This is a preprint of an article to appear in International J. Engng Sci.
More details at: http://dx.doi.org/10.1016/j.ijengsci.2005.10.001

On a generalised plane strain crack problem for inhomogeneous anisotropic elastic materials

David L. Clements [a,*], W.T. Ang [b]

[a] *School of Mathematics, University of Adelaide, SA 5005, Australia*

[b] *School of Mechanical and Aerospace Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798*

Abstract

A generalised plane strain crack problem is considered for a class of inhomogeneous anisotropic elastic materials. The problem is reduced to a boundary integral equation involving hypersingular integrals. The boundary integral equation may be solved numerically using standard procedures. Some crack problems for a particular inhomogeneous material are considered in detail and the stress intensity factors are obtained in order to assess the effect of the anisotropy and inhomogeneity on the stress field near the crack tips.

Key words: Boundary element method, cracks, anisotropy, hypersingular integrals, stress intensity factors.

PACS: : 62.20.Dc, 62.20.Mk

* Corresponding author

Email addresses: `david.clements@adelaide.edu.au` (David L. Clements), `mwtang@ntu.edu.sg` (W.T. Ang).

1 Introduction

The study of crack problems for inhomogeneous materials has received considerable attention in recent decades. This interest is related to the extensive use of composite materials in various engineering applications. In this connection many of the studies have been concerned with materials which are made up of two or more homogeneous parts and many problems have now been solved for materials of this type (see for example England [1], Rice and Sih [2] and Clements [3]). In comparison crack problems for materials in which the elastic moduli vary continuously with the spatial coordinates have received less attention. To some extent this is due to the inherent difficulties in solving boundary value problems for materials of this type. However in recent years some progress has been made with the analytical solution of particular problems for a restricted class of inhomogeneous materials (see for example Ang and Clements [4], Erdogan and Ozturk [5], Chen and Erdogan [6] and Clements, Ang and Kusuma [7]).

The current study is concerned with the solution of a crack problem for a class of inhomogeneous anisotropic elastic materials under generalised plane strain. The elastic moduli are assumed to vary continuously with two Cartesian coordinates. A boundary integral formulation is used to obtain a solution to the governing differential equations and this is then applied to the relevant crack problem. For certain variations in the elastic modulus this boundary integral equation has a relatively simple form and this case is used to obtain numerical results for a particular crack problem.

2 Statement of the problem

Referred to a Cartesian frame $Ox_1x_2x_3$ consider an anisotropic elastic body with a geometry that does not vary in the Ox_3 direction. Let the body occupy the region Ω with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves in the Ox_1x_2 plane. The material contains a plane crack which does not intersect with the boundary. The crack is defined to start at coordinates $\mathbf{A} = (a, b)$ and end at $\mathbf{B} = (c, d)$. The outer boundary is denoted by C and the crack surface will be referred to as D. Either the displacement or traction is specified at each point of the outer boundary C. The specified boundary displacement or traction on C is such that the crack opens and hence the crack faces D are taken to be traction free. The problem is to determine the stress and displacement throughout the elastic material, and to obtain values for the stress intensity factors at the tips of the plane crack.

3 Basic equations

The equilibrium equations governing small generalised plane deformations of an inhomogeneous anisotropic elastic material may be written in the form

$$\frac{\partial}{\partial x_j}\left[c_{ijkl}(\mathbf{x})\,\frac{\partial u_k(\mathbf{x})}{\partial x_l}\right] = 0, \tag{1}$$

where $i, j, k, l =$ 1,2,3, $\mathbf{x} = (x_1, x_2)$, u_k denotes the displacement, $c_{ijkl}(\mathbf{x})$ the elastic moduli and the repeated summation convention (summing from 1 to 3) is used for repeated Latin suffices. The stress displacement relations are given by

$$\sigma_{ij}(\mathbf{x}) = c_{ijkl}\,\frac{\partial u_k}{\partial x_l} \tag{2}$$

and the traction vector P_i on the boundary $\partial\Omega$ is defined as

$$P_i(\mathbf{x}) = \sigma_{ij}\,n_j = c_{ijkl}\,\frac{\partial u_k}{\partial x_l}\,n_j, \tag{3}$$

where $\mathbf{n} = (n_1, n_2)$ denotes the outward pointing normal to the boundary $\partial\Omega$.

For all points in Ω the coefficients $c_{ijkl}(\mathbf{x})$ are required to satisfy the usual symmetry condition

$$c_{ijkl} = c_{ijlk} = c_{jikl} = c_{klij} \tag{4}$$

and also sufficient conditions for the strain energy density to be positive. This requirement ensures that the system of partial differential equations is elliptic throughout Ω.

On the boundary C the displacement u_k is specified on $C^{(1)}$ and the traction P_i is specified on $C^{(2)}$ where $C = C^{(1)} \cup C^{(2)}$. Also the traction P_i is zero on D. A solution to (1) is sought which is valid in Ω and satisfies the specified boundary conditions on $\partial\Omega = C \cup D$.

4 Boundary integral equation

The coefficients in (1) are required to take the form

$$c_{ijkl}(\mathbf{x}) = c^{(0)}_{ijkl}\,g(\mathbf{x}), \tag{5}$$

where the $c^{(0)}_{ijkl}$ are constants and $g(x_1, x_2)$ is a twice differentiable function of the variables x_1 and x_2. Also in addition to the symmetry condition (4) the $c^{(0)}_{ijkl}$ are required to satisfy the additional condition

$$c^{(0)}_{ijkl} = c^{(0)}_{ilkj}. \tag{6}$$

Equation (1) may now be written in the form

$$c^{(0)}_{ijkl} \frac{\partial}{\partial x_j}\left(g \frac{\partial u_k}{\partial x_l}\right) = 0. \tag{7}$$

