A LAPLACE TRANSFORMATION DUAL-RECIPROCITY BOUNDARY ELEMENT METHOD FOR A CLASS OF TWO-DIMENSIONAL MICROSCALE THERMAL PROBLEMS

Whye-Teong Ang
Division of Engineering Mechanics
School of Mechanical and Production Engineering
Nanyang Technological University
50 Nanyang Avenue, Singapore 639798
Republic of Singapore
E-mail: mwtang@ntu.edu.sg

Abstract. The numerical solution of a two-dimensional thermal problem governed by a third order partial differential equation derived from a non-Fourier heat flux model which may account for thermal waves and/or microscopic effects is considered. A dual-reciprocity boundary element method is proposed for solving the problem in the Laplace transformation domain. The solution in the physical domain is recovered by a numerical inverse Laplace transformation technique.

Keywords:
Non-Fourier heat flux model, microscale thermal problems,
dual-reciprocity boundary element method, Laplace transformation
Short title:
LT-DRBEM for Microscale Thermal Problems

Note. This has been a draft of the article published in *Engineering Computations* **19** (2002) 467-478.

About the author. Whye-Teong Ang graduated with a doctorate degree in Applied Mathematics from the University of Adelaide in 1987. Presently, he is an Associate Professor in the Division of Engineering Mechanics, School of Mechanical and Production Engineering at Nanyang Technological University in Singapore. One of his main research interests is in scientific and engineering computation. To date, he has published about 50 papers in international journals on areas which include boundary element methods, stress analysis around cracks, thermo-elasticity and diffusion.

A LAPLACE TRANSFORMATION DUAL-RECIPROCITY BOUNDARY ELEMENT METHOD FOR A CLASS OF TWO-DIMENSIONAL MICROSCALE THERMAL PROBLEMS

Abstract. The numerical solution of a two-dimensional thermal problem governed by a third order partial differential equation derived from a non-Fourier heat flux model which may account for thermal waves and/or microscopic effects is considered. A dual-reciprocity boundary element method is proposed for solving the problem in the Laplace transformation domain. The solution in the physical domain is recovered by a numerical inverse Laplace transformation technique.

1 Introduction

The classical Fourier heat flux model assumes that the heat flux is instantaneously proportional to the temperature gradient of the thermal field. It leads to the physically undesirable conclusion that thermal waves travel at an infinite speed. A quite often used justification for the model is that the wave nature of heat conduction is important only over a very short period of time or under extreme conditions (such as at very low temperature or when heat flows at a very high rate), i.e. the model is applicable under "normal conditions" and is therefore adequate for thermal analysis in most (if not all) practical situations. Kaminski (1990) reported that thermal waves might possibly be observed in solids at normal room temperatures, however, and Yuen and Lee (1989) showed that they may possibly be significant over a relatively long period of time. In addition, microscopic phenomena such as phonon scattering and phonon-electron interaction which the Fourier heat flux model fails to capture play an influential role in heat conduction in many micro-engineering applications, e.g. in dielectric films, insulators and semiconductors. Thus, heat flux models which can account for thermal waves and/or microscopic effects have become increasingly important in recent years.

One such heat flux model is the dual-phase lag model proposed by Tzou (1995, 1997). It has been successfully employed in the analysis of certain microscale thermal problems. The thermal waves and microscopic effects are modeled by introducing small time delays required for the temperature gradient and the heat flux to be set up in the heat conduction process. More mathematically, for two-dimensional heat conduction in a thermally isotropic solid, if $q_1(x_1, x_2, t)$ and $q_2(x_1, x_2, t)$ are functions giving the components of the heat flux in the x_1 and x_2 directions respectively (x_1 and x_2 are the Cartesian coordinates and t denotes time) and if $T(x_1, x_2, t)$ is the temperature then

$$q_i(x_1, x_2, t + \tau_q) = -\varkappa \frac{\partial}{\partial x_i} \left[T(x_1, x_2, t + \tau_T) \right] \quad \text{for } t \geq 0, \tag{1}$$

where τ_q and τ_T are given positive constants of small magnitudes giving the phase lags of the heat flux components and the temperature gradient respectively and $\varkappa$ (assumed to be constant here) is the heat conductivity of the solid. In the classical Fourier heat flux model, the constants τ_q and τ_T are both taken to be zero. For further details on the Tzou dual-phase lag model, refer to Tzou (1995, 1997) and other relevant references there-in.

