On some contact problems for inhomogeneous anisotropic elastic materials

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Abstract

Some generalised plane strain contact problems are considered for a class of inhomogeneous anisotropic elastic materials for which the elastic moduli vary continuously with the spatial coordinates. Strip loading of a half-space and a layer on a rigid foundation are considered and integral expressions for the displacement and stress are obtained. Numerical results are obtained for some particular transversely isotropic and isotropic materials.

Key words: Contact problems; Anisotropy; Integral equations.
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1. Introduction

Contact problems involving the indentation of an elastic half-space by applied loads on its plane surface are the subject of an extensive literature. The greater part of this literature is concerned with either a homogeneous isotropic half-space or a half-space consisting of contiguous layers of homogeneous isotropic materials. Corresponding problems for anisotropic homogeneous materials have also attracted the attention of a number of authors and a considerable literature exists on this class of problems.

The solution of contact problems for an inhomogeneous half-space in which the elastic moduli vary continuously with the spatial coordinates generally presents considerable difficulties compared with the corresponding problems for homogeneous materials. Nevertheless a number of authors have succeeded in obtaining analytical solutions to contact problems involving this class of materials. Such analytical solutions as do exist are restricted to particular types of inhomogeneous materials. Thus, for example, in some of the early works in this area Gibson [1] and Gibson, Brown and Andrews [2] considered contact problems for an incompressible half-space in which the elastic moduli varied linearly

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with the perpendicular distance from the plane boundary of the half-space while Mossakovskii [3] considered problems where the elastic moduli varied exponentially with the perpendicular distance form the plane boundary. Other more recent examples of solutions to problems for half-spaces with continuously varying elastic moduli include the work of Clements and Ang [4], Azis and Clements [5] and Selvadurai [6]. The latter paper contains a number of references to papers which have addressed problems in this area in the latter half of the twentieth century.

The present work is concerned with the solution of some generalised plane contact problems for an anisotropic half-space in which the elastic parameters are a quadratic function of the spatial variables. The problems considered involve a specified displacement and a specified force on the boundary of the half-space. Solutions to these problems are obtained either in closed form or alternatively in terms of integrals which readily yield some analytical information regarding the solution and also numerical values for the displacement and stress. The analysis is for general anisotropy and through a limiting procedure also yields numerical results for the relevant class of isotropic materials. In the case of isotropic materials the class of materials to which the analysis applies has a Poisson's ratio of 1/4. The results obtained exhibit some similar characteristics to the results obtained by Gibson [1] and Gibson, Brown and Andrews [2] for inhomogeneous isotropic materials with a Poisson's ratio 1/2.

2. Statement of the problem

Referred to a Cartesian frame $Ox_1x_2x_3$ consider an anisotropic elastic body with a geometry that does not vary in the Ox_3 direction. Let the body occupy the region Ω which consists either of the half-space $x_2 > 0$ or the slab lying in the region $0 < x_2 < h$ where h is a constant. On the plane boundary $x_2 = 0$ either the displacement or stress is specified and the slab adheres to a rigid foundation so that for the slab the displacement is zero on $x_2 = h$. The problem is to determine the stress and displacement throughout the elastic material,

3. Basic equations

The equilibrium equations governing small generalised plane deformations of an inhomogeneous anisotropic elastic material may be written in the form

$$\frac{\partial}{\partial x_j}\left[c_{ijkl}(\mathbf{x})\,\frac{\partial u_k(\mathbf{x})}{\partial x_l}\right] = 0, \tag{1}$$

where i, j, k, l = 1,2,3, $\mathbf{x} = (x_1, x_2)$, u_k denotes the displacement, $c_{ijkl}(\mathbf{x})$ the elastic moduli and the repeated summation convention (summing from 1 to 3) is used for repeated Latin suffices. The stress displacement relations are given by

$$\sigma_{ij}(\mathbf{x}) = c_{ijkl}\,\frac{\partial u_k}{\partial x_l} \tag{2}$$

and the stress vector P_i on a boundary with outward pointing normal $\mathbf{n} = (n_1, n_2)$ is defined as

$$P_i(\mathbf{x}) = \sigma_{ij}\, n_j = c_{ijkl}\,\frac{\partial u_k}{\partial x_l}\, n_j, \tag{3}$$

For all points in Ω the coefficients $c_{ijkl}(\mathbf{x})$ are required to satisfy the usual symmetry condition

$$c_{ijkl} = c_{ijlk} = c_{jikl} = c_{klij} \tag{4}$$

and also sufficient conditions for the strain energy density to be positive. This requirement ensures that the system of partial differential equations is elliptic throughout Ω.

The coefficients in (1) are required to take the form

$$c_{ijkl}(\mathbf{x}) = c^{(0)}_{ijkl}\, g(\mathbf{x}), \tag{5}$$

where the $c^{(0)}_{ijkl}$ are constants and $g(x_1, x_2)$ is a twice differentiable function of the variables x_1 and x_2. Also in addition to the symmetry condition (4) the $c^{(0)}_{ijkl}$ are required to satisfy the additional condition

$$c^{(0)}_{ijkl} = c^{(0)}_{ilkj}. \tag{6}$$

Equation (17) may now be written in the form

$$c^{(0)}_{ijkl}\,\frac{\partial}{\partial x_j}\left(g\,\frac{\partial u_k}{\partial x_l}\right) = 0. \tag{7}$$

Following Azis and Clements [5] consider a transformation of the dependent variables in the form

$$u_k = g^{-1/2}\,\psi_k. \tag{8}$$

Use of (8) in (7) provides the equation

$$c^{(0)}_{ijkl}\left[g^{1/2}\frac{\partial^2\psi_k}{\partial x_j\partial x_l} + \frac{\partial g^{1/2}}{\partial x_j}\,\frac{\partial \psi_k}{\partial x_l} - \frac{\partial g^{1/2}}{\partial x_l}\,\frac{\partial \psi_k}{\partial x_j} - \psi_k\,\frac{\partial^2 g^{1/2}}{\partial x_j\partial x_l}\right] = 0, \tag{9}$$

where by virtue of (6) this equation reduces to

$$g^{1/2}\, c^{(0)}_{ijkl}\,\frac{\partial^2\psi_k}{\partial x_j\partial x_l} - \psi_k\, c^{(0)}_{ijkl}\,\frac{\partial^2 g^{1/2}}{\partial x_j\partial x_l} = 0. \tag{10}$$

Thus if

$$c^{(0)}_{ijkl}\,\frac{\partial^2\psi_k}{\partial x_j\partial x_l} = 0 \tag{11}$$

and

$$c^{(0)}_{ijkl}\,\frac{\partial^2 g^{1/2}}{\partial x_j\partial x_l} = 0, \tag{12}$$

then (10) will be satisfied. Thus when g satisfies the system (12) the transformation given by (8) transforms the linear system with variable coefficients (7) to the linear system with constant coefficients (11).

As a result of the symmetry property $c_{ijkl} = c_{klij}$ equation (12) consists of a system of six constant coefficients partial differential equations in the one dependent variable $g^{1/2}$. In general this system will be satisfied by a linear function of the two independent variables x_1, x_2. Thus $g(\mathbf{x})$ may be taken in the form

$$g(\mathbf{x}) = (\alpha x_1 + \beta x_2 + \gamma)^2 \,, \tag{13}$$

where α, β and γ are constants which may be used to fit the elastic moduli $c_{ijkl}(\mathbf{x}) = c^{(0)}_{ijkl}\, g(\mathbf{x})$ to given numerical data.

