A boundary element and radial basis function approximation method for a second order elliptic partial differential equation with general variable coefficients

Whye-Teong Ang
School of Mechanical and Aerospace Engineering,
Nanyang Technological University, Singapore 639798
E-mail: mwtang@ntu.edu.sg

Abstract

A numerical method based on boundary integral equation and radial basis function approximation is presented for solving boundary value problems governed by a second order elliptic partial differential equation with variable coefficients. The equation arises in the analysis of steady state anisotropic heat or mass diffusion in nonhomogeneous media with properties that vary according to general smoothly varying functions of space. The method requires only the boundary of the solution domain to be discretized into elements. To check the validity and accuracy of the numerical solution, some specific problems with known solutions are solved.

Keywords: Elliptic partial differential equation; Variable coefficients; Boundary element method; Radial basis functions; Anisotropic diffusion.

This is a preprint of an article accepted for publication in the open access peer review journal *Engineering Reports* published by Wiley and Sons. When the article is in print, it may be accessed by the following link: http://dx.doi.org/10.1002/eng2.12057.

1 Introduction

The analysis of two-dimensional steady state heat or mass diffusion in an anisotropic medium may be formulated in terms of boundary value problems governed by the second order linear elliptic partial differential equation (see, for example, Bera et al. [8] and Costa [13])

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial}{\partial x_i}(k_{ij}\frac{\partial u}{\partial x_j}) = 0, \tag{1}$$

where u is the temperature or concentration which is a function of the Cartesian coordinates x_1 and x_2 and k_{ij} are the diffusion or conduction coefficients satisfying the symmetric relation $k_{ij} = k_{ji}$ and the positive definiteness condition given by the strict inequality

$$\sum_{i=1}^{2}\sum_{j=1}^{2} k_{ij}\xi_i\xi_j > 0 \text{ for all real numbers } \xi_k \text{ such that } \xi_1^2 + \xi_2^2 \neq 0. \tag{2}$$

In nonhomogeneous media such as a functionally graded material, k_{ij} may be taken to be varying smoothly from point to point in space. Researchers such as AL-Jawary and Wrobel [1], Chakrabortya et al. [10], Fahmi [18, 19], Rangelov et al. [29] and Reutskiy [31], have proposed various numerical methods for the analyses of such nonhomogeneous materials.

Integral equation based methods like the boundary element method are of interest here as they offer certain advantages such as computational efficiency and accuracy in the treatment of the governing differential equations and the boundary conditions. Boundary element solutions for the boundary value problems governed by (1) are well established for homogeneous media, that is, for constant coefficients k_{ij} (see, for example, Clements [12] and Ooi et al. [28]). The development of boundary element methods for the more general case in which the coefficients k_{ij} are continuously varying functions of x_1 and x_2 is, however, a more challenging task.

In many papers on boundary integral methods for solving (1) for nonhomogeneous media, the diffusion or conduction coefficients are taken to be of the form $k_{ij} = \gamma_{ij} g(x_1, x_2)$, where γ_{ij} are constants, that is, the variation or grading of the coefficients are determined by just a single function $g(x_1, x_2)$. The perturbation boundary element approach proposed by Rangogni [30] may be used to solve approximately (1) for slightly varying coefficients k_{ij} with $g(x_1, x_2) = 1 + \epsilon f(x_1, x_2)$, where ϵ is a constant parameter of an extremely small magnitude and $f(x_1, x_2)$ is a given smoothly varying function. For heat conduction in isotropic solids, Ang et al. [6] and Clements [11] derived special fundamental solutions for (1) with $k_{ij} = \delta_{ij} X(x_1) Y(x_2)$ (where δ_{ij} is the Kronecker-delta and $X(x_1)$ and $Y(x_2)$ are given smoothly varying functions) and Kassab and Divo [25] introduced the idea of a generalised fundamental solution. With a suitable fundamental solution, a boundary integral equation needed for developing a boundary element method may then be derived for the partial differential equation (1).