Following Tzou (1997), we can expand the left and right hand sides of (1) respectively as Taylor-Maclaurin series about $\tau_q = 0$ and $\tau_T = 0$ and ignore second and higher order terms in τ_q and τ_T (i.e. assume that the phase lags are sufficiently small) to obtain the constitutive relation:

$$\begin{aligned} & q_i(x_1, x_2, t) + \tau_q \frac{\partial}{\partial t} \left[q_i(x_1, x_2, t) \right] \\ & = -\varkappa \frac{\partial}{\partial x_i} \left(T(x_1, x_2, t) + \tau_T \frac{\partial}{\partial t} \left[T(x_1, x_2, t) \right] \right) \quad \text{for } t \geq 0. \end{aligned} \tag{2}$$

The well-known Cattaneo-Vernotte constitutive relation for heat flux (Cattaneo, 1958; Vernotte, 1958) can be recovered from (2) by letting $\tau_T = 0$.

The use of (2) together with the energy equation

$$\rho c \frac{\partial}{\partial t}\left[T(x_1, x_2, t)\right] = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i} q_i(x_1, x_2, t) \text{ for } t \geq 0 \tag{3}$$

gives rise to

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\frac{\partial^2}{\partial x_i^2}[T(x_1, x_2, t)] + \tau_T \frac{\partial^3}{\partial t \partial x_i^2}[T(x_1, x_2, t)]\right) \\
&= \frac{\rho c}{\varkappa}\left\{\tau_q \frac{\partial^2}{\partial t^2}\left[T(x_1, x_2, t)\right] + \frac{\partial}{\partial t}\left[T(x_1, x_2, t)\right]\right\} \text{ for } t \geq 0,
\end{aligned} \tag{4}$$

where ρ and c are the volume density and the specific heat of the solid. According to Tzou (1997), the term containing the third order mixed partial derivative on the left hand side of (4) accounts for the effects of certain microscopic phenomena on heat conduction. On the other hand side, the term $\rho c \varkappa^{-1} \tau_q \partial^2 T / \partial t^2$ reveals the wave nature of heat conduction.

An important class of non-Fourier heat conduction problems is therefore to solve (4) in a two-dimensional region R bounded by a simple closed curve C subject to the initial-boundary conditions

$$T(x_1, x_2, 0) = f(x_1, x_2) \text{ for } (x_1, x_2) \in R, \tag{5}$$

$$\left.\frac{\partial T}{\partial t}\right|_{t=0} = g(x_1, x_2) \text{ for } (x_1, x_2) \in R, \tag{6}$$

$$T(x_1, x_2, t) = p(x_1, x_2, t) \text{ for } (x_1, x_2) \in C_1 \text{ and } t > 0, \tag{7}$$

$$H(x_1, x_2, t) = r(x_1, x_2, t) \text{ for } (x_1, x_2) \in C_2 \text{ and } t \geq 0, \tag{8}$$

where f, g, p and r are suitably given functions, C_1 and C_2 are non-intersecting curves such that $C_1 \cup C_2 = C$, the heat flux H is defined by $H(x_1, x_2, t) = q_1(x_1, x_2, t)n_1(x_1, x_2) + q_2(x_1, x_2, t)n_2(x_1, x_2)$ and $[n_1(x_1, x_2), n_2(x_1, x_2)]$ denotes the unit normal vector to C at the point (x_1, x_2) pointing away from R.

For the case in which $\tau_T = 0$, Manzari and Manzari (1998) had solved (4) numerically subject to suitable initial-boundary conditions using a finite-element method. For the case in which T depends on x_1 and t only, Dai and

Nassar (1999) had developed a finite difference scheme for solving numerically (4) in the region $0 < x_1 < l$ with T specified at $x_1 = 0$ and $x_1 = l$. More recently, Zhang and Zhao (2001) devised a finite-difference method to solve (4) approximately in a square region with the temperature completely specified on the boundary. In the present paper, a dual-reciprocity boundary element method (DRBEM) is proposed for the numerical solution of the problem defined by (4)-(8) after taking the Laplace transformation (LT) of the equations with respect to the time parameter t. A numerical technique for inverting LT is then employed to recover the physical solution. Such a Laplace transformation DRBEM approach had been previously used by Zhu, Satravaha and Lu (1994) and Zhu and Liu (1998) for the numerical solution of diffusion equations. The proposed method is applicable for arbitrary-shaped regions and mixed boundary conditions. It is applied to solve numerically a specific test problem.

2 Formulation in the LT domain

The Laplace transformation $\mathcal{L}$ of a function $\phi(x_1, x_2, t)$ with respect to the time parameter $t \geq 0$ is defined by

$$\mathcal{L}\{\phi(x_1, x_2, t); t \to s\} = \int_0^\infty \phi(x_1, x_2, t) \exp(-st) dt, \tag{9}$$

where s is the LT parameter. In the present paper, s is taken to be real.