Now substitution of (5) and (8) into (3) yields

$$P_i = -P^{[g]}_{ik}\, \psi_k + P^{[\psi]}_i\, g^{1/2}, \tag{14}$$

where

$$P^{[g]}_{ik}(\mathbf{x}) = c^{(0)}_{ijkl}\, \frac{\partial g^{1/2}}{\partial x_l}\, n_j, \tag{15}$$

$$P^{[\psi]}_{i}(\mathbf{x}) = c^{(0)}_{ijkl}\, \frac{\partial \psi_k}{\partial x_l}\, n_j. \tag{16}$$

Equation (11) has the general solution ((see Eshelby, Read and Shockley [7], Clements [8])

$$\psi_i = 2\Re\left[\sum_{\alpha=1}^{3} A_{i\alpha} f_\alpha(z_\alpha)\right], \tag{17}$$

where $\Re$ denotes the real part of a complex number, $f_\alpha(z_\alpha)$, $\alpha = 1, 2, 3$ are arbitrary analytic functions of the complex variables $z_\alpha = x_1 + \tau_\alpha x_2$, $\alpha = 1, 2, 3$ where τ_α are the three roots with positive imaginary part of the sextic in τ

$$|c^{(0)}_{i1k1} + c^{(0)}_{i2k1}\tau + c^{(0)}_{i1k2}\tau + c^{(0)}_{i2k2}\tau^2| = 0. \tag{18}$$

The $A_{i\alpha}$ occurring in (17) are the solutions of the system

$$\left(c^{(0)}_{i1k1} + c^{(0)}_{i2k1}\tau_\alpha + c^{(0)}_{i1k2}\tau_\alpha + c^{(0)}_{i2k2}\tau_\alpha^2\right) A_{k\alpha} = 0. \tag{19}$$

Use of (17) in (14) to (16) provides a representation for P_i in terms of the arbitrary functions $f_\alpha(z_\alpha)$ in the form

$$P_i = 2\Re\left[\sum_{\alpha=1}^{3} -P^{[g]}_{ik} A_{k\alpha} f_\alpha(z_\alpha) + g^{1/2} L_{ij\alpha} f'_\alpha(z_\alpha) n_j\right], \tag{20}$$

where primes denote differentiation with respect to the argument in question and

$$L_{ij\alpha} = (c^{(0)}_{ijk1} + \tau_\alpha c^{(0)}_{ijk2}) A_{k\alpha}, \tag{21}$$

From (19) and (21) it follows that

$$L_{i1\alpha} = \tau_\alpha L_{i2\alpha}. \tag{22}$$

Hence, in general it will only be necessary to consider the 3x3 matrix $[L_{i2\alpha}]$. The matrix $[L_{i1\alpha}]$ may then be determined through (22).

From (8) and (17) the displacement in terms of the functions $f_\alpha(z_\alpha)$ may be written in the form

$$u_k = 2g^{-1/2}\Re\left[\sum_{\alpha=1}^{3} A_{k\alpha} f_\alpha(z_\alpha)\right], \tag{23}$$

It is useful to have to have some alternative forms for equations (20) and (23). Let

$$\sum_{\alpha=1}^{3} A_{i\alpha} f_\alpha(z) = \theta_i(z), \tag{24}$$

where the $\theta_i(z)$, $k = 1, 2, 3$ are analytic functions of the complex variable z. The matrix $[A_{i\alpha}]$ is non-singular (see Stroh [9], Clements [8]) and hence from (24)

$$f_\alpha(z) = N_{i\alpha}\theta_i(z), \tag{25}$$

where

$$\delta_{ik} = \sum_{\alpha=1}^{3} A_{i\alpha} N_{\alpha j}, \tag{26}$$

where δ_{ij} is the Kronecker delta. Substitution of (25) into (23) and (20) yields

$$u_k = 2g^{-1/2}\Re\left[\sum_{\alpha=1}^{3} A_{k\alpha} N_{\alpha j}\theta_j(z_\alpha)\right], \tag{27}$$

$$P_i = 2\Re\left[\sum_{\alpha=1}^{3} -P_{ik}^{[g]} A_{k\alpha} N_{\alpha j}\theta_j(z_\alpha) + g^{1/2} L_{ij\alpha} N_{\alpha j}\theta_j'(z_\alpha) n_j\right]. \tag{28}$$

In particular, on $x_2 = 0$ (27) and (28) become

$$u_k = g^{-1/2}\left[\theta_k(x_1) + \overline{\theta}_k(x_1)\right], \tag{29}$$

$$P_i = -P_{ik}^{[g]}\left[\theta_k(x_1) + \overline{\theta}_k(x_1)\right] - g^{1/2}\left[C_{ik}\theta_k'(x_1) + \overline{C}_{ik}\overline{\theta}_k'(x_1)\right], \tag{30}$$

where the bar denotes the complex conjugate and

$$C_{ik} = \sum_{\alpha=1}^{3} L_{i2\alpha} N_{\alpha k}. \tag{31}$$

An alternative representation may be obtained by putting

$$\sum_{\alpha=1}^{3} L_{i2\alpha} f_\alpha(z) = \chi_i(z), \tag{32}$$

where the $\chi_k(z)$, $k = 1, 2, 3$ are analytic functions of the complex variable z. The matrix $[L_{i2\alpha}]$ is non-singular (see Stroh [9], Clements [8]) and hence from (32)

$$f_\alpha(z) = M_{i\alpha}\chi_i(z), \tag{33}$$

where

$$\delta_{ik} = \sum_{\alpha=1}^{3} L_{i2\alpha} M_{\alpha k}, \tag{34}$$

Substitution of (33) into (23) and (20) yields

$$u_k = 2g^{-1/2}\Re\left[\sum_{\alpha=1}^{3} A_{k\alpha} M_{\alpha j}\chi_j(z_\alpha)\right], \tag{35}$$

$$P_i = 2\Re\left[\sum_{\alpha=1}^{3} -P_{ik}^{[g]} A_{k\alpha} M_{\alpha r}\chi_r(z_\alpha) + g^{1/2} L_{ij\alpha} M_{\alpha r}\chi_r'(z_\alpha) n_j\right]. \tag{36}$$

In particular, on $x_2 = 0$ (35) and (36) become

$$u_k = g^{-1/2}\left[B_{kj}\chi_j(x_1) + \overline{B}_{kj}\overline{\chi}_j(x_1)\right], \tag{37}$$

$$P_i = -P_{ik}^{[g]}\left[B_{kr}\chi_r(x_1) + \overline{B}_{kr}\overline{\chi}_r(x_1)\right] - g^{1/2}\left[\chi_i'(x_1) + \overline{\chi}_i'(x_1)\right], \tag{38}$$

where

$$B_{kr} = \sum_{\alpha=1}^{3} A_{k\alpha} M_{\alpha r}. \tag{39}$$

4. A half-space with specified boundary displacement

Consider an inhomogeneous elastic half-space $x_2 > 0$ with the displacement u_k prescribed on the boundary $x_2 = 0$. The displacement and stress fields are required throughout the half-space. The boundary conditions on $x_2 = 0$ are

$$u_k(x_1, 0) = U_k(x_1), \tag{40}$$

where the $U_k(x_1), k = 1, 2, 3$ are given functions of x_1. For this problem the representation (8)-(12) is useful with $\theta_j(z)$ given by

$$\theta_j(z) = \frac{1}{2\pi}\int_0^\infty G_j(p)\exp(ipz)dp, \tag{41}$$

where the $G_j(p)$, $j = 1, 2, 3$ are functions of p which will be determined by the boundary conditions. From (11) and (40) it follows that

$$g^{-1/2}(x_1, 0)\Re\left[\frac{1}{\pi}\int_0^\infty G_k(p)\exp(ipx_1)dp\right] = U_k(x_1). \tag{42}$$

