A HYPERSINGULAR BOUNDARY INTEGRAL METHOD FOR QUASI-STATIC ANTIPLANE DEFORMATIONS OF AN ELASTIC BIMATERIAL WITH AN IMPERFECT AND VISCO-ELASTIC INTERFACE

Whye-Teong Ang[1] and **Hui Fan**
Division of Engineering Mechanics
School of Mechanical and Production Engineering
Nanyang Technological University
50 Nanyang Avenue, Singapore 639798
Republic of Singapore

Abstract. A hypersingular boundary integral method is proposed for the numerical solution of a quasi-static antiplane problem involving an elastic bimaterial with an imperfect interface. The interface exhibits visco-elastic behaviors and is modeled as comprising linear springs and dashpots. The proposed method is applied to solve a specific test problem.

Keywords: Quasi-static antiplane deformation, bimaterial, visco-elastic interface, hypersingular boundary integral method.

Note. This has been the original draft of the paper published in *Engineering Computations* **21** (2004) 529-539.

[1]Corresponding author. Fax: (65) 6791-1859. E-mail: mwtang@ntu.edu.sg

ABOUT THE AUTHORS

W. T. Ang graduated with a doctorate degree in Applied Mathematics from the University of Adelaide in 1987. Presently, he is an Associate Professor in the Division of Engineering Mechanics, School of Mechanical and Production Engineering at Nanyang Technological University in Singapore. One of his main research interests is in scientific and engineering computation. To date, he has published over 50 papers in international journals on areas which include boundary element methods, stress analysis around cracks, thermo-elasticity and diffusion.

H. Fan graduated with a doctorate degree in Mechanical Engineering from the University of Illinois at Chicago in 1989. Presently, he is an Associate Professor in the Division of Engineering Mechanics, School of Mechanical and Production Engineering at Nanyang Technological University in Singapore. To date, he has published over 40 papers in international journals on areas which include micromechanics, fracture mechanics and physics of microsystems.

A HYPERSINGULAR BOUNDARY INTEGRAL METHOD FOR QUASI-STATIC ANTIPLANE DEFORMATIONS OF AN ELASTIC BIMATERIAL WITH AN IMPERFECT AND VISCO-ELASTIC INTERFACE

Abstract. A hypersingular boundary integral method is proposed for the numerical solution of a quasi-static antiplane problem involving an elastic bimaterial with an imperfect interface. The interface exhibits visco-elastic behaviors and is modeled as comprising linear springs and dashpots. The proposed method is applied to solve a specific test problem.

1 Introduction

Composites which are made up of two or more dissimilar materials play an important role in modern technology. For example, media comprising a large number of very fine layers are employed in optical recording, and synthetic materials, such as plywood and fabric laminates, are widely used in the design and construction of modern aircrafts.

In many studies of those composites, the dissimilar materials are assumed to be perfectly joined or bonded to one another along their common boundaries (see e.g. Ang [1], Berger and Karageorghis [3], Clements [5] and Lee and Kim [9]). Nevertheless, such a perfect bond between the materials is only an idealization, as microscopic imperfections are bound to be present along the interfaces of the materials. Thus, in recent years, there is a growing interest among many researchers in the analyses and the modeling of microscopically imperfect interfaces (see e.g. Benveniste and Miloh [2], Fan and Sze [7], Fan and Wang [8] and other references therein).

The present paper is concerned with the numerical solution of a quasi-static antiplane problem involving an elastic bimaterial with a plane interface that is microscopically imperfect. As in Fan and Wang [8], the interface exhibits visco-elastic behaviors and is modeled as comprising linear springs and dashpots. The problem has practical applications in engineering. For example, epoxy which has a melting temperature in the range of about 340° to 380° Kelvin, when used as an adhesive to join together a pair of metals with a high melting temperature (e.g. aluminium with a melting temperature of 1000° Kelvin or thereabout), may form an imperfect and visco-elastic interface at room temperature (i.e. at around 300° Kelvin).

A boundary integral solution is derived for the problem under consideration. The path of the integration involved is over the interface and the exterior boundary of the bimaterial. The Green's function for the corresponding perfect interface is used in the boundary integral solution. With it, the relevant component of the antiplane shear stress is automatically continuous on the interface, as required. Thus, the only unknown function on the imperfect and visco-elastic interface is the displacement jump. At each and every point on the exterior boundary of the bimaterial, either the displacement or the traction is known. A differentiated form of the boundary integral solution is used to deal with the condition on the imperfect and visco-elastic interface. This gives rise to hypersingular boundary integral equations in the formulation of the problem. A simple numerical procedure for solving those equations is described and it is applied to solve a specific test problem.