If we apply $\mathcal{L}$ on (4)-(8), the problem under consideration is then to solve for $\widehat{T}(x_1, x_2, s)$ from

$$\sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}[\widehat{T}(x_1, x_2, s)] = \frac{\rho c s[s\tau_q + 1]}{\varkappa[s\tau_T + 1]}\widehat{T}(x_1, x_2, s) + \widehat{F}(x_1, x_2, s) \text{ in } R, \tag{10}$$

subject to

$$\widehat{T}(x_1, x_2, s) = \widehat{p}(x_1, x_2, s) \text{ for } (x_1, x_2) \in C_1, \tag{11}$$
$$\widehat{H}(x_1, x_2, s) = \widehat{r}(x_1, x_2, s) \text{ for } (x_1, x_2) \in C_2, \tag{12}$$

where $\widehat{T}(x_1,x_2,s)=\mathcal{L}\{T(x_1,x_2,t);t\rightarrow s\}$, $\widehat{H}(x_1,x_2,s)=\mathcal{L}\{H(x_1,x_2,t);t\rightarrow s\}$, $\widehat{p}(x_1,x_2,s)=\mathcal{L}\{p(x_1,x_2,t);t\rightarrow s\}$, $\widehat{r}(x_1,x_2,s)=\mathcal{L}\{r(x_1,x_2,t);t\rightarrow s\}$ and

$$\begin{aligned}\widehat{F}(x_1,x_2,s) &= \frac{\rho c}{\varkappa[s\tau_T+1]}\{-[s\tau_q+1]f(x_1,x_2)-\tau_q g(x_1,x_2)\\ &\quad +\frac{\varkappa\tau_T}{\rho c}\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}[f(x_1,x_2)]\}.\end{aligned} \tag{13}$$

3 LT-DRBEM

From (10), one may derive the integral equation

$$\begin{aligned}&\lambda(\xi_1,\xi_2)\widehat{T}(\xi_1,\xi_2,s)\\ &=\iint\limits_{R}\left\{\frac{\rho cs[s\tau_q+1]}{\varkappa[s\tau_T+1]}\widehat{T}(x_1,x_2,s)+\widehat{F}(x_1,x_2,s)\right\}\Phi(x_1,x_2,\xi_1,\xi_2)dx_1dx_2\\ &\qquad +\oint\limits_{C}\widehat{T}(\xi_1,\xi_2,s)\Gamma(x_1,x_2,\xi_1,\xi_2)dS(x_1,x_2)\\ &\qquad +\oint\limits_{C}\widehat{P}(\xi_1,\xi_2,s)\Phi(x_1,x_2,\xi_1,\xi_2)dS(x_1,x_2),\end{aligned} \tag{14}$$

where $\lambda(\xi_1,\xi_2)=0$ if $(\xi_1,\xi_2)\notin C\cup R$, $\lambda(\xi_1,\xi_2)=1$ if $(\xi_1,\xi_2)\in R$, $0<\lambda(\xi_1,\xi_2)<1$ if $(\xi_1,\xi_2)\in C$ and

$$\begin{aligned}\varkappa(s\tau_T+1)\widehat{P}(\xi_1,\xi_2,s) &= (s\tau_q+1)\widehat{H}(\xi_1,\xi_2,s)-\tau_q r(\xi_1,\xi_2,0)\\ &\quad -\varkappa\tau_T\sum_{i=1}^{2}n_i(\xi_1,\xi_2)\frac{\partial}{\partial\xi_i}[f(\xi_1,\xi_2)].\end{aligned}$$

$$\begin{aligned}\Phi(x_1,x_2,\xi_1,\xi_2) &= \frac{1}{4\pi}\ln\left[(x_1-\xi_1)^2+(y_2-\xi_2)^2\right],\\ \Gamma(x_1,x_2,\xi_1,\xi_2) &= \frac{n_1(x_1,x_2)(x_1-\xi_1)+n_2(x_1,x_2)(y_2-\xi_2)}{2\pi\left[(x_1-\xi_1)^2+(y_2-\xi_2)^2\right]}.\end{aligned} \tag{15}$$

For a guide on the derivation of (14), one may refer to Clements (1981).

The integral equation (14) may be used to devise a DRBEM for solving numerically (10) in R subject to (11)-(12) as follows.

