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Numerical Solution of the Heat Equation with Constrained Energy

Whye-Teong Ang
Division of Engineering Mechanics
School of Mechanical and Aerospace Engineering
Nanyang Technological University, Republic of Singapore
Phone: +65 67905937, Fax: +65 67911859, E-mail: mwtang@ntu.edu.sg

Abstract-A numerical method for the approximate solution of the one-dimensional heat equation subject to suitably prescribed initial-boundary condition and specification of energy is presented. It is applied to solve a specific test problem.

Keywords: heat equation, non-local condition, integro-differential formulation, numerical solution.

I. INTRODUCTION

Problems governed by partial differential equations of physics subject to non-classical conditions have been a subject of considerable interest. An example of such problems requires solving the one-dimensional heat equation (in non-dimensionalised form)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - s(x,t),\ \ x \in [0,\ 1], t \ge 0, \tag{1}$$

subject to the initial-boundary conditions

$$u(x,0) = f(x),\ \ x \in (0,1), \tag{2}$$

$$\alpha u(1,t) + \beta \left.\frac{\partial u}{\partial x}\right|_{x=1} = g(t),\ \ t > 0, \tag{3}$$

and the non-local condition

$$\int_0^{\ell} u(x,t)dx = E(t),\ \ t > 0, \tag{4}$$

where x and t are respectively the spatial and temporal variables, $u(x,t)$ is the unknown function (temperature) to be determined, $s(x,t)$, $f(x)$, $g(t)$ and $E(t)$ are well prescribed functions, α and β are suitably given constants and ℓ is a given constant such that $0<\ell\le 1$.

Note that (4) implies the total energy in the region $0<x<\ell$ is constrained. The special case in which $\alpha=1$ and $\beta=0$ gives the Dirichlet condition at the boundary $x=1$. The Neumann condition is applicable at $x=1$ if $\alpha=0$ and $\beta=1$.

Finite-difference methods for the numerical solution of the problem defined by (1)-(4) are given by various researchers, e.g. [DEH, 03] (for $\ell=1$ and Dirichlet condition at $x=1$), [CAN, 90] (for $s(x,t)=0$ and Neumann condition at $x=1$) and [CAN, 82] (for $s(x,t)=0$ and Dirichlet condition at $x=1$).

In the present paper, an alternative numerical method based on an integro-differential formulation of (1) and the use of local interpolating functions for approximating u is presented for solving the problem. The approach reduces (1)-(4) to an initial-value problem governed by a linear system of first order ordinary differential equations containing unknown functions of time t. To solve the intial-value problem numerically, the first order time derivatives of the unknown functions are approximated using linear functions of t. Numerical results for a specific test problem are presented.

II. INTEGRO-DIFFERENTIAL EQUATION

Through partial integrations of the heat equation (1) with respect to x, one may derive the integro-differential equation

$$\begin{aligned}2u(\xi,t) &= u(0,t) + u(1,t) + \xi p(t) + (\xi - 1)q(t)\\ &+ \int_0^1 |x-\xi|(\frac{\partial}{\partial t}[u(x,t)] - s(x,t))dx\end{aligned} \tag{5}$$

where p and q are the boundary flux functions defined by

$$p(t) = \left.\frac{\partial u}{\partial x}\right|_{x=0},\ \ q(t) = \left.\frac{\partial u}{\partial x}\right|_{x=1}. \tag{6}$$

The problem under consideration may be reformulated as one that requires finding $u(x,t)$ from (5) together with the initial-boundary conditions (2)-(4).

III. APPROXIMATION OF UNKNOWN FUNCTION

The unknown function $u(x,t)$ is approximated as

$$u(x,t) \approx \sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}\sigma_n(x)u_m(t), \tag{7}$$

where $u_m(t)=u(\xi_m,t)$, ξ_1, ξ_2, ..., ξ_{N-1} and ξ_N are N distinct well-spaced nodes selected from the interval [0,1] with $\xi_1=0$ and $\xi_N=1$, $\sigma_n(x)=1+|x-\xi_n|^{3/2}$ is the local interpolating function centred about ξ_n and c_{nm} are constant coefficients defined by

$$\sum_{k=1}^{N}\sigma_n(\xi_k)c_{pk} = \delta_{np} \tag{8}$$

with δ_{np} being the Kronecker-delta (that is, $\delta_{np}=1$ if $n=p$ and $\delta_{np}=0$ if $n\neq p$).

In general, $u_1(t)$, $u_2(t)$, …, $u_{N-1}(t)$ and $u_N(t)$ may be regarded as unknown functions. The integro-differential equation (5) and the approximation (7) can be used together with (2)-(4) to obtain a numerical procedure for determining the unknown functions.