2 The problem

With reference to an $0x_1x_2x_3$ Cartesian co-ordinate system, consider an isotropic body which is made up of two homogeneous materials having possibly different mechanical properties. The geometry of the body does not vary along the x_3-direction. On the $0x_1x_2$ plane, the materials are joined together along the straight line segment Γ which lies on part of the x_1-axis between the points $(a, 0)$ and $(b, 0)$ (where a and b are given real numbers such that $a < b$) and the exterior boundary of the body is the simple closed curve C. The curve C consists of two parts, namely C^+ which lies above the x_1-axis and C^- below the axis. A sketch of the geometry is given in Figure 1. The regions enclosed by $C^+ \cup \Gamma$ and $C^- \cup \Gamma$ are denoted by R^+ and R^- respectively.

The body is subject to an antiplane deformation such that the only non-zero component of the displacement is the one along the $0x_3$ direction and is

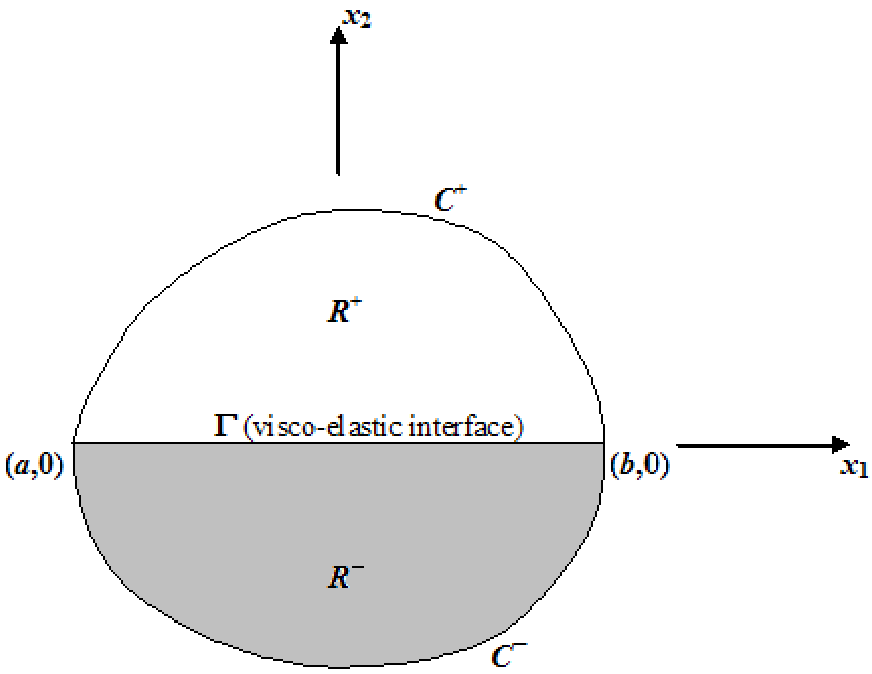

Figure 1: A finite elastic bimaterial with a viscoelastic interface.

given by the function $w(x_1, x_2, t)$, where t denotes time. Assuming that the materials in R^+ and R^- are linearly elastic, we find that the non-vanishing components of the Cartesian stress tensor are then given by

$$\sigma_{i3} = \sigma_{3i} = G^{\pm}\frac{\partial w}{\partial x_i} \ \text{ for } (x_1, x_2) \in R^{\pm}, \tag{1}$$

where G^+ and G^- are the shear moduli of the materials in R^+ and R^- respectively.

The bond between the materials in R^+ and R^- at the interface Γ is microscopically damaged and exhibits visco-elastic behaviors. One of the macroscopic models for such an imperfect and visco-elastic interface is given by (Fan and Wang [8])

$$\begin{aligned}\sigma_{23}(x_1,0^+,t) &= \sigma_{23}(x_1,0^-,t)\\ &= kr(x_1,t)+\eta\frac{\partial}{\partial t}r(x_1,t)\\ &\quad \text{for } x_1\in(a,b) \text{ and } t>0.\end{aligned} \tag{2}$$

where k and η are given coefficients and $r(x_1,t)=w(x_1,0^+,t)-w(x_1,0^-,t)$ is the jump in the displacement across the imperfect interface. If the interface is homogeneous then k and η are constants.

According to (2), the imperfect and visco-elastic interface is modeled as a distribution of linear springs and dashpots that are connected in parallel. Other more complicated visco-elastic models (such as those given in Shames and Cozzarelli [10]) may be used to describe the interface. However, for the purpose of illustrating how a hypersingular boundary integral method may be derived for solving problems involving imperfect and visco-elastic interfaces, we consider only the relatively simple visco-elastic model given by (2).

Since (2) contains a first order time derivative of $r(x_1,t)$, an initial condition is required. The initial condition is taken to be given by

$$r(x_1,0)=v(x_1) \quad \text{for } x_1\in(a,b), \tag{3}$$

where $v(x_1)$ is a suitably given function.