Discretize the boundary C by putting on it N well-spaced out points $(x_1^{(1)}, x_2^{(1)})$, $(x_1^{(2)}, x_2^{(2)})$, $\cdots$, $(x_1^{(N-1)}, x_2^{(N-1)})$ and $(x_1^{(N)}, x_2^{(N)})$ in an anticlockwise order. Define: $(x_1^{(N+1)}, x_2^{(N+1)}) = (x_1^{(1)}, x_2^{(1)})$. For $k = 1, 2, \cdots, N$, join the point $(x_1^{(k)}, x_2^{(k)})$ to $(x_1^{(k+1)}, x_2^{(k+1)})$ to form a directed line segment denoted by $C^{(k)}$. We approximate the curve C by:

$$C \simeq C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(N-1)} \cup C^{(N)}. \tag{16}$$

To treat the double integral over R in (14) using the dual-reciprocity method described in Brebbia and Nardini (1983) and Patridge and Brebbia (1990), we choose N collocation points on the boundary C and another L in the interior of R. Let us denote these collocation points by $(\xi_1^{(p)}, \xi_2^{(p)})$ for $p = 1, 2, \cdots, N + L$. For $k = 1, 2, \cdots, N$, we take $(\xi_1^{(k)}, \xi_2^{(k)})$ to be the midpoint of $C^{(k)}$. (The remaining L collocation points lie in the interior of R.) With these collocation points, we then make the approximation:

$$\frac{\rho c s[s\tau_q + 1]}{\varkappa[s\tau_T + 1]}\widehat{T}(x_1, x_2, s) + \widehat{F}(x_1, x_2, s) \simeq \sum_{j=1}^{N+L} \widehat{\mu}^{(j)}(s)\sigma^{(j)}(x_1, x_2), \tag{17}$$

where

$$\sigma^{(j)}(x_1, x_2) = 1 + \left([x_1 - \xi_1^{(j)}]^2 + [x_2 - \xi_2^{(j)}]^2\right) + \left([x_1 - \xi_1^{(j)}]^2 + [x_2 - \xi_2^{(j)}]^2\right)^{3/2}. \tag{18}$$

The local interpolating functions in (18) are those suggested by Zhang and Zhu (1994).

The approximation (17) allows the double integral in (14) to be transformed approximately into a line integral as follows:

$$\iint\limits_R \left\{\frac{\rho c s[s\tau_q + 1]}{\varkappa[s\tau_T + 1]}\widehat{T}(x_1, x_2, s) + \widehat{F}(x_1, x_2, s)\right\} \Phi(x_1, x_2, \xi_1, \xi_2) dx_1 dx_2$$
$$\simeq \sum_{j=1}^{N+L} \widehat{\mu}^{(j)}(s)\Psi^{(j)}(\xi_1, \xi_2), \tag{19}$$

where

$$\Psi^{(j)}(\xi_1,\xi_2) = \lambda(\xi_1,\xi_2)\theta^{(j)}(\xi_1,\xi_2) + \oint\limits_{C} \Phi(x_1,x_2,\xi_1,\xi_2)\beta^{(j)}(x_1,x_2)dS(x_1,x_2)$$
$$- \oint\limits_{C} \Gamma(x_1,x_2,\xi_1,\xi_2)\theta^{(j)}(x_1,x_2)dS(x_1,x_2),$$

$$\theta^{(j)}(x_1,x_2) = \frac{1}{4}\left([x_1-\xi_1^{(j)}]^2 + [x_2-\xi_2^{(j)}]^2\right)$$
$$+\frac{1}{16}\left([x_1-\xi_1^{(j)}]^2 + [x_2-\xi_2^{(j)}]^2\right)^2 + \frac{1}{25}\left([x_1-\xi_1^{(j)}]^2 + [x_2-\xi_2^{(j)}]^2\right)^{5/2},$$

$$\beta^{(j)}(x_1,x_2) = n_1(x_1,x_2)\frac{\partial}{\partial x_1}[\theta^{(j)}(x_1,x_2)] + n_2(x_1,x_2)\frac{\partial}{\partial x_2}[\theta^{(j)}(x_1,x_2)]. \quad (20)$$

For further details, refer to Brebbia and Nardini (1983) and Patridge and Brebbia (1990).