IV. NUMERICAL PROCEDURE

If (7) is substituted into (5) with $\xi=\xi_r$ for $r=1, 2,\ldots, N$, one obtains the system of ordinary differential equations

$$2u_r(t)+S_r(t)=u_1(t)+u_N(t)+\xi_r p(t)+(\xi_r-1)q(t) + \sum_{m=1}^{N} F_{rm}\frac{du_m}{dt}, \quad r=1,2,\ldots,N, \tag{9}$$

where

$$S_r(t)=\int_0^1 |x-\xi_r|\, s(x,t)dx \approx \sum_{m=1}^{N} F_{rm} s(\xi_m,t),$$

$$F_{rm}=\sum_{n=1}^{N} c_{nm}\Big(\frac{1}{2}((1-\xi_r)^2+\xi_r^2) + \frac{2}{5}((1-\xi_r)(1-\xi_n)^{5/2}+\xi_r\xi_n^{5/2}) - \frac{4}{35}((1-\xi_n)^{7/2}+\xi_n^{7/2})+\frac{8}{35}|\xi_r-\xi_n|^{7/2}\Big). \tag{10}$$

In general, the system (9) contains N+2 unknown functions of t as given by $u_1(t)$, $u_2(t)$, …, $u_{N-1}(t)$, $u_N(t)$, $p(t)$ and $q(t)$ but comprises only N equations. Two more equations are required to complete the system. They come from (3) and (4) as given respectively by

$$\alpha u_N(t)+\beta q(t)=g(t), \qquad \sum_{m=1}^{N} G_m u_m(t)=E(t), \tag{11}$$

where

$$G_m=\sum_{n=1}^{N} c_{nm}\Big(\ell+\frac{2}{5}\operatorname{sgn}(\ell-\xi_n)|\ell-\xi_n|^{5/2}+\frac{2}{5}\xi_n^{5/2}\Big). \tag{12}$$

The first order time derivatives of the unknown functions in (9) are approximated as linear functions of t over the time interval $[\tau, \tau+2\Delta t]$ as given by

$$\frac{d}{dt}(u_m(t)) \approx \frac{1}{(\Delta t)^2}\Big((t-\tau-\frac{3}{2}\Delta t)u_m(\tau) - 2(t-\tau-\Delta t)u_m(\tau+\Delta t) + (t-\tau-\frac{1}{2}\Delta t)u_m(\tau+2\Delta t)\Big), \quad t\in[\tau,\tau+2\Delta t], \tag{13}$$

where Δt is a very small positive constant.

TABLE I
A COMPARISON OF NUMERICAL AND EXACT SOLUTIONS

Point x	N=5,Δt=0.10	N=9,Δt=0.10	N=17,Δt=0.05	Exact
0.00	0.337679	0.334842	0.334174	0.333997
0.25	0.424551	0.422419	0.421968	0.421844
0.50	0.487809	0.486385	0.486095	0.486008
0.75	0.527955	0.527274	0.527137	0.527093

Substituting (13) into (9) and letting $t=\tau+j\Delta t$ (for j=1,2), one obtains

$$\begin{aligned} &2u_r(\tau+j\Delta t)+S_r(\tau+j\Delta t) \\ &= u_1(\tau+j\Delta t)+u_N(\tau+j\Delta t) \\ &+\xi_r p(\tau+j\Delta t)+(\xi_r-1)q(\tau+j\Delta t) \\ &+\frac{1}{\Delta t}\sum_{m=1}^{N}F_{rm}\big((j-\frac{3}{2})u_m(\tau)-2(j-1)u_m(\tau+\Delta t) \\ &+(j-\frac{1}{2})u_m(\tau+2\Delta t)\big), \quad r=1,2,\ldots,N,\ j=1,2. \end{aligned} \tag{14}$$

If $u_n(\tau)$ is assumed to be known for n=1,2,…,N, then (14) may be regarded as a system of 2N linear algebraic equations containing 2(N+2) unknowns given by $p(\tau+j\Delta t)$, $q(\tau+j\Delta t)$ and $u_n(\tau+j\Delta t)$ (n=1,2,…,N and j=1,2). Another 4 equations are obtained by letting $t=\tau+j\Delta t$ (for j=1,2) in (11). Thus,

$$\alpha u_N(\tau+j\Delta t)+\beta q(\tau+j\Delta t)=g(\tau+j\Delta t), \qquad \sum_{m=1}^{N}G_m u_m(\tau+j\Delta t)=E(\tau+j\Delta t),\ j=1,2. \tag{15}$$

The equations in (14) and (15) may be solved as follows. Work out $u_n(0)$ using the initial condition in (2) and let τ=0 in (14) and (15) to solve for $p(\Delta t)$, $q(\Delta t)$, $u_n(\Delta t)$, $p(2\Delta t)$, $q(2\Delta t)$ and $u_n(2\Delta t)$. With $p(2\Delta t)$, $q(2\Delta t)$ and $u_n(2\Delta t)$ now known, let $\tau=2\Delta t$ in (14) and (15) to solve for $p(3\Delta t)$, $q(3\Delta t)$, $u_n(3\Delta t)$, $p(4\Delta t)$, $q(4\Delta t)$ and $u_n(4\Delta t)$. Marching forward in time, one can let $\tau=4\Delta t$, $6\Delta t$, …, to solve for the unknown quantities at higher and higher time levels.