The antiplane deformation of the materials in R^+ and R^- is assumed to be in a quasi-static state, so that $w(x_1,x_2,t)$ is governed by the two-dimensional Laplace equation

$$\frac{\partial^2 w}{\partial x_k \partial x_k}=0 \text{ in } R^{\pm}. \tag{4}$$

Note that throughout the present paper the Einsteinian convention of summing over a repeated index is adopted for latin subscripts running from 1 to 2.

At each and every point on the exterior boundary $C=C^+\cup C^-$, either the displacement w or the traction $p=\sigma_{k3}n_k$ (but not both) is known. Note that $[n_1,n_2]$ is the unit normal vector to C which points away from the region R.

The problem of interest is then to solve (4) for w subject to the known boundary data on C and to the interface condition as given by (2) and (3).

3 Integral formulation

For a general point (ξ_1, ξ_2) such that $\xi_2 \neq 0$, the analysis in Clements [6] may be applied to (4) to derive the integral equations

$$\begin{aligned}\gamma^+(\xi_1,\xi_2)w(\xi_1,\xi_2,t) = &\int\limits_{C^+}[w(x_1,x_2,t)G^+n_k(x_1,x_2)\frac{\partial}{\partial x_k}\Phi(x_1,x_2,\xi_1,\xi_2)\\ &- \Phi(x_1,x_2,\xi_1,\xi_2)p(x_1,x_2,t)]ds(x_1,x_2)\\ &+\int_a^b[-w(x_1,0^+,t)G^+\left.\frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\right|_{x_2=0^+}\\ &+\Phi(x_1,0^+,\xi_1,\xi_2)\sigma_{23}(x_1,0^+,t)]dx_1, \qquad (5)\end{aligned}$$

and

$$\begin{aligned}\gamma^-(\xi_1,\xi_2)w(\xi_1,\xi_2,t) = &\int\limits_{C^-}[w(x_1,x_2,t)G^-n_k(x_1,x_2)\frac{\partial}{\partial x_k}\Phi(x_1,x_2,\xi_1,\xi_2)\\ &- \Phi(x_1,x_2,\xi_1,\xi_2)p(x_1,x_2,t)]ds(x_1,x_2)\\ &+\int_a^b[w(x_1,0^-,t)G^-\left.\frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\right|_{x_2=0^-}\\ &-\Phi(x_1,0^-,\xi_1,\xi_2)\sigma_{23}(x_1,0^-,t)]dx_1, \qquad (6)\end{aligned}$$

where $\gamma^+(\xi_1,\xi_2) = 0$ if $(\xi_1,\xi_2) \notin R^+ \cup C^+$, $\gamma^+(\xi_1,\xi_2) = 1$ if $(\xi_1,\xi_2) \in R^+$, $0 < \gamma^+(\xi_1,\xi_2) < 1$ if $(\xi_1,\xi_2) \in C^+$, $\gamma^-(\xi_1,\xi_2) = 0$ if $(\xi_1,\xi_2) \notin R^- \cup C^-$, $\gamma^-(\xi_1,\xi_2) = 1$ if $(\xi_1,\xi_2) \in R^-$, $0 < \gamma^-(\xi_1,\xi_2) < 1$ if $(\xi_1,\xi_2) \in C^-$ and

$$\Phi(x_1,x_2,\xi_1,\xi_2) = \frac{1}{2\pi G^\pm}\,\mathrm{Re}\{\ln(z-c)\} + \Phi^\pm(x_1,x_2,\xi_1,\xi_2) \text{ for } \pm x_2 > 0, \qquad (7)$$

with $z = x_1 + ix_2$, $c = \xi_1 + i\xi_2$, $i = \sqrt{-1}$ and $\Phi^\pm(x_1,x_2,\xi_1,\xi_2)$ being any arbitrary functions satisfying

$$\frac{\partial^2\Phi^\pm}{\partial x_k \partial x_k} = 0 \text{ for } (x_1,x_2) \in R^\pm. \qquad (8)$$

If we choose the functions $\Phi^{+}(x_1, x_2, \xi_1, \xi_2)$ and $\Phi^{-}(x_1, x_2, \xi_1, \xi_2)$ in such a way that (8) is satisfied together with

$$\Phi(x_1, 0^{+}, \xi_1, \xi_2) - \Phi(x_1, 0^{-}, \xi_1, \xi_2) = 0 \text{ for } -\infty < x_1 < \infty, \tag{9}$$

and

$$\begin{aligned} & G^{+} \frac{\partial}{\partial x_2} \Phi(x_1, x_2, \xi_1, \xi_2) \bigg|_{x_2=0^{+}} \\ = \; & G^{-} \frac{\partial}{\partial x_2} \Phi(x_1, x_2, \xi_1, \xi_2) \bigg|_{x_2=0^{-}} \quad \text{for } -\infty < x_1 < \infty, \end{aligned} \tag{10}$$