Following Azis and Clements [8] consider a transformation of the dependent variables in the form

$$u_k = g^{-1/2}\, \psi_k. \tag{8}$$

Use of (8) in (7) provides the equation

$$g^{1/2}\, c^{(0)}_{ijkl} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} + c^{(0)}_{ijkl} \frac{\partial g^{1/2}}{\partial x_j} \frac{\partial \psi_k}{\partial x_l} - c^{(0)}_{ijkl} \frac{\partial g^{1/2}}{\partial x_l} \frac{\partial \psi_k}{\partial x_j} - \psi_k\, c^{(0)}_{ijkl} \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = 0, \tag{9}$$

where by virtue of (6) this equation reduces to

$$g^{1/2}\, c^{(0)}_{ijkl} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} - \psi_k\, c^{(0)}_{ijkl} \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = 0. \tag{10}$$

Thus if

$$c^{(0)}_{ijkl} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} = 0 \tag{11}$$

and

$$c^{(0)}_{ijkl} \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = 0, \tag{12}$$

then (10) will be satisfied. Thus when g satisfies the system (12) the transformation given by (8) transforms the linear system with variable coefficients (7) to the linear system with constant coefficients (11).

As a result of the symmetry property $c_{ijkl} = c_{klij}$ equation (12) consists of a system of six constant coefficients partial differential equations in the one dependent variable $g^{1/2}$. In general this system will be satisfied by a linear function of the two independent variables x_1, x_2. Thus $g(\mathbf{x})$ may be taken in the form

$$g(\mathbf{x}) = (\alpha x_1 + \beta x_2 + \gamma)^2 , \tag{13}$$

where α, β and γ are constants which may be used to fit the elastic moduli $c_{ijkl}(\mathbf{x}) = c^{(0)}_{ijkl}\, g(\mathbf{x})$ to given numerical data.

Now substitution of (5) and (8) into (3) yields

$$P_i = -P^{[g]}_{ik}\, \psi_k + P^{[\psi]}_i\, g^{1/2}, \tag{14}$$

where

$$P^{[g]}_{ik}(\mathbf{x}) = c^{(0)}_{ijkl} \frac{\partial g^{1/2}}{\partial x_l}\, n_j, \tag{15}$$

$$P^{[\psi]}_i(\mathbf{x}) = c^{(0)}_{ijkl} \frac{\partial \psi_k}{\partial x_l}\, n_j. \tag{16}$$

A boundary integral equation for the solution of (11) with ψ_i given on $\partial\Omega_1$ and $P^{[\psi]}_i$ given on $\partial\Omega_2$ may be written in the form (see Clements [10])

$$\eta\, \psi_m(\mathbf{x}_0) = -\int_{\partial\Omega} \left[P^{[\psi]}_i(\mathbf{x})\, \Phi_{im}(\mathbf{x}, \mathbf{x}_0) - \psi_i(\mathbf{x})\, \Gamma_{im}(\mathbf{x}, \mathbf{x}_0) \right] ds(\mathbf{x}), \tag{17}$$

for $m = 1, 2, 3$, where $\mathbf{x}_0$ is the source point, $\eta = 0$ if $\mathbf{x}_0 \notin \Omega$, $\eta = 1$ if $\mathbf{x}_0 \in \Omega$ and $\eta = \frac{1}{2}$ if $\mathbf{x}_0 \in \partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at $\mathbf{x}_0$. The Φ_{im} in (17) is any solution of the equation

$$c_{ijkl}^{(0)} \frac{\partial^2 \Phi_{im}(\mathbf{x}, \mathbf{x}_0)}{\partial x_j \partial x_l} = \delta_{km} \delta(\mathbf{x} - \mathbf{x}_0) \tag{18}$$

and the Γ_{im} is given by

$$\Gamma_{im} = c_{ijkl}^{(0)} \frac{\partial \Phi_{km}}{\partial x_l} n_j. \tag{19}$$

For generalised plane problems with $\mathbf{x}_0 = (\xi_1, \xi_2,\ \mathbf{x} = (x_1, x_2)$, Φ_{im} and Γ_{im} are given by (see Clements [10] and Clements and Jones [9])

$$\Phi_{im}(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \Re \left[\sum_{\alpha=1}^{3} A_{i\alpha} N_{\alpha k} \log(z_\alpha - c_\alpha) \right] d_{km}, \tag{20}$$

$$\Gamma_{im}(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \Re \left[\sum_{\alpha=1}^{3} L_{ij\alpha} N_{\alpha k} (z_\alpha - c_\alpha)^{-1} \right] n_j d_{km}, \tag{21}$$

where $\Re$ denotes the real part of a complex number, $z_\alpha = x_1 + \tau_\alpha x_2$ and $c_\alpha = \xi_1 + \tau_\alpha \xi_2$, where τ_α are the three roots with positive imaginary part of the sextic in τ

$$|c_{i1k1}^{(0)} + c_{i2k1}^{(0)} \tau + c_{i1k2}^{(0)} \tau + c_{i2k2}^{(0)} \tau^2| = 0. \tag{22}$$

The $A_{i\alpha}$ occurring in (20) are the solutions of the system

$$\left(c_{i1k1}^{(0)} + c_{i2k1}^{(0)} \tau_\alpha + c_{i1k2}^{(0)} \tau_\alpha + c_{i2k2}^{(0)} \tau_\alpha^2 \right) A_{k\alpha} = 0. \tag{23}$$

Also the $N_{\alpha k}$, $L_{ij\alpha}$ and d_{km} are defined by

$$\delta_{ik} = \sum_{\alpha=1}^{3} A_{i\alpha} N_{\alpha k}, \tag{24}$$

$$L_{ij\alpha} = (c^{(0)}_{ijk1} + \tau_\alpha c^{(0)}_{ijk2}) A_{k\alpha}, \tag{25}$$

$$\delta_{im} = -\frac{1}{2}\imath \sum_{\alpha=1}^{3} \left\{ L_{i2\alpha} N_{\alpha k} - \overline{L}_{i2\alpha} \overline{N}_{\alpha k} \right\} d_{km}, \tag{26}$$

where the bar denotes the complex conjugate and $\imath$ denotes the square root of minus one.