Use of the inversion theorem for Fourier transforms provides

$$G_k(p) = \int_{-\infty}^{\infty} g^{1/2}(\xi_1, 0) U_k(\xi) \exp(-ip\xi) d\xi. \tag{43}$$

Substitution of (43) into (41) and changing the order of integration yields

$$\begin{aligned} \theta_j(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g^{1/2}(\xi, 0) U_k(\xi) d\xi \int_0^{\infty} \exp(ip(z - \xi)) dp, \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g^{1/2}(\xi, 0) U_k(\xi) d\xi}{\xi - z}. \end{aligned} \tag{44}$$

As a particular example consider the case when the surface displacement is given by

$$U_k(x_1) = \begin{cases} U_k^{(0)} (a^2 - x_1^2)^{1/2} & \text{for} \quad |x_1| < a, \\ 0 & \text{for} \quad |x_1| > a. \end{cases} \tag{45}$$

where the $U_k^{(0)}$ for $k = 1, 2, 3$ and a are constants. With $g(\mathbf{x})$ given by (13) equation (44) yields

$$\begin{aligned} \theta_k(z) &= \frac{U_k^0}{2\pi i} \int_{-a}^{a} \frac{(\alpha\xi + \gamma)(a^2 - \xi^2)^{1/2} d\xi}{\xi - z} \\ &= \frac{U_k^0}{i} \left[(\alpha z + \gamma)(z^2 - a^2)^{1/2} - \alpha(z^2 - a^2) - \gamma z \right]. \end{aligned} \tag{46}$$

Use of (46) in (27) and (28) provides expressions for the displacement vector u_k and the stress vector P_i throughout the the half-space when the surface displacement is given by (45)

Attention is now restricted to the case when $U_1 = U_3 = 0$ and $\alpha = 0$ so that equation (46) provides $\theta_1 = \theta_3 = 0$ and

$$\theta_2(z) = \frac{U_2^0 \gamma}{i} \left[(z^2 - a^2)^{1/2} - z \right], \tag{47}$$

$$\theta_2'(z) = \frac{U_2^0 \gamma}{i} \left[\frac{z}{(z^2 - a^2)^{1/2}} - 1 \right]. \tag{48}$$

Commonly problems of this type are applicable for half-spaces in which the planes $x_i = 0$, $i = 1, 2, 3$ are planes of elastic symmetry. If the anisotropy is restricted to materials which exhibit this symmetry the $c_{ijkl}^{(0)}$ with an odd number of ones, twos and threes in the suffices are zero. Thus from (15) and (13) (with $\alpha = 0$) on the boundary $x_2 = 0$ the matrix $[P_{ik}^{[g]}]$ has the form

$$[P_{ik}^{[g]}] = \begin{bmatrix} -\beta c_{1212}^{(0)} & 0 & 0 \\ 0 & -\beta c_{2222}^{(0)} & 0 \\ 0 & 0 & -\beta c_{3232}^{(0)} \end{bmatrix}. \tag{49}$$

Also in view of the symmetry it may be verified from the analysis of section 3 that the matrix $[C_{kj}]$ adopts the form (see Clements [8])

$$[C_{kj}] = \begin{bmatrix} ic_{11} & c_{12} & 0 \\ c_{21} & ic_{22} & 0 \\ 0 & 0 & ic_{33} \end{bmatrix}, \tag{50}$$

where the c_{ij}, $i, j = 1, 2$, and c_{33} are real.

Use of (47) and (48) in (29) and (30) provides

$$P_1(x_1) = \begin{cases} 2U_2^{(0)} c_{12}\gamma^2 \left[x_1(a^2 - x_1^2)^{-1/2}\right] & \text{for} \quad |x_1| < a, \\ 0 & \text{for} \quad |x_1| > a. \end{cases} \tag{51}$$

$$P_2(x_1) = \begin{cases} 2U_2^{(0)} \beta\gamma c_{2222}^{(0)} (a^2 - x_1^2)^{1/2} + 2U_2^{(0)}\gamma^2 c_{22} & \text{for} \quad |x_1| < a, \\ -2U_2^{(0)} c_{22}\gamma^2 \left[x_1(a^2 - x_1^2)^{-1/2} - 1\right] & \text{for} \quad |x_1| > a. \end{cases} \tag{52}$$

Let $\beta = m/\gamma$ where $m \geq 0$ is a constant. In terms of m and γ equation (5) together with (13) (with $\alpha = 0$) yields the elastic parameters in the form

$$\begin{aligned} c_{ijkl} &= c_{ijkl}^{(0)} \left[\gamma + \beta x_2\right] && (53) \\ &= c_{ijkl}^{(0)} \left[\gamma + \frac{m}{\gamma} x_2\right] && (54) \end{aligned}$$

and the normal surface force in the form

$$P_2(x_1) = \begin{cases} 2U_2^{(0)} m c_{2222}^{(0)} (a^2 - x_1^2)^{1/2} + 2U_2^{(0)}\gamma^2 c_{22} & \text{for} \quad |x_1| < a, \\ -2U_2^{(0)} c_{22}\gamma^2 \left[x_1(a^2 - x_1^2)^{-1/2} - 1\right] & \text{for} \quad |x_1| > a. \end{cases} \tag{55}$$

As $\beta \to 0$ for a fixed and finite $\gamma > 0$ it is apparent from (52) that, with the specified boundary displacement, the normal force over the contact region tends to a constant value and when $\beta = 0$ (so that the material is homogeneous) the normal force over the contact region assumes the constant value $2U_2^{(0)}\gamma^2 c_{22}$.

If γ is sufficiently small for terms of order γ^2 to be ignored then (52) yields

$$P_2(x_1) = \begin{cases} 2U_2^{(0)} \beta\gamma c_{2222}^{(0)} (a^2 - x_1^2)^{1/2} & \text{for} \quad |x_1| < a, \\ 0 & \text{for} \quad |x_1| > a. \end{cases} \tag{56}$$

Thus for a fixed finite β with γ sufficiently small for terms of order γ^2 to be ignored the surface force is concentrated over the contact region. Over that region the surface force is a multiple of the specified surface displacement. Also the surface force is order γ and hence tends to zero as $\gamma \to 0$.

For a fixed finite m as $\gamma \to 0$ it is apparent from (55) that

$$P_2(x_1) \to \begin{cases} 2U_2^{(0)} m c_{2222}^{(0)} (a^2 - x_1^2)^{1/2} & \text{for} \quad |x_1| < a, \\ 0 & \text{for} \quad |x_1| > a. \end{cases} \tag{57}$$

Thus if the elastic parameters have the form given by (54) then as $\gamma \to 0$ the boundary force outside the contact region tends to zero and the displacement over the contact region is a constant finite multiple of the specified surface displacement.