then the use of (5) and (6) yields (for $\xi_2 \neq 0$)

$$\begin{aligned} \gamma(\xi_1, \xi_2) w(\xi_1, \xi_2, t) = & \oint\limits_{C} [w(x_1, x_2, t) G(x_1, x_2) n_k(x_1, x_2) \frac{\partial}{\partial x_k} \Phi(x_1, x_2, \xi_1, \xi_2) \\ & - \Phi(x_1, x_2, \xi_1, \xi_2) p(x_1, x_2, t)] ds(x_1, x_2) \\ & - \int\limits_{a}^{b} r(x_1, t) G^{+} \frac{\partial}{\partial x_2} \Phi(x_1, x_2, \xi_1, \xi_2) \bigg|_{x_2=0^{+}} dx_1. \end{aligned} \tag{11}$$

where $\gamma(\xi_1, \xi_2) = \gamma^{+}(\xi_1, \xi_2) + \gamma^{-}(\xi_1, \xi_2)$, i.e. $\gamma(\xi_1, \xi_2) = 1$ if (ξ_1, ξ_2) lies inside R^{+} or R^{-}, $0 < \gamma(\xi_1, \xi_2) < 1$ if (ξ_1, ξ_2) lies on C^{+} or C^{-} $[\gamma(\xi_1, \xi_2) = 1/2$ if (ξ_1, ξ_2) lies on a smooth part of C^{+} or $C^{-}]$ and $G(x_1, x_2) = G^{\pm}$ if $(x_1, x_2) \in R^{\pm}$.

The functions $\Phi^{+}(x_1, x_2, \xi_1, \xi_2)$ and $\Phi^{-}(x_1, x_2, \xi_1, \xi_2)$ satisfying (8), (9) and (10) may be chosen to be given by (see Berger and Karageorghis [3])

$$\begin{aligned} & \Phi^{+}(x_1, x_2, \xi_1, \xi_2) \\ & = -\frac{G^{-} - G^{+}}{2\pi G^{+}(G^{-} + G^{+})} \operatorname{Re}\{H(-\xi_2) \ln(z - c) + H(\xi_2) \ln(z - \overline{c})\}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} & \Phi^{-}(x_1, x_2, \xi_1, \xi_2) \\ & = \frac{G^{-} - G^{+}}{2\pi G^{-}(G^{-} + G^{+})} \operatorname{Re}\{H(-\xi_2) \ln(\overline{z} - c) + H(\xi_2) \ln(\overline{z} - \overline{c})\}, \end{aligned} \tag{13}$$

where $H(x)$ is the Heaviside unit-step function and the bar denotes the complex conjugate of a complex number.

Now, from (11), we may derive

$$\begin{aligned}\sigma_{j3}(\xi_1,\xi_2,t) = G^+ \oint_C &[w(x_1,x_2,t)G(x_1,x_2)n_k(x_1,x_2)\frac{\partial^2}{\partial x_k \partial \xi_j}\Phi(x_1,x_2,\xi_1,\xi_2) \\ &- p(x_1,x_2,t)\frac{\partial}{\partial \xi_j}\Phi(x_1,x_2,\xi_1,\xi_2)]ds(x_1,x_2) \\ &- (G^+)^2 \int_a^b r(x_1,t)\left.\frac{\partial^2}{\partial x_2 \partial \xi_j}\Phi(x_1,x_2,\xi_1,\xi_2)\right|_{x_2=0^+} dx_1 \\ &\text{for } (\xi_1,\xi_2) \text{ in the interior of } R^+.\end{aligned} \tag{14}$$

Letting $\xi_2 \to 0^+$ in (14) with $j=2$ and using (2), we obtain

$$\begin{aligned}& kr(\xi_1,t) + \eta\frac{\partial}{\partial t}r(\xi_1,t) \\ = \; & G^+ \oint_C [w(x_1,x_2,t)G(x_1,x_2)n_k(x_1,x_2)\left.[\frac{\partial^2}{\partial x_k \partial \xi_2}\Phi(x_1,x_2,\xi_1,\xi_2)]\right|_{\xi_2=0} \\ & -p(x_1,x_2,t)\left.[\frac{\partial}{\partial \xi_2}\Phi(x_1,x_2,\xi_1,\xi_2)]\right|_{\xi_2=0}]ds(x_1,x_2) \\ & +\frac{G^+G^-}{\pi(G^+ + G^-)}\mathcal{H}\int_a^b \frac{r(x_1,t)}{(\xi_1 - x_1)^2}dx_1 \\ & \text{for } a < \xi_1 < b,\end{aligned} \tag{15}$$

where $\mathcal{H}$ denotes the integral over the interval $[a,b]$ is to be interpreted in the Hadamard finite-part sense which may be defined using

$$\mathcal{H}\int_a^b \frac{F(x)dx}{(x-\xi)^2} \stackrel{\text{def}}{=} \lim_{\sigma\to 0^+}[\int_a^b \frac{(x-\xi)^2F(x)dx}{[(x-\xi)^2+\sigma^2]^2} - \frac{\pi}{2\sigma}F(\xi)] \text{ for } a < \xi < b. \tag{16}$$

As the definition (16) for Hadamard finite-part integrals may not be the usual one used in the engineering mechanics literature, an explanation of how it arises is perhaps necessary and is as given as follows.

