.966400	0.973811	0.975744
$(0.3189, 0.7699)$	0.955593	0.963166	0.965003
$(0.6159, -0.2551)$	0.481295	0.474007	0.471336
$(0.7699, -0.3189)$	0.440700	0.434647	0.431801
$(0.2551, -0.6159)$	0.423296	0.413174	0.409806
$(0.3189, -0.7699)$	0.393844	0.384359	0.380789

For the purpose of obtaining some numerical results, the interior collocation points are generated using $P_1 = 4$ and $P_2 = 2$ as shown in Figure 3. The numerical values of the temperature at selected collocation points at $t = 1$, obtained using 36 boundary elements ($J = 2$) and time-step $\Delta t = 0.50$, are given in the second column of Table 3. The calculation is repeated using 72 boundary elements ($J = 4$) and $\Delta t = 0.25$. The numerical values thus

obtained are given in the third column of Table 3. The numerical values in the third column of Table 3 are significantly more accurate than those in the second column.

6 Summary and discussion

A numerical method is devised for the solution of a two-dimensional time-dependent heat conduction problem involving a bimaterial with an imperfect interface. It is based on an integro-differential formulation in which the conditions on the imperfect interface are satisfied through the use of a suitable Green's function for the corresponding steady-state problem. A dual-reciprocity boundary element method together with a time-stepping scheme is used to reduce the integro-differential equation approximately to a system of linear algebraic equations of the form $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is a known $N \times N$ matrix, $\mathbf{X}$ is an $N \times 1$ matrix containing the unknown parameters in the formulation and $\mathbf{B}$ is a given $N \times 1$ matrix. The system is to be solved at consecutive time levels.

In implementing the method, it is not necessary to discretise the region occupied by the bimaterial but only its exterior boundary. The imperfect interface is also divided into straight line segments but only for the purpose of calculating certain known line integrals which define the functions $\Psi_k^{\pm}(\xi, \eta)$ in (13). Note that the algebraic formulation $\mathbf{AX} = \mathbf{B}$ does not involve any unknown functions on the interface, as the imperfect interfacial conditions are automatically satisfied. In setting up the linear algebraic equations, the calculation of the matrix $\mathbf{A}$ and certain parameters related to the Green's function and the boundary elements takes up the main bulk of the computational time. Fortunately, the matrix $\mathbf{A}$ as well as all those parameters are independent of time and has to be calculated only once.

The proposed dual-reciprocity boundary element method is applied to solve two specific problems with known exact solutions. The numerical values of the temperature obtained at points close to the interface are found to

be more accurate than those computed using the more conventional dual-reciprocity boundary element method which does not make use of the special Green's function.

The Green's function employed here is limited to straight interfaces which are homogeneously imperfect. However, its ability to produce more accurate results, particularly at points near the interface, may be a compensation for this limitation. Furthermore, straight interfaces (both perfect and imperfect ones) are of both applied and theoretical interest in engineering studies and are analysed in numerous research papers, such as Ochiai *et al.* [23], Hasebe *et al.* [24] and Berger *et al.* [25], [26] (just to mention a few references). The steady-state Green's function for a homogeneously imperfect interface which has the shape of a circular arc can be obtained if the analysis in Fan and Wang [27] is used as a guide. The derivation of the Green's function in explicit form may be possible for only homogeneously imperfect interfaces with certain special geometries. In general, for any arbitrarily shaped and inhomogeneously imperfect interfaces, the appropriate Green's functions may have to be constructed numerically, however. The construction of such numerical Green's functions may be the subject of a future research paper. It may also be of interest here to note that numerical Green's functions have been successfully used for solving elasticity problems which involve cracks with general geometries and configurations (Telles *et al.* [28] and Ang and Telles [29]).

The numerical procedure described in the present paper employs only constant elements. For a more accurate numerical solution, higher order elements such as the discontinuous linear elements (e.g. Parí́s and Cañas [30] and Ang [19]) may be used. The setting up of the linear algebraic equations may be more algebraically more involved and computationally more time consuming for higher order elements.

Acknowledgement. The author would like to thank two anonymous reviewers for their comments and suggestions which gave rise to the opportunity

to revise and improve this paper.

References

[1] Ang WT. A crack in an anisotropic layered material under the action of impact loading. ASME Journal of Applied Mechanics 1988; **55**: 122-125.

[2] Berger JR. Karageorghis A. The method of fundamental solutions for heat conduction in layered materials. International Journal for Numerical Methods in Engineering 1999; **45**: 1681-1694.

[3] Clements DL. A crack in an anisotropic layered material. Rozprawy Inżynierskie 1979; **27**: 171-180.

[4] Lee BC. Kim ES. A simple and efficient method of analyzing mechanical behaviors of multi-layered orthotropic plates in rectangular shape. Journal of Micromechanics and Microengineering 1999; **9**: 385-393.