5 Solution of the problem

Use of (8) and (14) in (17) yields

$$\eta\, g^{1/2}(\mathbf{x}_0)\, u_m(\mathbf{x}_0) = -\int_{\partial\Omega} \Big\{ P_i(\mathbf{x}) \left[g^{-1/2}(\mathbf{x})\, \Phi_{im}(\mathbf{x}, \mathbf{x}_0) \right] - u_i(\mathbf{x}) \left[g^{1/2}(\mathbf{x})\, \Gamma_{im}(\mathbf{x}, \mathbf{x}_0) - P^{[g]}_{ki}(\mathbf{x})\, \Phi_{km}(\mathbf{x}, \mathbf{x}_0) \right] \Big\}\, ds(\mathbf{x}). \tag{27}$$

This equation provides a boundary integral equation which may be used to construct a system of linear equations to solve for the unknown u_m or P_m on the boundary, and then the values of $P_m(\mathbf{x})$ and $u_m(\mathbf{x})$ may be calculated at any point in Ω.

On the crack the coordinates x_1 and x_2 may be written in terms of a single parameter t in the form

$$x_1 = X_1(t) = [(c-a)t + (c+a)]/2 \ \text{ for } \ t \in [-1, 1], \tag{28}$$

$$x_2 = X_2(t) = [(d-b)t + (b+d)]/2 \ \text{ for } \ t \in [-1, 1]. \tag{29}$$

Thus, since the tractions P_i are zero over the crack faces, equation (27) provides

$$\eta\, g^{1/2}(\mathbf{x}_0)\, u_k(\mathbf{x}_0) = -\int_C \Big\{ P_n(\mathbf{x}) g^{-1/2}(\mathbf{x})\, \Phi_{nk}(\mathbf{x}, \mathbf{x}_0)$$

$$-u_n(\mathbf{x})\left[g^{1/2}(\mathbf{x})\,\Gamma_{nk}(\mathbf{x},\mathbf{x}_0)-P^{[g]}_{sn}(\mathbf{x})\,\Phi_{sk}(\mathbf{x},\mathbf{x}_0)\right]\Big\}\,ds(\mathbf{x})$$

$$+\frac{L}{2}\int_{-1}^{1}\left[g^{1/2}(\mathbf{X}(t))\,\Gamma_{nk}(\mathbf{X}(t),\mathbf{x}_0)\right.$$

$$\left.-P^{[g]}_{sn}(\mathbf{X}(t))\,\Phi_{sk}(\mathbf{X}(t),\mathbf{x}_0)\right]\Delta w_n(t)dt, \tag{30}$$

where $\mathbf{X}(t)=(X_1(t),X_2(t))$, $\Delta w_n=u_n^+-u_n^-$ and L is the length of the crack.

Now from (3), (5) and (27) an integral equation for P_i is given by

$$\begin{aligned}\eta P_i(\mathbf{x}_0)&=\eta\,c_{ijkl}\,\frac{\partial u_k}{\partial\xi_l}\,n_j=\eta\,c^{(0)}_{ijkl}\,g(\mathbf{x}_0)\frac{\partial u_k}{\partial\xi_l}\,n_j\\&=P^{[g]}_{ik}(\mathbf{x}_0)\int_{C\cup D}\Big\{P_n(\mathbf{x})\left[g^{-1/2}(\mathbf{x})\,\Phi_{nk}(\mathbf{x},\mathbf{x}_0)\right]\\&-u_n(\mathbf{x})\left[g^{1/2}(\mathbf{x})\,\Gamma_{nk}(\mathbf{x},\mathbf{x}_0)-P^{[g]}_{sn}(\mathbf{x})\,\Phi_{sk}(\mathbf{x},\mathbf{x}_0)\right]\Big\}\,ds(\mathbf{x})\\&-g^{1/2}(\mathbf{x}_0)\int_{C\cup D}\Big\{P_n(\mathbf{x})\left[g^{-1/2}(\mathbf{x})\,\Theta_{in}(\mathbf{x},\mathbf{x}_0)\right]\\&-u_n(\mathbf{x})\left[g^{1/2}(\mathbf{x})\,\Psi_{in}(\mathbf{x},\mathbf{x}_0)-P^{[g]}_{sn}(\mathbf{x})\,\Theta_{is}(\mathbf{x},\mathbf{x}_0)\right]\Big\}\,ds(\mathbf{x}),\end{aligned} \tag{31}$$

where

$$\Theta_{in}(\mathbf{x},\mathbf{x}_0)=c^{(0)}_{ijkl}\frac{\partial\Phi_{nk}}{\partial\xi_l}n_j, \tag{32}$$

$$\Psi_{in}(\mathbf{x},\mathbf{x}_0)=c^{(0)}_{ijkl}\frac{\partial\Gamma_{nk}}{\partial\xi_l}n_j. \tag{33}$$

Hence from (20) and (32)

$$\begin{aligned}\Theta_{in}(\mathbf{x},\mathbf{x}_0)&=-\frac{1}{2\pi}\Re\left[\sum_{\alpha=1}^{3}A_{n\alpha}N_{\alpha r}(c^{(0)}_{ijk1}+\tau_\alpha c^{(0)}_{ijk2})(z_\alpha-c_\alpha)^{-1}\right]n_jd_{rk}\\&=-\frac{1}{2\pi}\Re\left[\sum_{\alpha=1}^{3}S_{i\alpha n}(z_\alpha-c_\alpha)^{-1}\right],\end{aligned} \tag{34}$$

where

$$S_{i\alpha n}=A_{n\alpha}N_{\alpha r}(c^{(0)}_{ijk1}+\tau_\alpha c^{(0)}_{ijk2})n_jd_{rk} \tag{35}$$

and from (21) and (33)