From a mathematical standpoint, suitable fundamental solutions in analytic closed forms are, however, inherently difficult, if not impossible, to derive for general variable coefficients k_{ij}. If the fundamental solution for the special case where k_{ij} are constants (homogeneous media) is used instead to obtain an integral equation for (1) with k_{ij} given by smoothly varying functions, the resulting integral formulation contains not only a boundary integral but also a domain integral containing the unknown function u in the integrand. For $k_{ij} = \gamma_{ij} g(x_1, x_2)$ (with constant γ_{ij}), Ang et al. [5] and Tanaka et al. [33] applied the dual-reciprocity method proposed by Brebbia and Nardini [9] to approximate the resulting domain integral in terms of a boundary integral, in order to avoid the need to discretize the solution domain into elements.

In the current paper, we present a boundary element procedure for a more general steady state anisotropic diffusion equation by including a linear source term and taking the coefficients k_{ij} to be any general smoothly varying functions of space. No restrictive form (such as $k_{ij} = \gamma_{ij} g(x_1, x_2)$ with constant γ_{ij}, as assumed in, for example, Ang [4], Ang et al. [5], Dineva et al. [15] and Rangelov et al. [29]) is imposed on k_{11}, k_{12} and k_{22} here. The coefficients k_{11}, k_{12} and k_{22} may be individually given by any smoothly varying functions as long as they satisfy the positive definiteness condition (2) (for elliptic partial differential equations) in the solution domain.

Our solution approach here employs radial basis functions to approximate $u(x_1, x_2)$ and related functions in order to rewrite the governing partial differential equation as a constant coefficient linear elliptic partial differential equation which has a standard boundary integral equation. Unlike the dual-reciprocity boundary element approaches for nonhomogeneous media as in, for example, Ang et al. [5], Fahmy [17] and Tanaka et al. [33], the integral formulation here for the problem under consideration does not involve any domain integral. The radial basis function approximations and the discretization of the boundary integral equation together with the boundary conditions give rise to a system of linear algebraic equations for the approximate solution of the boundary value problem under consideration here. The numerical procedure does not require the entire solution domain to be discretized into elements. Only the boundary of the solution domain is discretized into elements. To check the validity and accuracy of the numerical procedure, specific problems with known solutions are solved numerically.

2 Boundary value problem

The boundary value problem of interest here is to solve the steady state anisotropic diffusion equation with a linear source term, as given by

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial}{\partial x_i}(k_{ij}(x_1,x_2)\frac{\partial u}{\partial x_j}) + q_1(x_1,x_2)u + q_2(x_1,x_2) = 0, \qquad (3)$$

in a two-dimensional bounded region R on the Ox_1x_2 plane subject to

$$\begin{aligned} u(x_1,x_2) &= r(x_1,x_2) \text{ for } (x_1,x_2)\in C_1, \\ \sum_{i=1}^{2}\sum_{j=1}^{2}k_{ij}(x_1,x_2)n_i(x_1,x_2)\frac{\partial u}{\partial x_j} &= s(x_1,x_2) \text{ for } (x_1,x_2)\in C_2, \end{aligned} \qquad (4)$$

where r and s are suitably prescribed functions, C_1 and C_2 are non-intersecting curves such that $C = C_1 \cup C_2$ is the curve bounding the region R and n_i is the x_i component of the outward unit normal vector to the curve C.

The coefficients q_i and k_{ij} in (3) are given by continuous functions of x_1 and x_2 in the solution domain. As mentioned earlier, k_{ij} may be given by any smoothly varying functions as long as the elliptic condition (2) is satisfied.

The special case of the boundary value problem for isotropic media (that is, the case of the Helmholtz equation where $k_{11} = k_{22}$ and $k_{12} = k_{21} = 0$) is considered in AL-Jawary and Wrobel [1]. In [1], the partial differential equation is formulated in terms of boundary-domain integro-differential equations and the resulting domain integral which has a weak singularity is approximated as boundary integral by using radial basis functions and the radial integration method in Gao [24]. A different approach based on boundary integral equation and radial basis function approximation, which does not involve any domain integral in the solution formulation, is presented here in Section 3 below for solving (3) numerically subject to the boundary conditions in (4).

3 Solution approach

3.1 Reformulation of the partial differential equations

For solving the boundary value problem, we rewrite the elliptic partial differential equation (3) as

$$\sum_{i=1}^{2}\sum_{j=1}^{2}(k_{ij}^{(0)}\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_i}(k_{ij}^{(1)}\frac{\partial u}{\partial x_j})) = -q_1(x_1,x_2)u - q_2(x_1,x_2), \qquad (5)$$

where $k_{ij}^{(0)}$ are constants which may be obtained by averaging k_{ij} uniformly over the solution domain R and $k_{ij}^{(1)} = k_{ij} - k_{ij}^{(0)}$ are, in general, functions that vary smoothly with x_1 and x_2 in the solution domain.