5. A half-space with specified boundary force

Consider an inhomogeneous elastic half-space $x_2 > 0$ with the stress vector P_i prescribed on the boundary $x_2 = 0$. The displacement and stress fields are required throughout the half-space. The boundary conditions on $x_2 = 0$ are

$$P_i(x_1, 0) = P_i(x_1) \quad \text{for} \quad i = 1, 2, 3, \tag{58}$$

where the $P_i(x_1), i = 1, 2, 3$ are given functions of x_1. For this problem the representation (35)-(39) is useful with $\chi_j(z)$ given by

$$\chi_j(z) = \frac{1}{2\pi} \int_0^\infty G_j(p) \exp(ipz) dp \quad \text{for} \quad j = 1, 2, 3, \tag{59}$$

where the $G_j(p)$, $j = 1, 2, 3$ are functions of p which will be determined by the boundary conditions. From (38), (59) and (58) it follows that

$$\Re \left[\frac{1}{\pi} \int_0^\infty \left[-P_{ik}^{[g]} B_{kr} - i g^{1/2} p \delta_{ir} \right] G_r(p) \exp(ipx_1) dp \right] = P_i(x_1). \tag{60}$$

If $g(\mathbf{x})$=$(\beta x_2 + \gamma)^2$ where $\beta \geq 0$ and $\gamma > 0$ are constants then g and $P_{ik}^{[g]}$ are constant on $x_2 = 0$ and thus use of the inversion formula for Fourier transforms provides

$$D_{ir} G_r(p) = \int_{-\infty}^\infty P_i(\xi) \exp(-ip\xi) d\xi, \tag{61}$$

where from (60) and (15)

$$\begin{aligned} D_{ir} &= -P_{ik}^{[g]} B_{kr} - i g^{1/2}(0) p \delta_{ir} \\ &= \beta c_{i2k2}^{(0)} B_{kr} - i\gamma p \delta_{ir}. \end{aligned} \tag{62}$$

Hence

$$G_r(p) = E_{ri}(p) \int_{-\infty}^\infty P_i(\xi) \exp(-ip\xi) d\xi, \tag{63}$$

where

$$\delta_{ij} = D_{ir} E_{rj}. \tag{64}$$

Use of (63) in (59) gives

$$\chi_r(z) = \frac{1}{2\pi} \int_0^\infty E_{rj}(p) \exp(ipz) dp \int_{-\infty}^\infty P_i(\xi) \exp(-ip\xi) d\xi. \tag{65}$$

Substitution of (65) into (35) gives an expression for the displacement throughout the half-space in the form

$$u_k = \frac{1}{\pi g^{1/2}} \Re \sum_{\alpha=1}^{3} A_{k\alpha} M_{\alpha r} \int_0^\infty E_{rj}(p) \exp(ipz_\alpha) dp \int_{-\infty}^\infty P_i(\xi) \exp(-ip\xi) d\xi. \tag{66}$$

In particular the displacement on the surface $x_2 = 0$ is given by

$$u_k(x_1, 0) = \frac{1}{\pi\gamma}\Re\left[B_{kr}\int_0^\infty E_{rj}(p)\exp(ipx_1)dp\int_{-\infty}^{\infty}P_i(\xi)\exp(-ip\xi)d\xi\right]. \quad (67)$$

Once the material constants $c^{(0)}_{ijkl}$, β and γ are known the constants $P^{[g]}_{ik}$, $A_{k\alpha}$, $L_{ij\alpha}$, $M_{\alpha r}$ and B_{kr} may be calculated from the analysis in section 3 and then the E_{ij} may be obtained through equations (62) and (64). Then when $P_i(x_1)$ is known (66) and (67) provide equations which are suitable for the calculation of the displacement throughout the half-space. Also substitution of (65) into (36) provides an expression from which the stress vector may be calculated.

Of particular interest is the case when the loading $P_i(x_1)$ on the surface of the half-space consists of a constant pressure $P^{(0)}_i$ over a strip of finite width so that

$$P_i(x_1) = \begin{cases} P^{(0)}_i & \text{for} \quad |x_1| < a, \\ 0 & \text{for} \quad |x_1| > a. \end{cases} \quad (68)$$

In this case (66) and (67) yield

$$u_k = \frac{2P^{(0)}_j}{\pi g^{1/2}}\Re\left[\sum_{\alpha=1}^{3}A_{k\alpha}M_{\alpha r}\int_0^\infty E_{rj}(p)\frac{\sin pa}{p}\exp(ipz_\alpha)dp\right]. \quad (69)$$

$$u_k(x_1, 0) = \frac{2P^{(0)}_j}{\pi\gamma}\Re\left[B_{kr}\int_0^\infty E_{rj}(p)\frac{\sin pa}{p}\exp(ipx_1)dp\right]. \quad (70)$$

Commonly problems of this type are applicable for half-spaces in which the planes $x_i = 0$, $i = 1, 2, 3$ are planes of elastic symmetry. If the anisotropy is restricted to materials which exhibit this symmetry the $c^{(0)}_{ijkl}$ with an odd number of ones, twos and threes in the suffices are zero. Thus from (15) and (13) (with $\alpha = 0$) on the boundary $x_2 = 0$ the matrix $[P^{[g]}_{ik}]$ has the form given by equation (49). Also in view of the symmetry it may be verified from the analysis of section 3 that the matrix $[B_{kj}]$ adopts the form (see Clements [8])

$$[B_{kj}] = \begin{bmatrix} ib_{11} & b_{12} & 0 \\ b_{21} & ib_{22} & 0 \\ 0 & 0 & ib_{33} \end{bmatrix}, \quad (71)$$

where the b_{ij}, $i, j = 1, 2$, and b_{33} are real. Therefore from (62) and (64) the matrices $[D_{ir}]$ and $[E_{rj}]$ take the forms

$$[D^{[g]}_{ik}] = \begin{bmatrix} i(\beta b_{11}c^{(0)}_{1212} - \gamma p) & \beta b_{12}c^{(0)}_{1212} & 0 \\ \beta b_{21}c^{(0)}_{2222} & i(\beta b_{22}c^{(0)}_{2222} - \gamma p) & 0 \\ 0 & 0 & i\beta b_{33}c^{(0)}_{3232} - i\gamma p \end{bmatrix}, \quad (72)$$

$$[E^{[g]}_{ik}] = \begin{bmatrix} \frac{i(\beta b_{22}c'_{2222}-\gamma p)}{D} & \frac{-(\beta b_{12}c'_{1212})}{D} & 0 \\ \frac{-(\beta b_{21}c'_{2222})}{D} & \frac{i(\beta b_{11}c'_{1212}-\gamma p)}{D} & 0 \\ 0 & 0 & \frac{1}{(i\beta b_{33}c'_{3232}-i\gamma p)} \end{bmatrix}, \quad (73)$$

where

$$D = -(\beta b_{11} c^{(0)}_{1212} - \gamma p)(\beta b_{22} c^{(0)}_{2222} - \gamma p) - (\beta b_{21} c^{(0)}_{2222})(\beta b_{12} c^{(0)}_{1212}) \tag{74}$$

Referred to the non-dimensional variables

$$x' = x_1/a, \quad p' = pa, \quad c'_{ijkl} = c^{(0)}_{ijkl}/C, \quad b'_{ij} = b_{ij}C,$$
$$\beta' = \beta a, \quad P'_i = P^{(0)}_i/C, \quad B'_{kr} = B_{kr}C, \quad E'_{rj}(p') = E_{rj}a \tag{75}$$

equation (70) may be written in the form

$$u_k(x', 0)/a = \frac{2P'_j}{\pi\gamma} \Re \left[B'_{kr} \int_0^\infty E'_{rj}(p') \frac{\sin p'}{p'} \exp(ip'x'_1) dp \right]. \tag{76}$$

where C is a reference pressure and γ is a non-dimensional constant.