From (19), by letting (ξ_1,ξ_2) be given by $(\xi_1^{(n)},\xi_2^{(n)})$ for $n = 1, 2, \cdots, N+L$ in (14), we obtain:

$$\lambda(\xi_1^{(n)},\xi_2^{(n)})T^{(n)}(s) = \sum_{j=1}^{N+L} \widehat{\mu}^{(j)}(s)\Psi^{(j)}\left(\xi_1^{(n)},\xi_2^{(n)}\right)$$
$$+\sum_{m=1}^{N} T^{(m)}(s) \int\limits_{C^{(m)}} \Gamma(x_1,x_2,\xi_1^{(n)},\xi_2^{(n)})dS(x_1,x_2)$$
$$+\sum_{m=1}^{N} P^{(m)}(s) \int\limits_{C^{(m)}} \Phi(x_1,x_2,\xi_1^{(n)},\xi_2^{(n)})dS(x_1,x_2)$$
$$\text{for } n = 1, 2, \cdots, N+L, \quad (21)$$

where $T^{(n)}(s) = \widehat{T}(\xi_1^{(n)},\xi_2^{(n)},s)$ and $P^{(m)}(s) = \widehat{P}(\xi_1^{(m)},\xi_2^{(m)},s)$. In deriving (21), we make the approximation:

$$\left.\begin{array}{l} \widehat{T}(x_1,x_2,s) \simeq T^{(m)}(s) \\ \widehat{P}(x_1,x_2,s) \simeq P^{(m)}(s) \end{array}\right\} \text{ for } (x_1,x_2) \in C^{(m)}. \quad (22)$$

Also, if we collocate (17) in a similar way, we obtain:

$$\sum_{j=1}^{N+L} \widehat{\mu}^{(j)}(s)\sigma^{(j)}(\xi_1^{(n)}, \xi_2^{(n)}) = \frac{\rho c s[s\tau_q + 1]}{\varkappa[s\tau_T + 1]} T^{(n)}(s) + \widehat{F}(\xi_1^{(n)}, \xi_2^{(n)}, s)$$
$$\text{for } n = 1,\ 2,\ \cdots, N + L. \tag{23}$$

Since either $T^{(m)}(s)$ or $P^{(m)}(s)$ (not both) is known from (11)-(12) for $m = 1, 2, \cdots, N$, we find that for a given s, (21) and (23) can be solved as a system of $2(N + L)$ linear algebraic equations for $2(N + L)$ unknowns given by $T^{(N+p)}$ for $p = 1, 2, \cdots, L$ (the values of the LT of the temperature at the interior collocation points), either $T^{(m)}(s)$ or $P^{(m)}(s)$ for $m = 1, 2, \cdots, N$ (the values of the LT of either the temperature or heat flux at the boundary points) and $\widehat{\mu}^{(j)}(s)$ for $j = 1, 2, \cdots, N + L$ (the unknown coefficients in the approximation (17)). Once $\widehat{\mu}^{(j)}(s)$ are determined, the LT of the temperature and its spatial partial derivatives, namely $\widehat{T}(x_1, x_2, s)$ and $\partial[\widehat{T}(x_1, x_2, s)]/\partial x_i$, can be computed approximately at any point $(x_1, x_2) \in R$ using (17).

4 LT inversion

The temperature $T(x_1, x_2, t)$ can be recovered from $\widehat{T}(x_1, x_2, s)$ by using a numerical LT inversion technique. A survey and comparison of some numerical LT inversion methods was given by Davies and Martin (1979). For the inversion of $\widehat{T}(x_1, x_2, s)$ here, we choose the numerical method due to Stehfest (1970), which is nowadays increasingly used in applied and engineering science for the numerical inversion of LT (e.g. Ang, 1988; Smith *et al.*, 1994; and Hemker, 1999).

Using the Stehfest's algorithm, we obtain the approximation:

$$T(x_1, x_2, t) \approx \frac{\ln(2)}{t} \sum_{n=1}^{2M} c_n \widehat{T}\left(x_1, x_2, \frac{n\ln(2)}{t}\right), \tag{24}$$

where M is a positive integer and

$$c_n = (-1)^{n+M} \sum_{m=[(n+1)/2]}^{\min(n,M)} \frac{m^M (2m)!}{(M-m)!m!(m-1)!(n-m)!(2m-n)!}, \tag{25}$$

where $[r]$ denotes the integer part of the real number r.

Most (if not all) numerical LT inversion techniques are highly susceptible to errors in the Laplace transforms. The Stehfest's algorithm is no exception. In theory, to obtain more accurate values of $T(x_1, x_2, t)$, a larger value of M (i.e. more terms) must be used in the LT inversion formula (24). Unfortunately, in practice, this is only true if we can guarantee that there is no error in the calculation of $\widehat{T}(x_1, x_2, s)$, as the magnitude of the coefficient c_n increases rapidly with increasing n.Thus, M cannot be selected to be as large as we like. On the other hand, taking M to be too small may give numerical results of lower accuracy. The optimum choice of M depends on the arithmetical precision of the computer as well as the accuracy of the numerical values of $\widehat{T}(x_1, x_2, s)$ (Stehfest, 1970).