$$
\begin{aligned}
\Psi_{in}(\mathbf{x},\mathbf{x}_0) &= \frac{1}{2\pi}\Re\left[\sum_{\alpha=1}^{2} L_{ns\alpha}N_{\alpha r}(c^{(0)}_{ijk1}+\tau_\alpha c^{(0)}_{ijk2})(z_\alpha - c_\alpha)^{-2}\right] n_s n_j d_{rk} \\
&= \frac{1}{2\pi}\Re\left[\sum_{\alpha=1}^{2} R_{i\alpha n}(z_\alpha - c_\alpha)^{-2}\right], \qquad (36)
\end{aligned}
$$

where

$$
R_{i\alpha n} = L_{ns\alpha}N_{\alpha r}(c^{(0)}_{ijk1}+\tau_\alpha c^{(0)}_{ijk2}) n_s n_j d_{rk}. \qquad (37)
$$

Now as $\mathbf{x}_0 = (\xi_1, \xi_2)$ approaches the crack, the integral over this crack in (30) must be interpreted as a Cauchy principal value integral. Hence differentiation of this integral (with respect to either ξ_1 or ξ_2) as $\mathbf{x}_0$ approaches the crack leads to a Hadamard finite-part integral.

On the crack the coordinates ξ_1 and ξ_2 may be written in terms of a single parameter s in the form

$$
\xi_1 = X_1(s) = [(c-a)s + (c+a)]/2 \ \text{ for } \ s \in [-1,1], \qquad (38)
$$

$$
\xi_2 = X_2(s) = [(d-b)s + (d+b)]/2 \ \text{ for } \ s \in [-1,1]. \qquad (39)
$$

Thus using equation (31) the traction-free condition $P_n = 0$ on the crack can be expressed as

$$
\begin{aligned}
\eta P_i(\mathbf{X}(s)) = P^{[g]}_{ik}(\mathbf{X}(s)) &\left[\int_C \left\{P_n(\mathbf{x})\left[g^{-1/2}(\mathbf{x})\,\Phi_{nk}(\mathbf{x},\mathbf{X}(s))\right]\right.\right. \\
&\left.\left. -u_n(\mathbf{x})\left[g^{1/2}(\mathbf{x})\,\Gamma_{nk}(\mathbf{x},\mathbf{X}(s)) - P^{[g]}_{sn}(\mathbf{x})\,\Phi_{sk}(\mathbf{x},\mathbf{X}(s))\right]\right\} ds(\mathbf{x}).\right. \\
&-\frac{L}{2}\int_{-1}^{1} g^{1/2}(\mathbf{X}(t))\,\Gamma_{nk}(\mathbf{X}(t),\mathbf{X}(s))\Delta w_n(t)dt
\end{aligned}
$$

$$+\frac{L}{2}\int_{-1}^{1} P_{sn}^{[g]}(\mathbf{X}(t))\,\Phi_{sk}(\mathbf{X}(t),\mathbf{X}(s))\Delta w_n(t)dt$$

$$-u_n(\mathbf{x})\left[g^{1/2}(\mathbf{x})\,\Psi_{in}(\mathbf{x},\mathbf{X}(s)) - P_{sn}^{[g]}(\mathbf{x})\,\Theta_{is}(\mathbf{x},\mathbf{X}(s))\right]\Big\}\,ds(\mathbf{x})$$

$$-\frac{L}{2}\int_{-1}^{1} g^{1/2}(\mathbf{X}(t))\,\Psi_{in}(\mathbf{X}(t),\mathbf{X}(s))\Delta w_n(t)dt$$

$$-P_{sn}^{[g]}(\mathbf{x}(t))\,\Theta_{is}(\mathbf{X}(t),\mathbf{X}(s))\Big]\,\Delta w_n(t)dt\Big]$$

$$= 0 \qquad \text{for} \quad -1 < s < 1, \tag{40}$$

where the integrals over the crack involving $\Gamma_{nk}(\mathbf{X}(t),\mathbf{X}(s))$ and $\Theta_{is}(\mathbf{X}(t),\mathbf{X}(s))$ are Cauchy principal value integrals and the integral involving $\Psi_{in}(\mathbf{X}(t),\mathbf{X}(s))$ is a Hadamard finite-part integral.

Now use of (28), (29), (38) and (39) in (34) and (36) yields

$$\Theta_{in}(\mathbf{X}(t),\mathbf{X}(s)) = \frac{h_{in}}{t-s}, \tag{41}$$

where

$$h_{in} = -\frac{1}{\pi}\Re\sum_{\alpha=1}^{2} S_{i\alpha n}\left[(c-a)+\tau_\alpha(d-b)\right]^{-1} \tag{42}$$

and

$$\Psi_{in}(\mathbf{X}(t),\mathbf{X}(s)) = \frac{k_{in}}{(t-s)^2}, \tag{43}$$

where

$$k_{in} = \frac{2}{\pi}\Re\sum_{\alpha=1}^{2} R_{i\alpha n}\left[(c-a)+\tau_\alpha(d-b)\right]^{-2}. \tag{44}$$

Also use of (28), (29), (38) and (39) in (20) and (21) yields

$$\Phi_{in}(\mathbf{X}(t),\mathbf{X}(s)) = \frac{1}{2\pi}d_{in}\,\log|t-s| + f_{in}, \tag{45}$$

where

$$f_{in} = \frac{1}{2\pi} \Re \sum_{\alpha=1}^{2} A_{i\alpha} N_{\alpha k} \, \log \left\{ \left[(c-a) + \tau_\alpha (d-b) \right] / 2 \right\} \, d_{kn} \tag{46}$$

and

$$\Gamma_{in}(\mathbf{X}(t), \mathbf{X}(s)) = \frac{q_{in}}{t-s}, \tag{47}$$

where

$$q_{in} = \frac{1}{\pi} \Re \sum_{\alpha=1}^{2} L_{ij\alpha} N_{\alpha k} \left[(c-a) + \tau_\alpha (d-b) \right]^{-1} n_j d_{kn}. \tag{48}$$