V. A SPECIFIC TEST PROBLEM

For a specific test problem, the given parameters in (1)-(4) are chosen to be

$$\begin{aligned} &s(x,t)=0,\ f(x)=\cos(\frac{\pi x}{3})+\sin(\frac{\pi x}{6}),\ \alpha=1,\ \beta=0, \\ &g(t)=\frac{1}{2}\exp(-\frac{\pi^2 t}{9})+\frac{1}{2}\exp(-\frac{\pi^2 t}{36}),\ \ell=\frac{1}{2}, \\ &E(t)=\frac{3}{2\pi}\exp(-\frac{\pi^2 t}{36})(\exp(-\frac{\pi^2 t}{12})-\sqrt{6}-\sqrt{2}+4). \end{aligned} \tag{16}$$

The exact solution is given by

$$u(x,t)=\cos(\frac{\pi x}{3})\exp(-\frac{\pi^2 t}{9})+\sin(\frac{\pi x}{6})\exp(-\frac{\pi^2 t}{36}) \tag{17}$$

To execute the numerical procedure in Section IV, the collocation points are selected to be given by $\xi_k=(k-1)/(N-1)$ for k=1,2,…,N. Table I compares 3 sets of numerical values of u at 4 selected points and at t=1 with the exact solution. The numerical values agree well with the exact ones. Furthermore, as N is increased and/or as Δt

is reduced, there is an obvious improvement in the accuracy of the numerical solution.

Plots of the boundary flux functions $p(t)$ and $q(t)$ obtained numerically by solving (14) and (15) with N=17 and Δt=0.05 are given in Fig. 1 for $0<t<2$. The graphs for the approximately determined $p(t)$ and $q(t)$ are visually indistinguishable from those obtained from the exact solution (17). In Fig. 1, the numerical and exact values of $p(t)$ and $q(t)$ mainly agree to three significant figures.

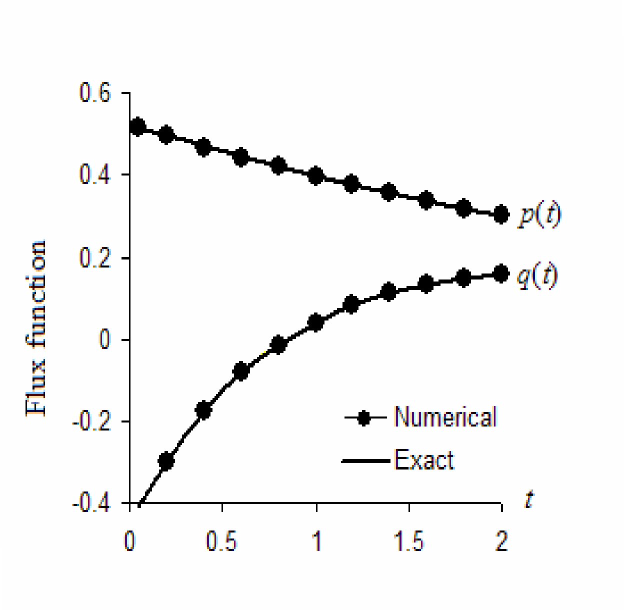

Fig. 1. Plots of numerical and exact flux functions.

Related papers accepted for publication in journals

A related paper (also on the heat or diffusion equation) dealing with conditions more general than those given by (3) and (4) has recently been accepted for publication. (W. T. Ang, Numerical solution of a non-classical parabolic problem: an integro-differential approach, to appear in *Applied Mathematics and Computation*.)

Another related paper which applies the method of solution presented here to solve the one-dimensional wave equation with non-local condition has also been recently accepted for publication in the IMACS journal *Applied Numerical Mathematics*. (W. T. Ang, A numerical method for the wave equation subject to a non-local conservation condition, to appear in *Applied Numerical Mathematics*.)

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W. T. Ang
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VI. CONCLUSION

A numerical method has been successfully developed and implemented for solving the one-dimensional heat equation subject to a non-local condition. It (the method) uses chosen local interpolating functions to reduce a suitable integro-differential equation into a system of first order ordinary differential equations. This results in a formulation containing the boundary heat fluxes, allowing for easy treatment of boundary conditions involving fluxes. If the boundary fluxes are not known a priori, they are unknown functions which can be directly and accurately determined from the formulation. It is not necessary to approximate the boundary fluxes using any finite-difference formula. Numerical results obtained for a specific test problem (and some other problems not reported herein) indicate that the temperature and the boundary heat fluxes can be computed accurately by the numerical procedure here.

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