Firstly, we note that it is well established that

$$\frac{d}{d\xi}\left(\mathcal{C}\int_a^b \frac{F(x)dx}{x-\xi}\right) = \mathcal{H}\int_a^b \frac{F(x)dx}{(x-\xi)^2} \text{ for } a < \xi < b, \tag{17}$$

where $\mathcal{C}$ denotes that the integral is to be interpreted in the Cauchy principal sense (see, for example, Chen and Hong [4] and other references therein).

Secondly, if Cauchy principal integrals are defined using

$$\mathcal{C}\int_a^b \frac{F(x)dx}{x-\xi} \stackrel{\text{def}}{=} \lim_{\sigma\to 0^+}\int_a^b \frac{(x-\xi)F(x)dx}{(x-\xi)^2+\sigma^2} \text{ for } a < \xi < b, \tag{18}$$

then differentiating both sides of (18) with respect to ξ and using (17) can be shown to give rise to the definition (16).

Perhaps the more commonly used definition of Cauchy principal integrals in the literature is the one given by

$$\mathcal{C}\int_a^b \frac{F(x)dx}{x-\xi} \stackrel{\text{def}}{=} \lim_{\epsilon\to 0^+}[\int_a^{\xi-\epsilon} \frac{F(x)dx}{x-\xi} + \int_{\xi+\epsilon}^b \frac{F(x)dx}{x-\xi}] \text{ for } a < \xi < b. \tag{19}$$

The two limits on the right hand sides of (18) and (19) can, however, be shown to be equal to each other, if we expand $F(x)$ as a Taylor series about $x = \xi$. Thus, (18) and (19) are equivalent definitions.

4 A numerical procedure

We shall now use (11) and (15) to derive a numerical procedure for solving the problem described in Section 2 above.

The boundary C is discretized into N straight line elements denoted by $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N-1)}$ and $C^{(N)}$. Across each of these elements, the displacement and the traction are approximated as spatially independent functions. More specifically, we make the approximation:

$$\left.\begin{array}{r} w(x_1,x_2,t) \simeq w^{(n)}(t) \\ p(x_1,x_2,t) \simeq p^{(n)}(t) \end{array}\right\} \text{ for } (x_1,x_2) \in C^{(n)}. \tag{20}$$

Note that, for a given n, either $w^{(n)}(t)$ or $p^{(n)}(t)$ (not both) is known from the boundary condition of the problem under consideration. Thus, in (20), there are N unknown functions of t to be determined.

The interval $[a, b]$ representing the imperfect interface is divided into M subintervals $[x^{(0)}, x^{(1)}]$, $[x^{(1)}, x^{(2)}]$, $\cdots$, $[x^{(M-2)}, x^{(M-1)}]$ and $[x^{(M-1)}, x^{(M)}]$. The displacement jump $r(x_1, t)$ over the interface is approximated using

$$r(x_1, t) \simeq r^{(m)}(t) \text{ for } x_1 \in [x^{(m-1)}, x^{(m)}], \tag{21}$$

where $r^{(m)}(t)$ are unknown functions to be determined.

To deal with the time derivative in (15), we also make the following approximations:

$$\begin{aligned} \frac{d}{dt}r^{(m)}(t) &\simeq \frac{r^{(m)}(t+\frac{1}{2}\Delta t) - r^{(m)}(t-\frac{1}{2}\Delta t)}{\Delta t}, \\ r^{(m)}(t) &\simeq \frac{1}{2}[r^{(m)}(t+\frac{1}{2}\Delta t) + r^{(m)}(t-\frac{1}{2}\Delta t)], \end{aligned} \tag{22}$$

where $\Delta t > 0$ is a small time-step.