Perhaps the best way to choose the optimum M is through testing the computer code of (24) on inverting $\widehat{T}(x_1, x_2, s)$ obtained by the DRBEM from specific problems with known exact solutions. In general, one may determine the reliability of the numerical results obtained by starting the inversion of $\widehat{T}(x_1, x_2, s)$ with $M = 2$ (say) and performing the same inversion again and again using increasingly larger values of M. Initially, as M increases, convergence should be observed in the numerical results, if everything is in order. However, when M is greater than a certain value, the LT inversion returns results that fail to show any convergent pattern. Alternatively, one may also choose to invert $\widehat{T}(x_1, x_2, s)$ using two or more different LT inversion techniques in order to assess whether the numerical values of the temperature obtained can be accepted.

5 A specific example

Let us take the region R to be $R = \{(x_1, x_2) : x_1^2 + x_2^2 < 1,\ x_1 > 0,\ x_2 > 0\}$ and a special case of the governing equation (4) with $\rho c = 1$, $\varkappa = 1$, $\tau_T = 1/2$ and $\tau_q = 1/4$, i.e.

$$\sum_{i=1}^{2}\left(\frac{\partial^2}{\partial x_i^2}[T(x_1,x_2,t)]+\frac{1}{2}\frac{\partial^3}{\partial t\partial x_i^2}[T(x_1,x_2,t)]\right)$$
$$=\frac{1}{4}\frac{\partial^2}{\partial t^2}\left[T(x_1,x_2,t)\right]+\frac{\partial}{\partial t}\left[T(x_1,x_2,t)\right]. \tag{26}$$

We shall apply the LT-DRBEM to solve (26) in R subject to the initial-boundary conditions:

$$T(x_1,x_2,0)=(1+x_1)\sin(\sqrt{\frac{3}{2}}x_2)+x_1x_2 \text{ for } (x_1,x_2)\in R, \tag{27}$$
$$\left.\frac{\partial T}{\partial t}\right|_{t=0}=-(1+x_1)\sin(\sqrt{\frac{3}{2}}x_2) \text{ for } (x_1,x_2)\in R, \tag{28}$$
$$T(0,x_2,t)=\exp(-t)\sin(\sqrt{\frac{3}{2}}x_2) \text{ for } x_2\in[0,1],\ t>0, \tag{29}$$
$$T(x_1,0,t)=0 \text{ for } x_1\in[0,1],\ t>0, \tag{30}$$

$$H(x_1,x_2,t)=-\frac{2}{3}\left\{x_1\sin(\sqrt{\frac{3}{2}}x_2)\right.$$
$$\left.+\sqrt{\frac{3}{2}}(1+x_1)x_2\cos(\sqrt{\frac{3}{2}}x_2)\right\}\exp(-t)-2x_1x_2$$
$$\text{for } x_1^2+x_2^2=1,\ x_1>0,\ x_2>0,\ t\geq 0. \tag{31}$$

It is easy to verify by direct substitution that the exact solution of the test problem is

$$T(x_1,x_2,t)=(1+x_1)\exp(-t)\sin(\sqrt{\frac{3}{2}}x_2)+x_1x_2. \tag{32}$$

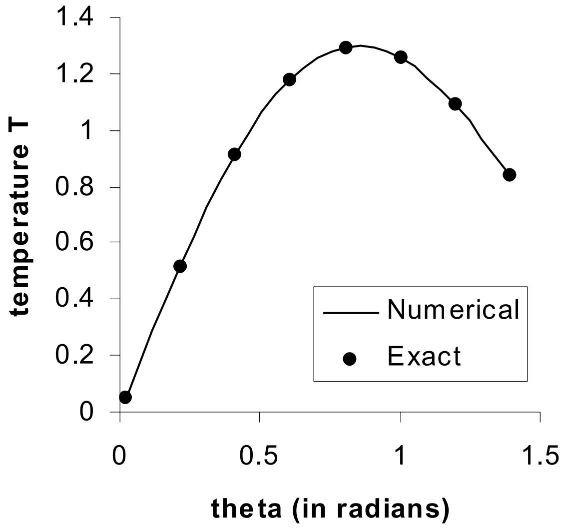
We apply the LT-DRBEM described in Section 2 to solve (26) numerically subject to (27)-(31) in order to assess the accuracy of the method through comparing the numerical results with the exact solution (32). We invert $\widehat{T}(x_1,x_2,s)$ using $M=2$, 4, 6 and 8 to recover approximately the temperature $T(x_1,x_2,t)$. We find that the numerical results obtained using