Now use of (41), (43), (45) and (47) in (40) yields

$$\begin{aligned}
& P_{ik}^{[g]}(\mathbf{X}(s)) \left[\int_C \left\{ P_n(\mathbf{x}) \left[g^{-1/2}(\mathbf{x}) \, \Phi_{nk}(\mathbf{x}, \mathbf{X}(s)) \right] \right. \right. \\
& -u_n(\mathbf{x}) \left. \left[g^{1/2}(\mathbf{x}) \, \Gamma_{nk}(\mathbf{x}, \mathbf{X}(s)) - P_{sn}^{[g]}(\mathbf{x}) \, \Phi_{sk}(\mathbf{x}, \mathbf{X}(s)) \right] \right\} ds(\mathbf{x}). \\
& -\frac{L}{2} \int_{-1}^{1} g^{1/2}(\mathbf{X}(t)) \, q_{nk} (t-s)^{-1} \Delta w_n(t) dt \\
& \qquad\qquad + \frac{L}{2} \int_{-1}^{1} P_{sn}^{[g]}(\mathbf{X}(t)) \, \left[(1/2\pi) d_{sk} \log|t-s| + f_{sk} \right] \Delta w_n(t) dt \\
& -g^{1/2}(\mathbf{X}(s)) \left[\int_C \left\{ P_n(\mathbf{x}) \left[g^{-1/2}(\mathbf{x}) \, \Theta_{in}(\mathbf{x}, \mathbf{X}(s)) \right] \right. \right. \\
& -u_n(\mathbf{x}) \left. \left[g^{1/2}(\mathbf{x}) \, \Psi_{in}(\mathbf{x}, \mathbf{X}(s)) - P_{sn}^{[g]}(\mathbf{x}) \, \Theta_{is}(\mathbf{x}, \mathbf{X}(s)) \right] \right\} ds(\mathbf{x}) \\
& -\frac{L}{2} \int_{-1}^{1} g^{1/2}(\mathbf{X}(t)) \, k_{in} (t-s)^{-2} \Delta w_n(t) dt \\
& \qquad\qquad + \frac{L}{2} \int_{-1}^{1} P_{sn}^{[g]}(\mathbf{X}(t)) \, h_{is} (t-s)^{-1} \Delta w_n(t) dt \\
& = 0 \qquad \text{for} \quad -1 < s < 1,
\end{aligned} \tag{49}$$

where the integrals involving the terms $(t-s)^{-1}$ are Cauchy principal value integrals and the integral involving $(t-s)^{-2}$ is a Hadamard finite-part integral.

Equations (30) and (49) are used for the numerical solution of the problem. Following Kaya and Erdogan [11] and Ang [12], let

$$g^{1/2}(\mathbf{X}(t))\Delta w_n(t) \simeq \sqrt{1-t^2}\sum_{\beta=1}^{J}\alpha_{\beta n}U_{\beta-1}(t) \ \ (2NJ \ \text{ unknowns}), \tag{50}$$

$$C \simeq C_1 \cup C_2 \cup \cdots \cup C_M,$$

$$u \simeq u^{(m)} \text{ constant over } C_m, \quad P \simeq P^{(m)} \text{ constant over } C_m$$

where $U_\beta(t)$ denotes the Chebyshev polynomial of the second kind. Letting $\mathbf{x}_0^{(m)} = (\xi_1^{(m)}, \xi_2^{(m)})$ be the midpoint of C_m, the equation (30) may be approximated by (for $m = 1, 2, \cdots, M$)

$$\begin{aligned}
\eta g^{1/2}(\mathbf{x}_0^{(m)})u_k^{(m)} = &-\sum_{r=1}^{M}\Bigg\{P_n^{(r)}\int\limits_{C_r} g^{-1/2}(\mathbf{x})\,\Phi_{nk}(\mathbf{x},\mathbf{x}_0^{(m)})ds(\mathbf{x}) \\
&-u_n^{(r)}\int\limits_{C_r}\left[g^{1/2}(\mathbf{x})\,\Gamma_{nk}(\mathbf{x},\mathbf{x}_0^{(m)}) - P_{sn}^{[g]}(\mathbf{x})\,\Phi_{sk}(\mathbf{x},\mathbf{x}_0^{(m)})\right]ds(\mathbf{x})\Bigg\} \\
&+\frac{L}{2}\sum_{\beta=1}^{J}\alpha_{\beta n}\int\limits_{-1}^{1}\Big[\Gamma_{nk}(\mathbf{X}(t),\mathbf{x}_0^{(m)}) \\
&+Q_{sn}^{[g]}(\mathbf{X}(t))\,\Phi_{sk}(\mathbf{X}(t),\mathbf{x}_0^{(m)})\Big]U_{\beta-1}(t)\sqrt{1-t^2}\,dt,
\end{aligned} \tag{51}$$

where

$$Q_{sn}^{[g]}(\mathbf{x}) = -g^{-1/2}(\mathbf{x})\,P_{sn}^{[g]}(\mathbf{x}). \tag{52}$$

There are $2J + 2M$ unknowns in (51) and the integrals over C_m can be evaluated numerically using standard techniques for the boundary element method (see Clements and Jones [9]). The integrals in (51) over $(-1, 1)$ can be evaluated numerically by using expression (25.4.40) in Abramowitz and Stegun [13]. Equation (51) consists of $2M$ equations since $m = 1, 2, \cdots, M$ and $k = 1, 2$.