With (20), (21) and (22), if we let (ξ_1, ξ_2) in (11) be given by $(\xi_1^{(j)}, \xi_2^{(j)})$ [the midpoint of $C^{(j)}$], we obtain the approximation:

$$\begin{aligned} \frac{1}{2}w^{(j)}(t) &= \sum_{n=1}^{N} w^{(n)}(t) \int\limits_{C^{(n)}} G(x_1, x_2) n_k(x_1, x_2) \frac{\partial}{\partial x_k} \Phi(x_1, x_2, \xi_1^{(j)}, \xi_2^{(j)}) ds(x_1, x_2) \\ &- \sum_{n=1}^{N} p^{(n)}(t) \int\limits_{C^{(n)}} \Phi(x_1, x_2, \xi_1^{(j)}, \xi_2^{(j)}) ds(x_1, x_2) \\ &- \frac{G^+}{2} \sum_{m=1}^{M} [r^{(m)}(t+\frac{1}{2}\Delta t) + r^{(m)}(t-\frac{1}{2}\Delta t)] \\ &\qquad \times \int\limits_{x^{(m-1)}}^{x^{(m)}} \left. \frac{\partial}{\partial x_2} \Phi(x_1, x_2, \xi_1^{(j)}, \xi_2^{(j)}) \right|_{x_2 = 0^+} dx_1 \\ &\qquad\qquad \text{for } j = 1,\ 2, \cdots, N. \end{aligned} \tag{23}$$

Similarly, if we let $\xi_1 = \overline{x}^{(j)}$ (the midpoint of the subinterval $[x^{(j-1)}, x^{(j)}]$)

in (15), we obtain

$$
\begin{aligned}
&\frac{k}{2}[r^{(j)}(t+\frac{1}{2}\Delta t)+r^{(j)}(t-\frac{1}{2}\Delta t)]+\frac{\eta[r^{(j)}(t+\frac{1}{2}\Delta t)-r^{(j)}(t-\frac{1}{2}\Delta t)]}{\Delta t} \\
= \quad & G^{+}\sum_{n=1}^{N}w^{(n)}(t)\int\limits_{C^{(n)}}G(x_1,x_2)n_k(x_1,x_2) \\
&\qquad\qquad \times [\frac{\partial^2}{\partial x_k\partial\xi_2}\Phi(x_1,x_2,\xi_1,\xi_2)]\bigg|_{(\xi_1,\xi_2)=(\overline{x}^{(j)},0)}ds(x_1,x_2) \\
&-G^{+}\sum_{n=1}^{N}p^{(n)}(t)\int\limits_{C^{(n)}}[\frac{\partial}{\partial\xi_2}\Phi(x_1,x_2,\xi_1,\xi_2)]\bigg|_{(\xi_1,\xi_2)=(\overline{x}^{(j)},0)}ds(x_1,x_2) \\
&+\frac{G^{+}G^{-}}{2\pi(G^{+}+G^{-})}\sum_{m=1}^{M}[r^{(m)}(t+\frac{1}{2}\Delta t)+r^{(m)}(t-\frac{1}{2}\Delta t)] \\
&\qquad\qquad \times\left\{\frac{1}{\overline{x}^{(j)}-x^{(m)}}-\frac{1}{\overline{x}^{(j)}-x^{(m-1)}}\right\} \\
&\qquad \text{for } j=1,\ 2,\cdots,M. \qquad (24)
\end{aligned}
$$

If $r^{(j)}(t-\frac{1}{2}\Delta t)$ are assumed known, (23) and (24) constitutes a system of $N+M$ linear algebraic equations from which we may solve for the $N+M$ unknowns given by $r^{(m)}(t+\frac{1}{2}\Delta t)$ $(m=1,\ 2,\ \cdots,\ M)$ and by either $w^{(n)}(t)$ or $p^{(n)}(t)$ $(n=1,\ 2,\ \cdots,\ N)$. Once the unknowns are determined, we may compute the displacement $w(\xi_1,\xi_2,t)$ at any point $(\xi_1,\xi_2)\in R^{+}\cup R^{-}$ using (11).

At $t=\frac{1}{2}\Delta t$, since $r^{(j)}(0)$ are known from the initial condition (3) [i.e. $r^{(j)}(0)=v(\overline{x}^{(j)})$], we may solve (23) and (24) for $r^{(m)}(\Delta t)$ and for either $w^{(n)}(\frac{1}{2}\Delta t)$ or $p^{(n)}(\frac{1}{2}\Delta t)$. With $r^{(m)}(\Delta t)$ determined, we may then let $t=\frac{3}{2}\Delta t$ to solve (23) and (24) for $r^{(m)}(2\Delta t)$ and for either $w^{(n)}(\frac{3}{2}\Delta t)$ or $p^{(n)}(\frac{3}{2}\Delta t)$.Thus, we may solve (23) and (24) for the unknowns at different consecutive time levels given by $t=\frac{1}{2}(2q-1)\Delta t$ $(q=1,\ 2,\ \cdots)$.

5 Numerical example

For the mere purpose of obtaining some numerical results in order to assess the validity and the accuracy of the method presented in Section 4, a specific problem with a known exact solution, as described below, is used as a test problem.