With the assumed elastic symmetry the plane and antiplane problems separate out and, from equation (76), the antiplane displacement on the surface $x_2 = 0$ is given by

$$u_3(x', 0)/aP'_3 = \frac{2}{\pi\gamma} \left[b'_{33} \int_0^\infty \frac{1}{\beta' b'_{33} c'_{3232} - \gamma p'} \frac{\sin p'}{p'} \cos(p'x') dp' \right], \tag{77}$$

In the case of a normal load with $P'_1 = 0$ and $P'_3 = 0$ the normal displacement is given by

$$u_2(x', 0)/aP'_2 = \frac{-2}{\pi\gamma} \int_0^\infty F(p') \frac{\sin p'}{p'} \cos(p'x') dp', \tag{78}$$

where

$$F(p') = \frac{-[b'_{21}(\beta' b'_{12} c'_{1212}) + b'_{22}(\beta' b'_{11} c'_{2222} - \gamma p')]}{(\beta' b'_{11} c'_{1212} - \gamma p')(\beta' b'_{22} c'_{2222} - \gamma p') + \beta' b'_{21} c'_{2222} \beta' b'_{12} c'_{1212}}. \tag{79}$$

Let

$$\beta' = \frac{m'}{\gamma} \tag{80}$$

where m' is a constant. In terms of m' and γ equation (5) together with (13) (with $\alpha = 0$) yields the elastic parameters in the form

$$\frac{c_{ijkl}}{C} = c'_{ijkl} \left[\gamma + \frac{m'}{\gamma} \frac{x_2}{a} \right] \tag{81}$$

and the antiplane surface displacement in the form

$$u_3(x', 0)/aP'_3 = \frac{2}{\pi} \left[b'_{33} \int_0^\infty \frac{1}{m' b'_{33} c'_{3232} - \gamma^2 p'} \frac{\sin p'}{p'} \cos(p'x') dp' \right], \tag{82}$$

For a fixed m' as $\gamma \to 0$

$$\begin{aligned} u_3(x', 0)/aP'_3 \quad &\to \quad \frac{2}{\pi} \int_0^\infty \frac{1}{m' c'_{3232}} \frac{\sin p'}{p'} \cos(p'x') dp' \\ &= \quad \begin{cases} \frac{1}{m' c'_{3232}} & \text{for} \quad |x'| < 1, \\ 0 & \text{for} \quad |x'| > 1. \end{cases} \end{aligned} \tag{83}$$

For the purpose of obtaining some numerical values for the antiplane surface displacement the relevant constants c'_{ijkl} for a particular transversely isotropic material with the x_1 axis normal to the transverse planes are chosen as $c'_{1111} = 18.1$, $c'_{2222} = 16.2$, $c'_{1212} = c'_{1122} = c'_{1133} = c'_{1313} = 6.9$, $c'_{3232} = 5.4$. With these values of the c'_{ijkl} the analysis of section 3 may be used to determine the values of b'_{ij} as $b'_{11} = --0.1227$, $b'_{12} = 0.0416$, $b'_{21} = -0.01426$, $b'_{22} = -0.1297$ and $b'_{33} = -0.1638$.

For the material with these constants the the antiplane surface displacement $u_3(x', 0)/aP'_3$ given by (82) is plotted in Figure 1 for various values of β' and γ with $\beta'\gamma = m' = 1$. The plotted values indicate the movement as $\gamma \to 0$ of the surface displacement towards the values given by (83) in which in this particular case $1/(m'c'_{3232}) = 1/5.4 = 0.1852$.

Similarly in terms of m' and γ equation (78) yields the normal surface displacement in the form

$$u_2(x', 0)/aP'_2 = \frac{-2}{\pi} \int_0^\infty F'(p') \frac{\sin p'}{p'} \cos(p'x')dp', \tag{84}$$

where

$$F'(p') = \frac{-[b'_{21}(m'b'_{12}c'_{1212}) + b'_{22}(m'b'_{11}c'_{1212} - \gamma^2 p')]}{(m'b'_{11}c'_{1212} - \gamma^2 p')(m'b'_{22}c'_{2222} - \gamma^2 p') + m'^2 b'_{21}c'_{2222}b'_{12}c'_{1212}}. \tag{85}$$

For a fixed m' as $\gamma \to 0$

$$\begin{aligned} u_2(x', 0)/aP'_2 \quad &\to \quad \frac{2}{m'c'_{2222}\pi} \int_0^\infty \frac{\sin p'}{p'} \cos(p'x')dp' \\ &= \begin{cases} \frac{1}{m'c'_{2222}} & \text{for} \quad |x'| < 1, \\ 0 & \text{for} \quad |x'| > 1. \end{cases} \end{aligned} \tag{86}$$

For the purpose of obtaining some numerical values for the normal surface displacement the constants c'_{ijkl} for a particular transversely isotropic material are chosen as $c'_{1111} = 5$, $c'_{2222} = 16.2$, $c'_{1212} = c'_{1122} = c'_{1133} = c'_{1313} = 2$, $c'_{3232} = 5.4$. With these values of the c'_{ijkl} the values of b_{ij} are $b'_{11} = --0.3787$, $b'_{12} = 0.0909$, $b'_{21} = -0.0909$ $b'_{22} = -0.2104$ and $b'_{33} = -0.3043$.

For the material with these constants the surface displacement $u_2(x', 0)/aP'_2$ is plotted in Figure 2 for various values of β' and γ with $\beta'\gamma = m' = 1$. The plotted values indicate the movement as $\gamma \to 0$ of the surface displacement towards the values given by (86) in which in this particular case $1/(m'c'_{2222}) = 1/16.2 = 0.0617$.

6. An elastic layer on a rigid foundation

Consider an inhomogeneous elastic layer occupying the region $0 < x_2 < h$ with the stress vector P_i prescribed on the boundary $x_2 = 0$. The displacement and stress fields are required throughout the layer. The layer adheres to a rigid foundation on $x_2 = h$. The boundary conditions are

$$P_i(x_1, 0) = P_i(x_1), \tag{87}$$

$$u_i(x_1, h) = 0, \tag{88}$$

where the $P_i(x_1), i = 1, 2, 3$ are given functions of x_1. For this problem the representation (35)-(39) is useful with $\chi_j(z)$ given by

$$\chi_j(z) = \frac{1}{2\pi}\int_0^\infty [G_j(p)\exp(ipz) + H_j(p)\exp(-ipz)]\,dp \text{ for } j = 1, 2, 3, \tag{89}$$

where the $G_j(p)$ and $H_j(p)$, $j = 1, 2, 3$ are functions of p which will be determined by the boundary conditions. From (38), (87) and (89) it follows that

$$\frac{-1}{\pi}\Re\int_0^\infty [(P_{ik}^{[g]}B_{kr} + ig^{1/2}p\delta_{ir})G_r(p) + (\overline{P}_{ik}^{[g]}\overline{B}_{kr} + ig^{1/2}p\delta_{ir})\overline{H}_r(p)]\exp(ipx_1)dp = P_i(x_1). \tag{90}$$

From (37), (88) and (89) it follows that

$$\frac{1}{\pi g^{1/2}}\Re\sum_{\alpha=1}^{3}\int_0^\infty [(A_{k\alpha}M_{\alpha j})G_j(p)\exp(i\tau_\alpha ph) - (\overline{A}_{k\alpha}\overline{M}_{\alpha j})\overline{H}_j(p)\exp(i\overline{\tau}_\alpha ph)]\exp(ipx_1)dp = 0. \tag{91}$$