In a similar manner the discretised formulation of equation (49) may be obtained in the form

$$
\begin{aligned}
&P_{ik}^{[g]}(\mathbf{X}(s)) \left[\sum_{m=1}^{M} \left\{ P_n^{(m)} \int_{C_m} \left[g^{-1/2}(\mathbf{x})\, \Phi_{nk}(\mathbf{x}, \mathbf{X}(s)) \right] ds(\mathbf{x}) \right.\right. \\
&\left.\left. -u_n^{(m)} \int_{C_m} \left[g^{1/2}(\mathbf{x})\, \Gamma_{nk}(\mathbf{x}, \mathbf{X}(s)) - P_{sn}^{[g]}(\mathbf{x})\, \Phi_{sk}(\mathbf{x}, \mathbf{X}(s)) \right] \right\} ds(\mathbf{x}). \right. \\
&+\pi \frac{L}{2} \sum_{\beta=1}^{J} \alpha_{\beta n} q_{nk} T_{\beta}(s) \\
&-\frac{L}{2} \sum_{\beta=1}^{J} \alpha_{\beta n} \int_{-1}^{1} Q_{sn}^{[g]}(\mathbf{X}(t)) \left\{ \frac{d_{sk}}{2\pi} \log|t-s| + f_{sk} \right\} U_{\beta-1}(t) \sqrt{1-t^2}\, dt \\
&-g^{1/2}(\mathbf{X}(s)) \left[\sum_{m=1}^{M} \left\{ P_n^{(m)} \int_{C_m} \left[g^{-1/2}(\mathbf{x})\, \Theta_{in}(\mathbf{x}, \mathbf{X}(s)) \right] ds(\mathbf{x}) \right.\right. \\
&\left.\left. -u_n^{(m)} \int_{C_m} \left[g^{1/2}(\mathbf{x})\, \Psi_{in}(\mathbf{x}, \mathbf{X}(s)) - P_{sn}^{[g]}(\mathbf{x})\, \Theta_{is}(\mathbf{x}, \mathbf{X}(s)) \right] \right\} ds(\mathbf{x}) \right. \\
&+\pi \frac{L}{2} \sum_{r=1}^{J} \alpha_{rn} k_{in} r U_{r-1}(s) \\
&-\frac{L}{2} \sum_{r=1}^{J} \alpha_{rn} \int_{-1}^{1} \left[Q_{sn}^{[g]}(\mathbf{X}(t))\, h_{is}(t-s)^{-1} \right] U_{r-1}(t) \sqrt{1-t^2}\, dt \\
&= 0 \qquad \text{for} \quad -1 < s < 1. \qquad (53)
\end{aligned}
$$

where a result given in Kaya and Erdogan [11] has been used to evaluate the integral involving the term $(t-s)^{-2}$ in equation (49).

In order to generate the extra $2J$ equations required to solve the system, equation (53) may be evaluated at J points on the crack (for instance, setting $s = s_p = \cos([2p-1]\pi/[2J])$, $(p = 1, 2, \cdots, J)$ for the crack). Thus the total number of unknowns $2J + 2M$ is equal to the number of linear algebraic equations, and the unknowns can be determined.

6 Stress intensity factors and crack energy

For the Chebyschev polnomials $U_\alpha(t)$ use of contour integration provides

$$\int_{-1}^{1} \frac{U_\alpha(t)\sqrt{1-t^2}}{(t-s)^2}\, dt \to \frac{\pi U_\alpha(1)}{\sqrt{2(s-1)}} \quad \text{as} \quad s \to 1+ \quad \text{for} \quad \alpha = 0,1,2\ldots \quad (54)$$

$$\int_{-1}^{1} \frac{U_\alpha(t)\sqrt{1-t^2}}{(t-s)^2}\, dt \to \frac{\pi U_\alpha(-1)}{\sqrt{2(s+1)}} \quad \text{as} \quad s \to -1- \quad \text{for} \quad \alpha = 0,1,2\ldots (55)$$

From (40) and (53) it is apparent that the left hand side of (53) provides an expression for the $P_i(\mathbf{X}(s))$ for all real s and it therefore follows from (54), (55) and (53) that as $s \to 1+$

$$P_i(\mathbf{X}(s)) \to \frac{L}{2} g^{1/2}(\mathbf{X}(1)) \sum_{r=1}^{J} \alpha_{rn} k_{in} \left[\frac{\pi\, U_{r-1}(1)}{\sqrt{2(s-1)}} \right] \quad (56)$$

and as $s \to -1-$

$$P_i(\mathbf{X}(s)) \to \frac{L}{2} g^{1/2}(\mathbf{X}(-1)) \sum_{r=1}^{J} \alpha_{rn} k_{in} \left[\frac{\pi\, U_{r-1}(-1)}{\sqrt{-2(s+1)}} \right]. \quad (57)$$

Let $s = 1 + \delta_1$ where $\delta_1 > 0$ is small. Then from (38) and (39)

$$X_1(1+\delta_1) = X_1(1) + \delta_1 \frac{dX_1}{ds} = X_1(1) + \frac{c-a}{2}\delta_1, \quad (58)$$

$$X_2(1+\delta_1) = X_2(1) + \delta_1 \frac{dX_2}{ds} = X_2(1) + \frac{d-b}{2}\delta_1. \quad (59)$$

Let

$$(r_1)^2 = [X_1(1+\delta_1) - X_1(1)]^2 + [X_2(1+\delta_1) - X_2(1)]^2 \quad (60)$$

so that from (58), (59) and (60)

$$r_1 = L\delta_1/2, \qquad s - 1 = \delta_1 = 2r_1/L. \quad (61)$$

Similarly let $s = -1 - \delta_2$ where $\delta_2 > 0$ is small. Then from (38) and (39)

$$X_1(-1-\delta_2) = X_1(-1) - \delta_2 \frac{dX_1}{ds} = X_1(-1) - \frac{c-a}{2}\delta_2, \tag{62}$$

$$X_2(-1-\delta_2) = X_2(-1) - \delta_2 \frac{dX_2}{ds} = X_2(-1) - \frac{d-b}{2}\delta_2. \tag{63}$$