We choose the constants G^+ and G^- in (1) to be given by

$$G^+ = 1/5 \text{ and } G^- = 1/2, \tag{25}$$

and take the regions R^+ and R^- to be

$$\begin{aligned} R^+ &= \{(x_1, x_2) : \ 0 < x_1 < 1, \ 0 < x_2 < 1/2\}, \\ R^- &= \{(x_1, x_2) : \ 0 < x_1 < 1, \ -1/2 < x_2 < 0\}. \end{aligned} \tag{26}$$

Furthermore, we require the unknown functions $\sigma_{23}(x_1, x_2, t)$ and $r(x_1, t)$ (the displacement jump) to satisfy the interface condition

$$\begin{aligned} \sigma_{23}(x_1, 0^+, t) &= \sigma_{23}(x_1, 0^-, t) \\ &= \frac{5}{4} r(x_1, t) + \frac{1}{2}\frac{\partial}{\partial t} r(x_1, t) \\ &\quad \text{for } x_1 \in (0, 1) \text{ and } t > 0, \end{aligned} \tag{27}$$

with the initial condition

$$r(x_1, 0) = \exp(-x_1) + \frac{4}{3}x_1 \ \text{ for } x_1 \in (0, 1). \tag{28}$$

A solution of (4) which satisfies (27) and (28) [with G^+ and G^- as given by (25)] is

$$\begin{aligned} w(x_1, x_2, t) &= H(x_2)\{[2\cos(x_2) + 5\sin(x_2)]\exp(-x_1 - \frac{1}{2}t) \\ &\quad + (1 + 5x_2)x_1 \exp(-t)\} \\ &\quad + H(-x_2)\{[\cos(x_2) + 2\sin(x_2)]\exp(-x_1 - \frac{1}{2}t) \\ &\quad + (-\frac{1}{3} + 2x_2)x_1 \exp(-t)\}. \end{aligned} \tag{29}$$

To devise a test problem, we use (29) to generate boundary values of the displacement w on the sides $x_2 = \pm 1/2$, $0 < x_1 < 1$, and boundary values of the traction p on $x_1 = 0$, $-1/2 < x_2 < 1/2$ and also on $x_1 = 1$, $-1/2 < x_2 < 1/2$. The numerical procedure in Section 4 is then applied to solve (4) subject to the boundary data thus generated and the interface condition (27) with the initial condition (28). If the procedure really works, we should be able to recover the solution (29) and the displacement jump $r(x_1, t)$ approximately at selected time levels. From (29), the exact $r(x_1, t)$ is given by

$$r(x_1, t) = \exp(-x_1 - \frac{1}{2}t) + \frac{4}{3}x_1 \exp(-t). \tag{30}$$

We discretize each side of the bimaterial into L equal length boundary elements (so that $N = 4L$) and the interface $[0, 1]$ into M equal subintervals. Furthermore, we require a boundary element to lie completely in either R^+ or R^-. One of the endpoints of a boundary element may lie on the interface between R^+ and R^-, however. Thus, L must be selected to be an even integer.

Table 1. A comparison of the numerical and exact values of the interfacial displacement jump $r(x_1, t)$ at $t = 0.7500$ and at various points on the interface.

(x_1, x_2)	$(N, M) = (24, 5)$ $\Delta t = 0.3000$	$(N, M) = (72, 15)$ $\Delta t = 0.1000$	Exact
$(0.1000, 0)$	0.6963	0.6874	0.6849
$(0.3000, 0)$	0.7004	0.6986	0.6981
$(0.5000, 0)$	0.7326	0.7320	0.7318
$(0.7000, 0)$	0.7831	0.7825	0.7822
$(0.9000, 0)$	0.8529	0.8473	0.8463

In Table 1, we present the numerical values of the displacement jump $r(x_1, t)$ at $t = 0.7500$ and selected points on the interface, as obtained by using $(N, M) = (24, 5)$ with $\Delta t = 0.3000$ and $(N, M) = (72, 15)$ with $\Delta t = 0.1000$. Even with a relatively coarse discretization of the exterior boundary and the interface with $(M, N) = (24, 5)$ (i.e. with the exterior boundary replaced by boundary elements each of length about 0.1667 units and the interface divided into subintervals of size 0.2000 units) and a relatively large time-step of 0.3000, the numerical values obtained show reasonably good agreement with the exact solution. It is obvious that there is a significant improvement in the accuracy of the numerical results when the discretization of the exterior boundary and the interface of the bimaterial is refined and when the size of the time-step used is reduced.