Motivated by the analysis in Ang [4] and Dobroskok and Linkov [16], we introduce the substitution

$$u(x_1,x_2) = v(x_1,x_2) + w(x_1,x_2), \qquad (6)$$

where v is chosen to be related to u by

$$\sum_{i=1}^{2}\sum_{j=1}^{2}(k_{ij}^{(0)}\frac{\partial^2 v}{\partial x_i \partial x_j}+\frac{\partial}{\partial x_i}(k_{ij}^{(1)}\frac{\partial u}{\partial x_j}))=-q_1(x_1,x_2)u-q_2(x_1,x_2), \quad (7)$$

and w satisfies the constant coefficient linear partial differential equation

$$\sum_{i=1}^{2}\sum_{j=1}^{2}k_{ij}^{(0)}\frac{\partial^2 w}{\partial x_i \partial x_j}=0. \quad (8)$$

It is easy to show that (3) is satisfied by (6), (7) and (8).

Our solution approach is as follows. We employ a meshless technique based on radial basis functions to approximate (7) in terms of a system of linear algebraic equations. We discretize the boundary integral equation for (8), as in Clements [12], into linear algebraic equations. For the numerical solution of the boundary value problem in Section 2, we solve the resulting linear algebraic equations by taking into account the boundary conditions in (4).

3.2 Radial basis function approximation

For the meshless technique for approximating (7), we choose P well spaced out collocation points in $R \cup C$, denoting the chosen points by $(\xi_1^{(1)}, \xi_2^{(1)})$, $(\xi_1^{(2)}, \xi_2^{(2)})$, $\cdots$, $(\xi_1^{(P-1)}, \xi_2^{(P-1)})$ and $(\xi_1^{(P)}, \xi_2^{(P)})$, where $\xi_i^{(j)}$ is the x_i coordinate of the j-th collocation point.

We make the approximation

$$\sum_{j=1}^{2}(k_{ij}^{(0)}\frac{\partial v}{\partial x_j}+k_{ij}^{(1)}(x_1,x_2)\frac{\partial u}{\partial x_j})\simeq\sum_{r=1}^{P}a_i^{(r)}\rho^{(r)}(x_1,x_2), \quad (9)$$

where $a_i^{(r)}$ are constant coefficients and $\rho^{(r)}(x_1,x_2)$ is a radial basis function centered about $(\xi_1^{(r)}, \xi_2^{(r)})$. One may refer to Dehghan and Mohammadi [14] and Fasshauer [22] for details on radial basis functions.

Substituting (9) into (7), we obtain

$$\sum_{r=1}^{P}\sum_{i=1}^{2}a_i^{(r)}\frac{\partial}{\partial x_i}(\rho^{(r)}(x_1,x_2))=-q_1(x_1,x_2)u-q_2(x_1,x_2). \quad (10)$$

If we make the approximations

$$
\begin{aligned}
u(x_1, x_2) &\simeq \sum_{r=1}^{P} b^{(r)} \rho^{(r)}(x_1, x_2), \\
v(x_1, x_2) &\simeq \sum_{r=1}^{P} c^{(r)} \rho^{(r)}(x_1, x_2),
\end{aligned} \tag{11}
$$

collocate (11) at $(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1,\ 2,\ \cdots,\ P$ and invert the resulting linear equations for the constants $b^{(r)}$ and $c^{(r)}$, we obtain

$$
\begin{aligned}
u(x_1, x_2) &\simeq \sum_{r=1}^{P} \sum_{m=1}^{P} \varphi^{(rm)} u^{(m)} \rho^{(r)}(x_1, x_2), \\
v(x_1, x_2) &\simeq \sum_{r=1}^{P} \sum_{m=1}^{P} \varphi^{(rm)} v^{(m)} \rho^{(r)}(x_1, x_2),
\end{aligned} \tag{12}
$$

where $u^{(m)} = u(\xi_1^{(m)}, \xi_2^{(m)})$, $v^{(m)} = v(\xi_1^{(m)}, \xi_2^{(m)})$ and $\varphi^{(rm)}$ are defined by

$$
\sum_{m=1}^{P} \varphi^{(rm)} \rho^{(s)}(\xi_1^{(m)}, \xi_2^{(m)}) = \begin{cases} 1 & \text{if} \quad r = s, \\ 0 & \text{if} \quad r \neq s. \end{cases} \tag{13}
$$