[5] Benveniste Y. Miloh T. Imperfect soft and stiff interfaces in two-dimensional elasticity. Mechanics of Materials 2001; **33**: 309-323.

[6] Fan H. Sze KY. On the decay of end effects in conduction phenomena: A sandwich strip with imperfect interfaces of low or high conductivity. Mechanics of Materials 2001; **33**: 363-370.

[7] Benveniste Y. On the decay of end effects in conduction phenomena: A sandwich strip with imperfect interfaces of low or high conductivity. Journal of Applied Physics 1999; **86**: 1273-1279.

[8] Torquato S. Rintoul MD. Effect of the interface on the properties of composite materials. Physical Review Letters 1995; **75**: 4067-4070.

[9] Ang WT. Choo KK. Fan H. A Green's function for steady-state two-dimensional isotropic heat conduction across a homogeneously imperfect

interface. Communications in Numerical Methods in Engineering 2004; **20**: 391-399.

[10] Berger JR. Boundary element analysis of anisotropic bimaterials with special Green's functions. Engineering Analysis with Boundary Elements 1994; **14**: 123-131.

[11] Clements DL. Fundamental solutions for second order linear elliptic partial differential equations. Computational Mechanics 1998; **22**: 26-31.

[12] Clements DL, Haselgrove, M. The boundary integral equation method for the solution of problems involving elastic slabs. International Journal for Numerical Methods in Engineering 1985; **21**: 663-670.

[13] Cruse TA, Ewing AP, Wikswo JP. Green's function formulation of Laplace's equation for electromagnetic crack detection. Computational Mechanics 1999; **23**: 420-429.

[14] Ang WT. Fan H. A hypersingular boundary integral formulation for heat conduction across an imperfect interface. In: Gallego R, Aliabadi MH, editors. Advances in Boundary Element Techniques IV. London: University of London; 2003.

[15] Brebbia CA. Nardini D. Dynamic analysis in solid mechanics by an alternative boundary element procedure. International Journal of Soil Dynamics and Earthquake Engineering 1983; **2**: 228-233.

[16] Partridge PW, Brebbia CA. The dual reciprocity boundary element method for the Helmholtz equation. In: Brebbia CA, Choudouet-Miranda A, editors. Proceedings of the International Boundary Elements Symposium. Berlin: Computational Mechanics Publications/Springer; 1990: pp. 543-555.

[17] Zhu SP. Satravaha P. Lu XP. Solving linear diffusion equations with the dual reciprocity method in Laplace space. Engineering Analysis with Boundary Elements 1994; **13**: 1-10.

[18] Profit A. Chen K. Amini S. On a DRBEM model for time-dependent PDEs in semi-conductor simulation. Technical Report CMS-99-1; Department of Computer and Mathematical Sciences; The University of Salford; 1999.

[19] Ang WT. A time-stepping dual-reciprocity boundary element method for anisotropic heat diffusion subject to specification of energy. Applied Mathematics and Computation 2005; **162**: 661-678.

[20] Ang WT. Ang KC. Dehghan M. The determination of a control parameter in a two-dimensional diffusion equation using a dual-reciprocity boundary element method. International Journal of Computer Mathematics 2003; **80**: 65-74.

[21] Clements DL. Boundary Value Problems Governed by Second Order Elliptic Systems. London: Pitman; 1981.

[22] Zhang Y. Zhu S. On the choice of interpolation functions used in the dual-reciprocity boundary-element method. Engineering Analysis with Boundary Elements 1994; **13**: 387-396.

[23] Ochiai S. Tanaka H. Kimura S. Tanaka M. Hojo M. Okuda K. Modeling of residual stress-induced stress-strain behavior of unidirectional brittle fiber/brittle matrix composite with weak interface. Composites Science and Technology 2003; **63**: 1027-1040.

[24] Hasebe N. Kato S. Ueda A. Nakamura T. Stress analysis of bimaterial strip with debondings under tension. JSME International Journal Series A-Mechanics and Material Engineering 1996; **39**: 157-165.

[25] Berger JR. Tewary VK. Elastic Green's function for a damaged interface in anisotropic bimaterials. Journal of Materials Research 1995; **11**: 537-544.

[26] Berger JR. Skilowitz J. Tewary VK. Green's function for steady state heat conduction in a bimaterial composite solid. Computational Mechanics 2000; **25**: 627-634.

[27] Fan H. Wang GF. Screw dislocation interacting with an imperfect interface. Mechanics of Materials 2003; **35**: 943-953.

[28] Telles JCF. Castor GS. Guimarães S. A numerical Green's function approach for boundary elements applied to fracture mechanics. International Journal for Numerical Methods in Engineering 1995; **38**: 3259-3274.

[29] Ang WT. Telles JCF. A numerical Green's function for multiple cracks in anisotropic bodies. Journal of Engineering Mathematics 2004; **49**: 197-207.

[30] París F. Cañas J. Boundary Element Method : Fundamentals and Applications. Oxford: Oxford University Press; 1997.