Table 2. A comparison of the numerical and exact values of displacement w at the interior point $(0.2500, 0.3500)$ and at selected time t.

t	$(N, M) = (24, 5)$ $\Delta t = 0.3000$	$(N, M) = (72, 15)$ $\Delta t = 0.1000$	Exact
0.1500	3.1923	3.1885	3.1879
0.4500	2.6764	2.6734	2.6729
0.7500	2.2506	2.2484	2.2481
1.0500	1.8978	1.8962	1.8960
1.3500	1.6043	1.6032	1.6031
1.6500	1.3593	1.3585	1.3584
1.9500	1.1540	1.1534	1.1533

Table 3. A comparison of the numerical and exact values of $(\sigma_{13}, \sigma_{23})$ at the interior point $(0.2500, 0.3500)$ and at selected time t.

t	Numerical $(\sigma_{13}, \sigma_{23})$	Exact $(\sigma_{13}, \sigma_{23})$
0.1500	$(-0.04651, 0.7943)$	$(-0.04585, 0.7948)$
0.4500	$(-0.09686, 0.6577)$	$(-0.09622, 0.6583)$
0.7500	$(-0.1254, 0.5470)$	$(-0.1249, 0.5475)$
1.0500	$(-0.1390, 0.4567)$	$(-0.1386, 0.4571)$
1.3500	$(-0.1427, 0.3826)$	$(-0.1424, 0.3829)$
1.6500	$(-0.1399, 0.3216)$	$(-0.1396, 0.3218)$
1.9500	$(-0.1330, 0.2711)$	$(-0.1329, 0.2712)$

Still using $(N, M) = (24, 5)$ with $\Delta t = 0.3000$ and $(N, M) = (72, 15)$ with $\Delta t = 0.1000$, we may compute the displacement w at selected time levels t and at chosen points $(\xi_1, \xi_2) \in R^+ \cup R^-$ using (11), once the unknowns in (23) and (24) are all determined. The numerical values of w at the interior point $(0.2500, 0.3500)$ are compared with the exact solution at selected time levels in Table 2. The numerical results in Table 2 are observed to converge to the exact values when the number of subintervals of the interface and the boundary elements is increased and the time-step reduced.

Lastly, we discretize (14) to compute approximately the stress σ_{j3} at interior points in R^{+}. In Table 3, we compare the numerical values of $(\sigma_{13}, \sigma_{23})$, as obtained using $(N, M) = (72, 15)$ with $\Delta t = 0.1000$, with the exact ones at the interior point $(0.2500, 0.3500)$ and at selected time levels t. The numerical and the exact values of $(\sigma_{13}, \sigma_{23})$ are in good agreement with each other.

6 Conclusion

A hypersingular boundary integral method is proposed for the numerical solution of a quasi-static antiplane problem involving a finite bimaterial with a microscopically imperfect and visco-elastic interface. The condition on the interface between the dissimilar materials is expressed in terms of a hypersingular boundary integral equation which contains the interfacial displacement jump and its time derivative. The method approximates the time derivative of the displacement jump using a finite-difference method. This leads to a time-stepping scheme which requires the solution of a system of linear algebraic equations of the form $\mathbf{AX} = \mathbf{B}$ at consecutive time levels. The elements in the square matrix $\mathbf{A}$ are independent of t and therefore have to be evaluated only once, while the matrix $\mathbf{B}$ has to be re-computed at every time level.

The proposed method is applied to solve numerically a specific test problem. For the test problem, the unknown functions like the interfacial displacement jump, the displacement and the stress, are computed numerically and found to be in good agreement with the known exact solution. The accuracy of the numerical solution is also found to improve when the calculation is repeated using a more refined discretization of the exterior boundary and the interface of the bimaterial together with a smaller time-step.

References

[1] Ang, W. T. (1988), A crack in an anisotropic layered material under the action of impact loading, ASME Journal of Applied Mechanics, Vol. 55, pp. 122-125.

[2] Benveniste, Y. and Miloh, T. (2001), Imperfect soft and stiff interfaces in two-dimensional elasticity, Mechanics of Materials, Vol. 33, pp. 309-323.

[3] Berger, J. R. and Karageorghis, A. (1999), The method of fundamental solutions for heat conduction in layered materials, International Journal for Numerical Methods in Engineering, Vol. 45, pp. 1681-1694.

[4] Chen, J. T. and Hong, H. K. (1999), Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series, Applied Mechanics Review, Vol. 52, pp. 17-33.

[5] Clements, D. L. (1979), A crack in an anisotropic layered materials, Rozprawy Inżynierskie, Vol. 27, pp. 171-180.

[6] Clements, D. L. (1981), Boundary Value Problems Governed by Second Order Elliptic Systems, Pitman, London.

[7] Fan, H. and Sze, K. Y. (2001), A micro-mechanics model for imperfect interface in dielectric materials, Mechanics of Materials, Vol. 33, pp. 363-370.

[8] Fan, H. and Wang, G. F. (2003), Interaction between a screw dislocation and viscoelastic interfaces, International Journal of Solids and Structures, Vol. 40, pp. 763-776.

[9] Lee, B. C. and Kim, E. S. (1999), A simple and efficient method of analyzing mechanical behaviors of multi-layered orthotropic plates in rectangular shapes, Journal of Micromechanics and Microengineering, Vol. 9, pp. 385-393.

[10] Shames, I. H. and Cozzarelli, F. A. (1992), Elastic and Inelastic Stress Analysis, Prentice-Hall, New Jersey.