If we substitute (12) into (9) and collocate at $(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1,\ 2,\ \cdots,\ P$, we obtain

$$
\sum_{r=1}^{P} a_i^{(r)} \rho^{(r)}(\xi_1^{(n)}, \xi_2^{(n)}) = \sum_{m=1}^{P} (\theta_i^{(nm)} v^{(m)} + \phi_i^{(nm)} u^{(m)}) \text{ for } n = 1, 2, \cdots, P, \tag{14}
$$

where

$$
\begin{aligned}
\theta_i^{(nm)} &= \sum_{j=1}^{2} k_{ij}^{(0)} \sum_{r=1}^{P} \varphi^{(rm)} \left. \frac{\partial}{\partial x_j} (\rho^{(r)}(x_1, x_2)) \right|_{(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})}, \\
\phi_i^{(nm)} &= \sum_{j=1}^{2} k_{ij}^{(1)}(\xi_1^{(n)}, \xi_2^{(n)}) \sum_{r=1}^{P} \varphi^{(rm)} \left. \frac{\partial}{\partial x_j} (\rho^{(r)}(x_1, x_2)) \right|_{(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})}.
\end{aligned} \tag{15}
$$

Inversion of (14) gives

$$a_i^{(r)} = \sum_{m=1}^{P} (\gamma_i^{(rm)} v^{(m)} + \beta_i^{(rm)} u^{(m)}), \tag{16}$$

where

$$\begin{aligned} \gamma_i^{(rm)} &= \sum_{n=1}^{P} \theta_i^{(nm)} \varphi^{(rn)}, \\ \beta_i^{(rm)} &= \sum_{n=1}^{P} \phi_i^{(nm)} \varphi^{(rn)}. \end{aligned} \tag{17}$$

If we substitute (16) into (10) and collocate the resulting equation at each of the chosen collocation points, we obtain

$$q_1^{(n)} u^{(n)} + \sum_{m=1}^{P} (\mu^{(nm)} v^{(m)} + \omega^{(nm)} u^{(m)}) = -q_2^{(n)} \text{ for } n = 1, 2, \cdots, P, \tag{18}$$

where $q_i^{(n)} = q_i(\xi_1^{(n)}, \xi_2^{(n)})$ and

$$\begin{aligned} \mu^{(nm)} &= \sum_{r=1}^{P} \sum_{i=1}^{2} \gamma_i^{(rm)} \frac{\partial}{\partial x_i} (\rho^{(r)}(x_1, x_2)) \bigg|_{(x_1,x_2)=(\xi_1^{(n)},\xi_2^{(n)})}, \\ \omega^{(nm)} &= \sum_{r=1}^{P} \sum_{i=1}^{2} \beta_i^{(rm)} \frac{\partial}{\partial x_i} (\rho^{(r)}(x_1, x_2)) \bigg|_{(x_1,x_2)=(\xi_1^{(n)},\xi_2^{(n)})}. \end{aligned} \tag{19}$$

The linear algebraic equations in (18) with unknowns $u^{(m)}$ and $v^{(m)}$ ($m = 1, 2, \cdots, P$) may be regarded as a radial basis function approximation of the partial differential equation (7).

In the formulation above, the radial basis function $\rho^{(r)}(x_1, x_2)$ is required to be partially differentiable once with respect to x_1 or x_2. We may take $\rho^{(r)}(x_1, x_2)$ to be the well known multiquadric radial basis function given by (see Ferreira [23] and Sarra [32])

$$\rho^{(r)}(x_1, x_2) = \sqrt{1 + (x_1 - \xi_1^{(r)})^2 + (x_2 - \xi_2^{(r)})^2}. \tag{20}$$