Let

$$(r_2)^2 = [X_1(-1-\delta_2) - X_1(-1)]^2 + [X_2(-1-\delta_2) - X_2(-1)]^2 \tag{64}$$

so that from (62), (63) and (64)

$$r_2 = L\delta_2/2, \qquad -s-1 = \delta_2 = 2r_2/L. \tag{65}$$

Hence from (56) and (57) it follows that the mode I and mode II stress intensity factors for the crack are given by

$$K_I^+ = \lim_{r_1 \to 0^+} (r_1)^{1/2} P_2(\mathbf{X}(1+2r_1/L) \tag{66}$$

$$= \frac{\pi}{4}(L)^{3/2} g^{1/2}(\mathbf{X}(1)) \sum_{r=1}^{J} \alpha_{rn} k_{2n} U_{r-1}(1), \tag{67}$$

$$K_{II}^+ = \lim_{r_1 \to 0^+} (r_1)^{1/2} P_1(\mathbf{X}(1+2r_1/L) \tag{68}$$

$$= \frac{\pi}{4}(L)^{3/2} g^{1/2}(\mathbf{X}(1)) \sum_{r=1}^{J} \alpha_{rn} k_{1n} U_{r-1}(1), \tag{69}$$

$$K_I^- = \lim_{r_2 \to 0^+} (r_2)^{1/2} P_2(\mathbf{X}(-1-2r_2/L) \tag{70}$$

$$= \frac{\pi}{4}(L)^{3/2} g^{1/2}(\mathbf{X}(-1)) \sum_{r=1}^{J} \alpha_{rn} k_{2n} U_{r-1}(-1), \tag{71}$$

$$K_{II}^- = \lim_{r_2 \to 0^+} (r_2)^{1/2} P_1(\mathbf{X}(-1-2r_2/L) \tag{72}$$

$$= \frac{\pi}{4}(L)^{3/2} g^{1/2}(\mathbf{X}(-1)) \sum_{r=1}^{J} \alpha_{rn} k_{1n} U_{r-1}(-1), \tag{73}$$

where K_I^+ and K_{II}^+ denote the mode I and mode II stress intensity factors at the end (c, d) of the crack and K_I^- and K_{II}^- denote the mode I and mode II stress intensity factors at the end (a, b) of the crack.

The crack energy U is given by the integral

$$U = \frac{1}{2} \int_{\partial\Omega} \sigma_{ij} u_i n_j ds \tag{74}$$

and since the traction $P_i = \sigma_{ij} n_j$ is zero over the surface of the cracks this reduces to

$$U = \frac{1}{2} \int_{C} \sigma_{ij} u_i n_j ds. \tag{75}$$

7 Numerical results

Numerical values for the stress intensity factors and crack energy for some particular crack problems are given in Tables 1, 2 and 3.

Table 1 provides the non-zero stress intensity factors and the crack energy for a homogeneous anisotropic material containing a crack lying along the x_1 axis between $(a/l, b/l) = (0.5, 0)$ and $(c/l, d/l) = (-0.5, 0)$ where l is a reference length (see Figure 1). The elastic moduli are given by $c_{ijkl}/p_0 = c_{ijkl}^{(0)}/p_0$ where p_0 is a reference stress and the non-zero elastic constants $c_{ijkl}^{(0)}/p_0$ take the values $c_{1111}^{(0)}/p_0 = 6.14$, $c_{1122}^{(0)}/p_0 = 1.89$, $c_{1212}^{(0)}/p_0 = 1.89$, $c_{2222}^{(0)}/p_0 = 5.96$. The material is under biaxial tension so that the sides $x_1/l = -h$ and $x_1/l = h$ are subjected to a constant applied normal stress $\sigma_{11}/p_0 = 1$ and the sides $x_2/l = -h$ and $x_2/l = h$ are subjected to a constant applied normal stress $\sigma_{22}/p_0 = 1$ where p_0 is a constant reference stress.

The reference crack energy U_0 in the tables is the energy of the corresponding crack in an infinite homogeneous anisotropic material under biaxial tension with the same elastic constants as given in the previous paragraph.

The values of the stress intensity factors in the tables may be compared with the stress intensity factors for a corresponding crack in an infinite homogeneous anisotropic material under biaxial tension. The relevant stress intensity factors may be obtained from the results in Stroh [14]. Specifically, the non-zero stress intensity factors are $K_I^-/p_0 = 0.5$ and $K_I^+/p_0 = 0.5$

Tables 2 and 3 give the stress intensity factors and crack energy for a single crack along the line $x_2/l = 2$ from $(a/l, b/l) = (4.5, 2)$ to $(c/l, d/l) = (3.5, 2)$ in an inhomogeneous anisotropic material lying in the region $0 < x_1 < 8,\ 0 < x_2 < 4$ (see Figure 2). The inhomogeneous material has the elastic moduli $c_{ijkl}/p_0 = c_{ijkl}^{(0)}(c_0 + c_1x_1 + c_2x_2)^2/p_0$ where $c_{1111}^{(0)}/p_0 = 6.14$, $c_{1122}^{(0)}/p_0 = 1.89$, $c_{1212}^{(0)}/p_0 = 1.89$, $c_{2222}^{(0)}/p_0 = 5.96$.

Table 2 shows the stress intensity factors and crack energy for the case when the outer boundary C is subjected to a constant applied normal stress $\sigma_{22}/p_0 = 1$ over the sides $x_2/l = 4$ and $x_2/l = 0$ and the sides $x_1/l = 0$ and $x_1/l = 8$ are traction free. Table 3 provides results for the case when the sides $x_1/l = 0$ and $x_1/l = 8$ are subjected to a constant applied normal stress $\sigma_{11}/p_0 = 1$ and the sides $x_2/l = 4$ and $x_2/l = 0$ are subjected to a constant applied normal stress $\sigma_{22}/p_0 = 1$.

8 Final remarks

A boundary element method has been obtained for the solution of a generalised plane strain crack problem for a class of inhomogeneous materials. The analysis is restricted to a single plane crack but the extension of the analysis to several non-interacting plane cracks is straightforward.