$M = 4$ do not differ significantly from those calculated using $M = 6$ and are in good agreement with the exact solution. However, due to the precision in the computing machine, the LT inversion fails to deliver accurate solution for $M = 8$. In fact, for $M = 8$, the numerical results depart "wildly" from the actual ones. At the point $(0.25, 0.50)$, the numerical values of the temperature T calculated using $M = 4$ [i.e. 8 terms in the formula (24)] are compared with the exact solution at selected time t in Table 1. The numerical values in the second column of the table are obtained by discretizing the boundary into 60 elements ($N = 60$) and choosing 40 well spaced out collocation points ($L = 40$) in the interior of the solution domain. The third column contains numerical values obtained using $N = 120$ and $L = 80$. It is obvious from the table that the numerical values of the temperature agree well with the exact ones and there is significant improvement in the accuracy of the numerical results when we double the number of boundary elements and interior collocation points used.

Table 1. Numerical and exact values of $T(0.25, 0.50, t)$ at selected time t.

t	$N = 60$ $L = 40$	$N = 120$ $L = 80$	Exact
0.10	0.7746	0.7752	0.7751
0.20	0.7126	0.7133	0.7133
0.30	0.6566	0.6573	0.6573
0.40	0.6060	0.6066	0.6066
0.50	0.5601	0.5607	0.5608
0.60	0.5185	0.5191	0.5193
0.70	0.4809	0.4815	0.4818
0.80	0.4469	0.4475	0.4478
0.90	0.4162	0.4167	0.4171
1.00	0.3884	0.3889	0.3893

Figure 1

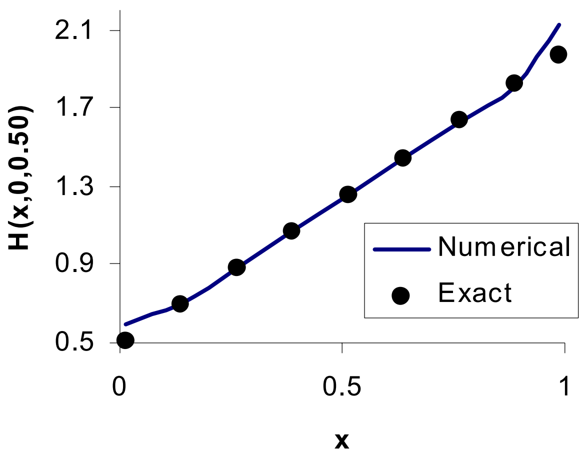


On the circular part of the boundary of the solution, i.e. at $x_1^2 + x_2^2 = 1$, $x_1 > 0$, $x_2 > 0$, the heat flux is specified and the temperature is unknown. In Figure 1, we plot the numerical and the exact values of the temperature at $t = 0.50$ against the angle $\theta = \arctan(x_2/x_1)$, where (x_1, x_2) is a point on the circular part of the boundary. The numerical values are obtained by using $N = 120$, $L = 80$ and $M = 4$. The graphs of the numerical and the exact temperature on the boundary are in good agreement with each other.

On the boundary $x_2 = 0$, $0 < x_1 < 1$, the temperature is specified. We can solve for the LT of $\partial T/\partial n$ on this part of the boundary, i.e. for $\widehat{P}(x_1, 0, s)$ and hence from (15) for the LT of the heat flux across the boundary, i.e. for $\widehat{H}(x_1, 0, s)$. Once $\widehat{H}(x_1, 0, s)$ is obtained, it can be inverted to obtain the boundary heat flux $H(x_1, 0, t)$. Discretizing the boundary into 120 elements, employing 80 collocation points in the interior of the solution domain and using 8 terms in the approximate LT inversion formula, we obtain numerical values of H on $x_2 = 0$ for $0 < x_1 < 1$ at time $t = 0.50$. The numerical and the exact values of $H(x, 0, 0.50)$ are plotted against x for $0 < x < 1$ in Figure

2. [The exact value of H on $x_2 = 0$ for $0 < x_1 < 1$ varies linearly across the line according to $H(x_1, 0, t) = \sqrt{(2/3)}(1 + x_1)\exp(-t)$.] From Figure 2, the numerical and exact values of the boundary heat flux agree well except for points very close to the sharp corners $(0, 0)$ and $(1, 0)$. The accuracy of the numerical calculation near the corner points can be improved by employing more and finer boundary elements near those points.