Equation (91) may be written in the form

$$\frac{1}{\pi g^{1/2}}\Re\int_0^\infty [R_{kj}(p)G_j(p) - S_{kj}(p)\overline{H}_j(p)]\exp(ipx_1)dp = 0, \tag{92}$$

where

$$R_{kj}(p) = \sum_{\alpha=1}^{3} A_{k\alpha}M_{\alpha j}\exp(i\tau_\alpha ph), \tag{93}$$

$$S_{kj}(p) = \sum_{\alpha=1}^{3} \overline{A}_{k\alpha}\overline{M}_{\alpha j}\exp(i\overline{\tau}_\alpha ph). \tag{94}$$

Equation (92) will be satisfied if

$$\overline{H}_j(p) = -T_{jn}(p)R_{nl}(p)G_l(p), \tag{95}$$

where

$$S_{kj}(p)T_{jn}(p) = \delta_{kn}. \tag{96}$$

Substitution of (95) in (90) provides

$$\frac{1}{\pi}\Re\int_0^\infty [F_{ir}(p)G_r(p)]\exp(ipx_1)dp = P_i(x_1), \tag{97}$$

where

$$F_{ir}(p) = -(P_{ik}^{[g]}B_{kr} + ig^{1/2}p\delta_{ir}) - (\overline{P}_{ik}^{[g]}\overline{B}_{km} + ig^{1/2}p\delta_{im})T_{mk}(p)R_{kr}(p). \quad (98)$$

Use of the inversion formula for Fourier transforms now provides

$$F_{ir}(p)G_r(p) = \int_{-\infty}^{\infty} P_i(\xi)\exp(-ip\xi)d\xi. \quad (99)$$

If $P_i(x_1)$ satisfies (68) then

$$\begin{aligned} F_{ir}(p)G_r(p) &= P_i^{(0)}\int_{-a}^{a}\exp(-ip\xi)d\xi \\ &= 2P_i^{(0)}\frac{\sin pa}{p}. \quad (100) \end{aligned}$$

Hence

$$G_n(p) = 2U_{ni}(p)P_i^{(0)}\frac{\sin pa}{p}, \quad (101)$$

where

$$U_{ni}(p)F_{ir}(p) = \delta_{nr}. \quad (102)$$

Hence

$$\overline{H}_j(p) = -2T_{jk}(p)R_{kl}(p)U_{li}(p)P_i^{(0)}\frac{\sin pa}{p}. \quad (103)$$

The surface displacement is given by

$$\begin{aligned} u_k(x_1,0) &= \Re\frac{1}{\pi\gamma}\int_0^{\infty}[B_{kr}G_r(p) - \overline{B}_{kr}\overline{H}_r(p)]\exp(ipx_1)dp \\ &= \Re\frac{2P_i^{(0)}}{\pi\gamma}\int_0^{\infty}[B_{kn} - \overline{B}_{kr}T_{rl}(p)R_{ln}(p)]U_{ni}(p)\frac{\sin pa}{p}\exp(ipx_1)dp. \quad (104) \end{aligned}$$

With the same elastic symmetry as in the previous section the plane and antiplane parts of the problem separate with the antiplane surface displacement given by

$$u_3(x_1,0) = \Re\frac{2}{\pi\gamma}\int_0^{\infty}[B_{33} - \overline{B}_{33}T_{33}R_{33}]U_{33}P_3^{(0)}\frac{\sin pa}{p}\exp(ipx_1)dp, \quad (105)$$

where

$$\begin{aligned} B_{33} &= ib_{33}, & (106) \\ R_{33} &= ib_{33}\exp(i\tau_1 ph), & (107) \\ S_{33} &= -ib_{33}\exp(i\overline{\tau}_1 ph), & (108) \\ T_{33} &= 1/(S_{33}) = i(b_{33})^{-1}\exp(-i\overline{\tau}_1 ph), & (109) \\ F_{33} &= (i\beta c_{3232}^{(0)}b_{33} - i\gamma p) + (i\beta c_{3232}^{(0)}b_{33} - i\gamma p)\exp(i(\tau_1 - \overline{\tau}_1)ph), & (110) \\ U_{33} &= 1/F_{33}. & (111) \end{aligned}$$

Use of (106) - (111) in (105) provides

$$u_3(x_1,0) = \frac{2b_{33}}{\pi}\int_0^\infty \frac{[1-\exp(i(\tau_1-\overline{\tau}_1)ph)]\sin(pa)\cos(px_1)P_3^{(0)}}{(\beta c_{3232}^{(0)}b_{33}-\gamma p)[1+\exp(i(\tau_1-\overline{\tau}_1)ph)]p}dp, \quad (112)$$

Referred to the non-dimensional coordinates of the previous section and with $h = ah'$ equation (112) becomes

$$\frac{u_3(x',0)}{aP_3'} = \frac{2b_{33}'}{\pi\gamma}\int_0^\infty \frac{[1-\exp(i(\tau_1-\overline{\tau}_1)p'h')]\sin(p')\cos(p'x')}{(\beta' b_{33}' c_{3232}'-\gamma p')[1+\exp(i(\tau_1-\overline{\tau}_1)p'h')]p'}dp'. \quad (113)$$

Setting $\beta' = m'/\gamma$ where m' is a non-dimensional positive constant equation (113) yields

$$\frac{u_3(x',0)}{aP_3'} = \frac{2b_{33}'}{\pi}\int_0^\infty \frac{[1-\exp(i(\tau_1-\overline{\tau}_1)p'h')]\sin(p')\cos(p'x')}{(m' b_{33}' c_{3232}'-\gamma^2 p')[1+\exp(i(\tau_1-\overline{\tau}_1)p'h')]p'}dp'. \quad (114)$$

Since the imaginary part of τ_1 is chosen to be positive it follows from equation (112) that as $\gamma \to 0$ and $h \to \infty$

$$\begin{aligned}\frac{u_3(x_1,0)}{aP_3'} &\to \frac{2}{\pi}\int_0^\infty \frac{\sin(p')\cos(p'x')}{m' c_{3232}' p'}dp', \\ &= \begin{cases} \frac{1}{m' c_{3232}'} & \text{for} \quad |x'| < 1 \\ 0 & \text{for} \quad |x'| > 1 \end{cases} \end{aligned} \quad (115)$$

7. An isotropic half-space and layer on a rigid foundation

For isotropic materials the non-zero elastic coefficients c_{ijkl} may be expressed in terms of the two Lamé coefficients λ and μ where

$$\lambda = \lambda^{(0)}g(\mathbf{x}), \qquad \mu = \mu^{(0)}g(\mathbf{x}), \quad (116)$$

where $\lambda^{(0)}$ and $\mu^{(0)}$ are constants. The relevant coefficients c_{ijkl} are related to λ and μ by the equations

$$c_{1111} = c_{2222} = c_{3333} = \lambda + 2\mu, \quad (117)$$
$$c_{1122} = c_{1133} = c_{2233} = \lambda, \quad (118)$$
$$c_{1212} = c_{1313} = c_{2323} = \mu. \quad (119)$$

The coefficients $c_{ijkl}^{(0)}$ may be expressed in terms of the two constants $\lambda^{(0)}$ and $\mu^{(0)}$ by the equations

$$c_{1111}^{(0)} = c_{2222}^{(0)} = c_{3333}^{(0)} = \lambda^{(0)} + 2\mu^{(0)}, \quad (120)$$
$$c_{1122}^{(0)} = c_{1133}^{(0)} = c_{2233}^{(0)} = \lambda^{(0)}, \quad (121)$$
$$c_{1212}^{(0)} = c_{1313}^{(0)} = c_{2323}^{(0)} = \mu^{(0)}. \quad (122)$$