The class of materials for which the analysis holds is restricted in two ways. Firstly, the elastic moduli are constrained by the symmetry condition (6). As a result the elastic modulus relating the stress $\sigma_{\alpha\alpha}$ for $\alpha = 1, 2, 3$ to the strain $\epsilon_{\beta\beta}$ for $\beta = 1, 2, 3$ ($\beta \neq \alpha$) is equal to the elastic modulus relating the shear stress $\sigma_{\alpha\beta}$ to the shear strain $\epsilon_{\alpha\beta}$. In the case of isotropic materials (which for the practical purposes of numerical calculations is a limiting case of the current analysis) the consequence of the symmetry condition (6) is that the Lamé parameters λ and μ are equal which provides a Poisson's ratio of 0.25. Secondly, the functional form of the elastic moduli is, in general, required to be of the multi-parameter form given by equations (5) and (13).

Although these constraints on the elastic moduli limit the application of the analysis it remains applicable to a significant class of materials. For example in the area of geomechanics a Poisson's ratio of 0.25 is a common value for rock materials (see Manolis and Shaw [15] and Turcotte and Schubert [16]). Also geotechnical analysis of certain subterraean regions (see for example Ward, Burland and Gallois [17]) indicates that the elastic parameters of such regions may be closely approximated by a multi-parameter form of the type given by (5), (6) and (13) with appropriate values of the constants $c_{ijkl}^{(0)}$, α, β and γ (see Azis and Clements [8]).

Acknowlegments

The authors are indebted to Nyoman Widana for his assistance with the numerical computations reported in this paper and also to the referee for some comments which led to an improved revised version of the paper.

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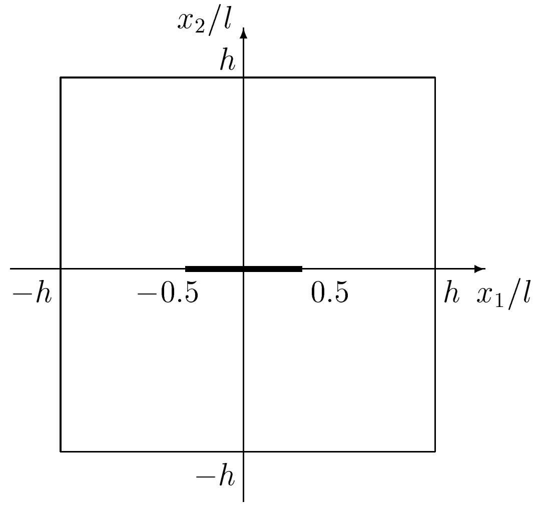

Fig. 1. Geometry for a crack in a homogeneous material.

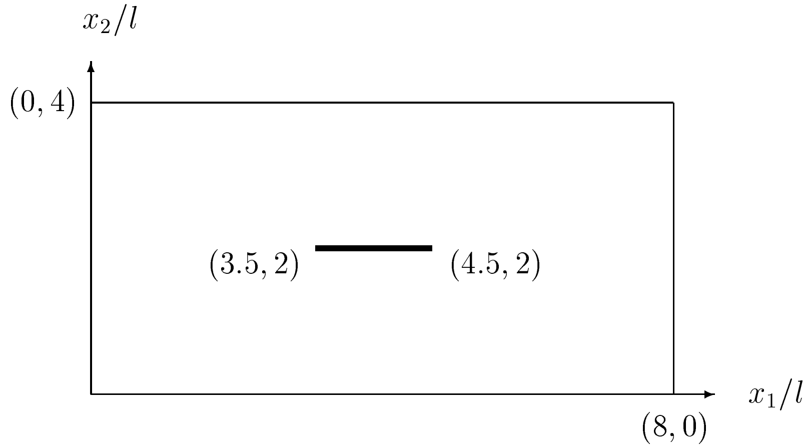

Fig. 2. Geometry for a crack in an inhomogeneous material.

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Table 1

Crack energy and stress intensity factors for a homogeneous material.

h	U/U_0	K_I^-/p_0	K_I^+/p_0
8	1.005	0.502	0.502
7	1.007	0.503	0.503
6	1.009	0.504	0.504
5	1.013	0.506	0.506
4	1.021	0.510	0.510
3	1.037	0.518	0.518
2	1.084	0.541	0.541
1	1.353	0.657	0.657

Table 2

Stress intensity factors and crack energy for a crack in an inhomogeneous material with moduli $c_{ijkl}/p_0 = c_{ijkl}^{(0)}(c_0+c_1x_1+c_2x_2)^2/p_0$ under uniaxial stress.

c_0	c_1	c_2	L	a	K_I^-	K_I^+	K_{II}^-	K_{II}^+	U/U_0
1.0	0.0	0.0	1.0	3.5	0.5192	0.5192	-0.0002	0.0002	1.0395
1.0	0.1	0.0	1.0	3.5	0.5238	0.5344	0.0257	0.0296	0.5410
1.0	0.0	0.1	1.0	3.5	0.5264	0.5264	0.0084	-0.0084	0.7326

Table 3

Stress intensity factors and crack energy for a crack in an inhomogeneous material with moduli $c_{ijkl}/p_0 = c^{(0)}_{ijkl}(c_0 + c_1 x_1 + c_2 x_2)^2/p_0$ under biaxial stress.

c_0	c_1	c_2	L	a	K_I^-/p_0	K_I^+/p_0	K_{II}^-/p_0	K_{II}^+/p_0	U/U_0
1.0	0.0	0.0	1.0	3.5	0.5187	0.5187	-0.0001	0.0001	1.0385
1.0	0.1	0.0	1.0	3.5	0.5179	0.5284	0.0123	0.0142	0.5349
1.0	0.0	0.1	1.0	3.5	0.5026	0.5026	0.0131	-0.0131	0.6991