Figure 2



6 Conclusion

In the present studies, a LT-DRBEM is proposed for the numerical solution of a two-dimensional micro-scale thermal problem. In the LT domain, the method reduces the problem under consideration into a system of linear algebraic equations of the form $\mathbf{AX} = \mathbf{B}$. It is necessary to solve the system several times for different values of the LT parameter s in order to apply a LT inversion technique to recover the physical solution. A significantly large amount of computational time is taken up in the setting up of the matrix $\mathbf{A}$. However, fortunately, most of the elements in $\mathbf{A}$ are independent of s and therefore have to be evaluated only once. The proposed method is applied to solve numerically a specific test problem. For the test problem, the temperature at a selected interior point and at some boundary points as well as the heat flux across a certain part of the boundary is computed numerically and found to be in good agreement with the known exact solution. The accuracy of the numerical solution is also found to improve when the number of boundary elements and collocation points used is increased.

References

[1] Abramowitz, M. and Stegun, I. A. (1970), Handbook of Mathematical Functions, Dover, New York.

[2] Ang, W. T. (1988), A crack in an anisotropic layered material under the action of impact loading, ASME Journal of Applied Mechanics, Vol. 55, pp. 122-125.

[3] Brebbia, C. A. and Nardini, D. (1983), Dynamic analysis in solid mechanics by an alternative boundary element procedure, International Journal of Soil Dynamics and Earthquake Engineering, Vol. 2, pp. 228-233.

[4] Cattaneo, C. (1958), A form of heat conduction equation which eliminates the paradox of instantaneous propagation, Compte Rendus, Vol. 247, pp. 431-433.

[5] Clements, D. L. (1981), Boundary Value Problems Governed by Second Order Elliptic Systems, Pitman, London.

[6] Dai, W. and Nassar, R. (1999), A finite difference scheme for solving solving the heat transport equation at the microscale, Numerical Methods for Partial Differential Equations, Vol. 15, pp. 697-708.

[7] Davies, B. and Martin, B. (1979), Numerical inversion of the Laplace transform: a survey and comparison of methods, Journal of Computational Physics, Vol. 33, pp. 1-32.

[8] Hemker, C. J. (1999), Transient well flow in vertically heterogeneous aquifer, Journal of Hydrology, Vol. 225, pp. 1-18.

[9] Kaminski, W. (1990), Hyperbolic heat conduction for materials with a nonhomogeneous inner structure, ASME Journal of Heat Transfer, Vol. 112, pp. 555-560.

[10] Manzari, M. T. and Manzari, M. T. (1998), A mixed approach to finite-element analysis of hyperbolic heat conduction problems, International Journal for Numerical Methods in Heat and Fluid Flow, Vol. 8, pp. 83-97.

[11] Patridge, P. W. and Brebbia, C. A. (1990), "The dual reciprocity boundary element method for the Helmholtz equation", in Brebbia, C. A. and Choudouet-Miranda, A. (Eds), Proceedings of the International Boundary Elements Symposium, Computational Mechanics Publications/Springer, Berlin, pp. 543-555.

[12] Smith, R. S., Edwards, R. N. and Beselli, G. (1994), An automatic technique for presentation of coincident-loop, impulse-response, transient electromagnetic data, Geophysics, Vol. 59, pp. 1542-1550.

[13] Stehfest, H. (1970), Numerical inversion of the Laplace transform, Communications of ACM, Vol. 13, pp. 47-49 (see also p624).

[14] Tzou, D. Y. (1995), A unified field approach for heat conduction from micro- to macro-scales, ASME Journal of Heat Transfer, Vol. 117, pp. 8-16.

[15] Tzou, D. Y. (1997), Macro- to Microscale Heat Transfer: The Lagging Behavior, Taylor and Francis, Bristol.

[16] Vernotte, P. (1958), Les paradoxes de la théorie continue de l'equation de la chaleur, Compte Rendus, Vol. 246, pp. 3154-3155.

[17] Yuen, W. W. and Lee, S. C. (1989), Non-Fourier heat conduction in a semi-infinite solid subjected to oscillatory surface thermal disturbances, ASME Journal of Heat Transfer, Vol. 11, pp. 178-181.

[18] Zhang, J. and Zhao, J. J. (2001), Unconditionally stable finite difference scheme and iterative solution of two-dimensional microscale heat transport equation, Journal of Computational Physics, Vol. 170, pp. 261-275.

[19] Zhang, Y. and Zhu, S. (1994), On the choice of interpolation functions used in the dual-reciprocity boundary-element method, Engineering Analysis with Boundary Elements, Vol. 13, pp. 387-396.

[20] Shu, S. P., Satravaha, P. and Lu, X. P. (1994), Solving linear diffusion equations with the dual reciprocity method in Laplace space, Engineering Analysis with Boundary Elements, Vol. 13, pp. 1-10.

[21] Zhu, S. P. and Liu, H. W. (1998), On the application of multiquadric bases in conjunction with the LTDRM to solve nonlinear diffusion equations, Applied Mathematics and Computation, Vol. 96, pp. 161-175.