The coefficients c_{ijkl} must satisfy the condition (6) which requires that $c_{ijkl} = c_{ilkj}$ so that from (118), (119), (121) and (122) it follows that

$$\lambda = \mu, \qquad \lambda^{(0)} = \mu^{(0)}. \tag{123}$$

Young's modulus E and Poisson's ratio ν are related to the Lamé coefficients by the equations

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{124}$$

Hence, by virtue of equation (123) it follows that

$$E = \frac{5\lambda}{2}, \qquad \nu = \frac{1}{4}. \tag{125}$$

Thus the analysis of the previous two sections may be applied to an inhomogeneous isotropic material for which the Poisson's ratio ν is 1/4 (a frequently occurring value in rock materials - see Turcotte [10]) and the inhomogeneity is specified by either Young's modulus or one of the Lamé constants in the forms

$$E = E^{(0)}(\gamma + \beta x_2), \qquad \lambda = \lambda^{(0)}(\gamma + \beta x_2), \qquad \mu = \mu^{(0)}(\gamma + \beta x_2), \tag{126}$$

where these alternative forms are related by

$$E = 5\lambda/2, \qquad \lambda = \mu, \tag{127}$$

so that

$$E^{(0)} = 5\lambda^{(0)}/2, \qquad \lambda^{(0)} = \mu^{(0)}. \tag{128}$$

For the isotropic case the sextic (18) has equal roots. Also the plane and antiplane parts of the problem uncouple to form two separate problems. In the plane case the two rows of the matrices $[A_{k\alpha}]$ and $[L_{i2\alpha}]$ associated with the plane problem are linearly dependent and hence their inverses do not exist. Thus for the plane problems the analysis of the previous three sections is no longer valid. In order to employ the analysis of the previous sections to obtain numerical results for the isotropic case the numerical values of the isotropic elastic moduli are perturbed slightly from their exact isotropic values in order to obtain distinct roots for the sextic (18). Specifically the relevant elastic parameters c'_{ijkl} are chosen in the form

$$c'_{1111} = c'_{3333} = 3\lambda', \tag{129}$$
$$c'_{2222} = 3\lambda' + \epsilon, \tag{130}$$
$$c'_{1122} = c'_{1133} = c'_{2233} = \lambda', \tag{131}$$
$$c'_{1212} = c'_{1313} = c'_{2323} = \lambda'. \tag{132}$$

where $\lambda' = \lambda^{(0)}/C$ and ϵ is a small non-dimensional constant. As $\epsilon \to 0$ numerical values obtained from the relevant equations for the displacement and stress vector in the previous three sections tend to the values for an isotropic material with Poisson's ratio 1/4.

For illustrative purposes the values of the constants in (129) to (132) are chosen to be $\lambda = 5.4$ and $\epsilon = 0.001$. With this choice the analysis of section 3 yields the numerical values

$\tau_1 = i, \quad b_{11} = -0.13888, \quad b_{12} = 0.04629,$

$b_{21} = -0.04629, \ ; \ b_{22} = -0.138888, \ b_{33} = -0.18518.$

Numerical values of the surface normal displacement for an inhomogeneous isotropic half-space with the above elastic moduli were calculated using equation (84). The results are shown in Figure 3. They may be compared with the corresponding results for an anisotropic half space displayed in Figure 2. The results in Figure 2 are for an anisotropic material whose constants c'_{ijkl} are all less than or equal to the corresponding isotropic constants as given by equations (129) to (132) with $\lambda = 5.4$. For the anisotropic material $c'_{2222} = 16.2$ while from (130) $c'_{2222} = 16.201$ in the isotropic case. Thus from equation (86) both the anisotropic and isotropic surface displacement could be expected to be virtually identical as $\gamma \to 0$. The Figures 2 and 3 verify that this is the case. For larger values of γ the lower elastic moduli in the anisotropic case give rise to a larger surface displacement than for the corresponding isotropic half-space.

Numerical values of the surface antiplane displacement for an isotropic layer on a rigid base with the above elastic moduli were calculated using equation (114) for various values of the layer thickness and with $m' = \beta\gamma = 1$ and $\gamma = 0.0001$. The results are shown in Figure 4. The results indicate that for sufficiently small layer thickness the antiplane surface displacement is positive over the contact region and increases as $|x'|$ increases from zero to one. Outside the contact region the surface displacement becomes negative and tends towards zero as $|x'| \to 0$. For this isotropic material the soft upper layers can easily move laterally and under the specified boundary conditions on $x_2 = 0$ and $x_2 = h$ this capability facilitates a sharp change in the antiplane displacement from a small positive value to a smaller negative value in the vicinity of the edge of the loaded region. This feature of the displacement profiles in Figure 4 is similar to profiles obtained by Gibson, Brown and Andrews [2] for a related problem involving an incompressible isotropic half-space with a linearly varying elastic modulus.

8. Final remarks

Some contact problems have been considered for an anisotropic half-space and layer in which the elastic moduli vary quadratically with the spatial variables. For the class of problems considered formulas for the displacement are given in either closed form or in integral formulations which readily yield numerical values for particular problems and provide closed form formulas in limiting cases. The analysis can be used to consider the corresponding contact problems for isotropic materials as a particular case of the general anisotropic analysis. The problems considered exhibit a number of characteristics which are similar to the features observed by Gibson [1] and Gibson, Brown and Andrews [2] for the corresponding problems for an inhomogeneous incompressible isotropic half space and layer in which the elastic moduli vary linearly with a spatial variable.

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List of Figures

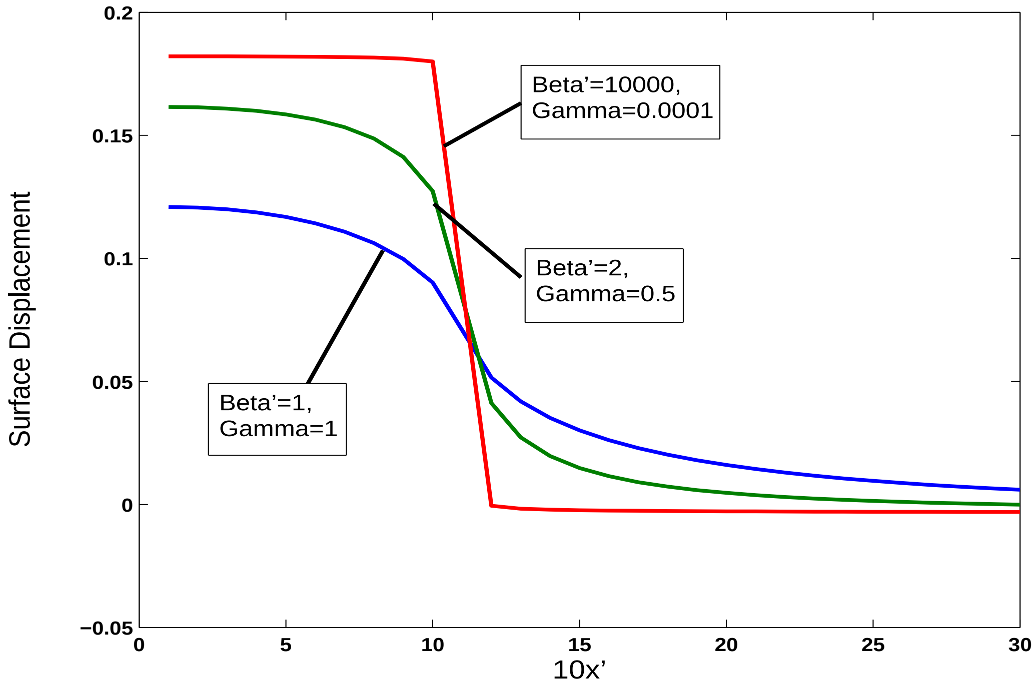

Figure 1: Antiplane surface displacement $u_3(x', 0)/aP'_3$.

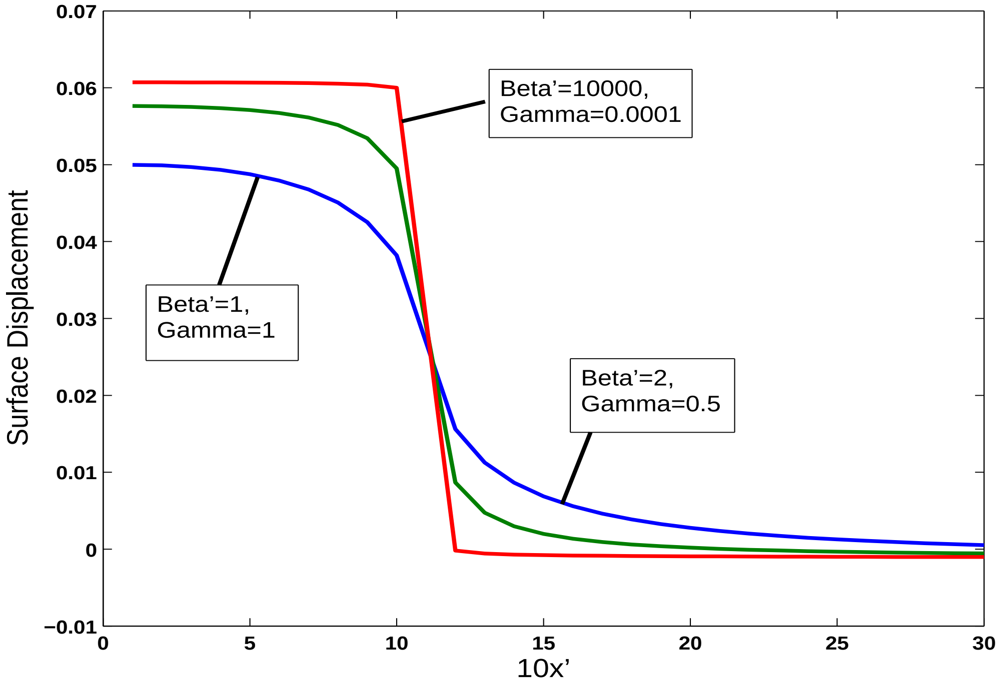

Figure 2: Normal surface displacement $u_2(x', 0)/aP_2'$.

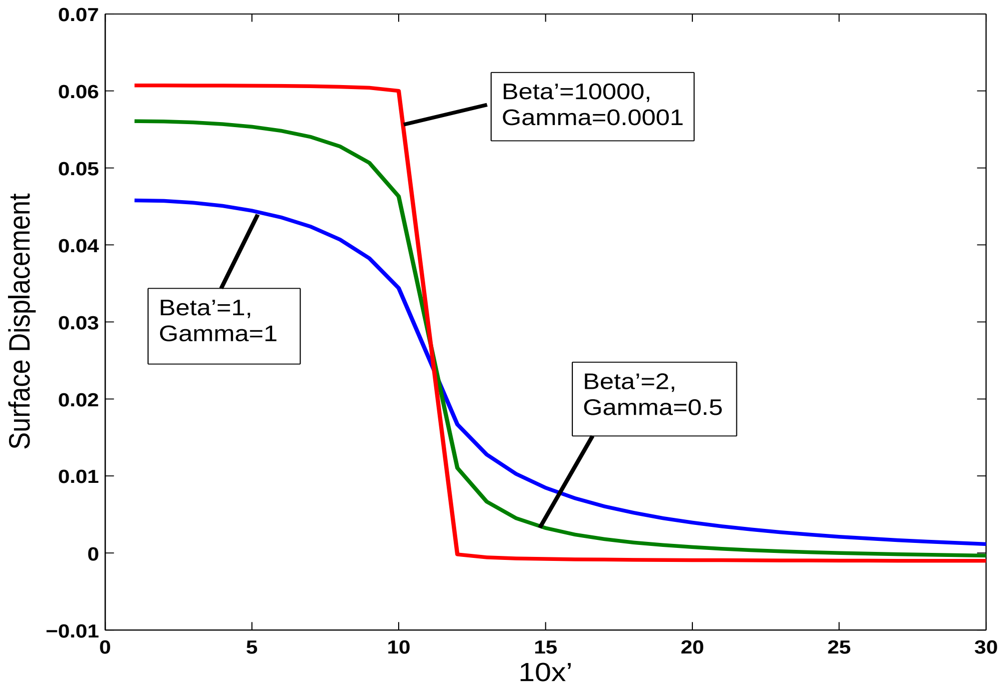

Figure 3: Normal surface displacement $u_2(x', 0)/aP'_2$ for an isotropic material.

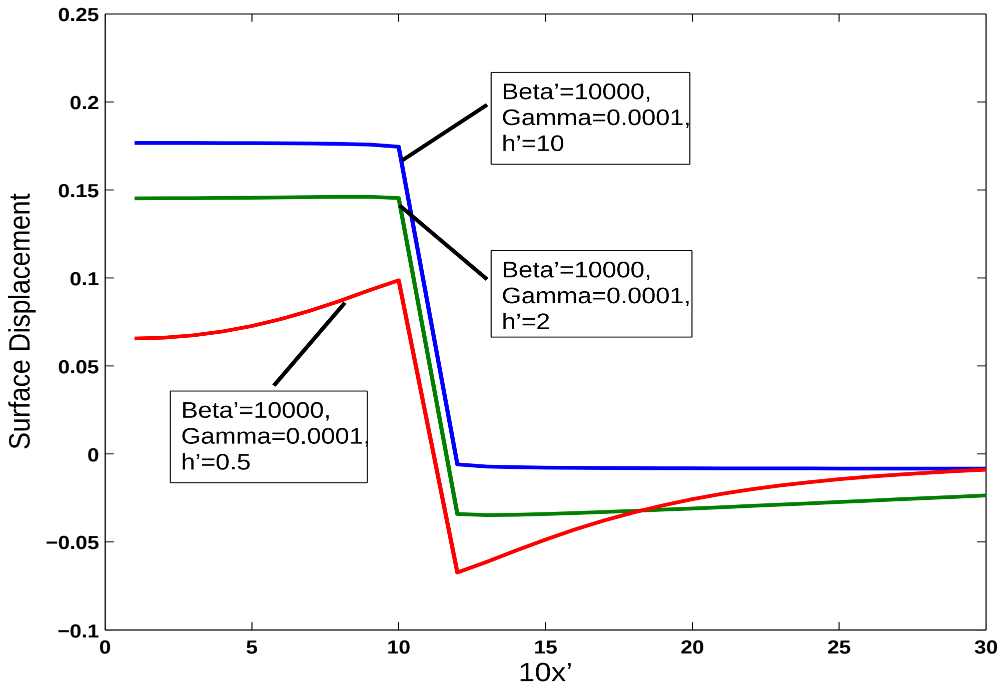

Figure 4: Antiplane surface displacement $u_3(x', 0)/aP_3'$ for an isotropic material.