Alternatively, one may use the radial basis function proposed in Zhang and Zhu [34], that is,

$$\rho^{(r)}(x_1, x_2) = 1 + (x_1 - \xi_1^{(r)})^2 + (x_2 - \xi_2^{(r)})^2 + ((x_1 - \xi_1^{(r)})^2 + (x_2 - \xi_2^{(r)})^2)^{3/2}. \tag{21}$$

3.3 Boundary integral approximation

The elliptic partial differential equation (8) can be recast into the boundary integral equation (Clements [12])

$$\begin{aligned}
&\lambda(\xi_1, \xi_2) w(\xi_1, \xi_2) \\
&= \int_C (\Gamma(x_1, x_2, \xi_1, \xi_2) w(x_1, x_2) \\
&- \Phi(x_1, x_2, \xi_1, \xi_2) \sum_{i=1}^{2} \sum_{j=1}^{2} k_{ij}^{(0)} n_i(x_1, x_2) \frac{\partial}{\partial x_j}(w(x_1, x_2))) ds(x_1, x_2), \qquad (22)
\end{aligned}$$

where $\lambda(\xi_1, \xi_2)$ is such that $\lambda(\xi_1, \xi_2) = 1$ if (ξ_1, ξ_2) lies in the interior of the solution domain R bounded by the curve C and $\lambda(\xi_1, \xi_2) = 1/2$ if (ξ_1, ξ_2) lies on a smooth part of the curve C, and

$$\begin{aligned}
\Phi(x_1, x_2, \xi_1, \xi_2) &= \frac{1}{2\pi\sqrt{k_{11}^{(0)} k_{22}^{(0)} - (k_{12}^{(0)})^2}} \text{Re}\{\ln(x_1 - \xi_1 + \tau[x_2 - \xi_2])\}, \\
\Gamma(x_1, x_2, \xi_1, \xi_2) &= \frac{1}{2\pi\sqrt{k_{11}^{(0)} k_{22}^{(0)} - (k_{12}^{(0)})^2}} \text{Re}\left\{\frac{L(x_1, x_2)}{(x_1 - \xi_1 + \tau[x_2 - \xi_2])}\right\}, \\
L(x_1, x_2) &= (k_{11}^{(0)} + \tau k_{12}^{(0)}) n_1(x_1, x_2) + (k_{21}^{(0)} + \tau k_{22}^{(0)}) n_2(x_1, x_2), \\
\tau &= \frac{-k_{12}^{(0)} + i\sqrt{k_{11}^{(0)} k_{22}^{(0)} - (k_{12}^{(0)})^2}}{k_{22}^{(0)}} \quad (i = \sqrt{-1}), \qquad (23)
\end{aligned}$$

where Re denotes the real part of a complex number. Note that $k_{11}^{(0)} k_{22}^{(0)} - (k_{12}^{(0)})^2 > 0$.

If we substitute (6) into (22), we obtain

$$\begin{aligned}
&\lambda(\xi_1, \xi_2)(u(\xi_1, \xi_2) - v(\xi_1, \xi_2)) \\
&= \int_C (\Gamma(x_1, x_2, \xi_1, \xi_2)(u(x_1, x_2) - v(x_1, x_2)) \\
&- \Phi(x_1, x_2, \xi_1, \xi_2)(p(x_1, x_2) - t(x_1, x_2))) ds(x_1, x_2), \qquad (24)
\end{aligned}$$

where

$$\begin{aligned} p(x_1,x_2) &= \sum_{i=1}^{2}\sum_{j=1}^{2} k_{ij}^{(0)} n_i(x_1,x_2)\frac{\partial}{\partial x_j}(u(x_1,x_2)), \\ t(x_1,x_2) &= \sum_{i=1}^{2}\sum_{j=1}^{2} k_{ij}^{(0)} n_i(x_1,x_2)\frac{\partial}{\partial x_j}(v(x_1,x_2)). \end{aligned} \quad (25)$$

To approximate (24), we discretize the boundary C into M straight line elements denoted by $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(M-1)}$ and $C^{(M)}$. The collocation point $(\xi_1^{(m)}, \xi_2^{(m)})$ for the meshless technique in Subsection 3.2 is taken to be the midpoint of $C^{(m)}$ for $m = 1, 2, \cdots , M$, that is, the first M collocation points are the midpoints of the M straight line elements. The remaining collocation points are in the interior of the solution domain of the boundary value problem under consideration. If the number of interior collocation points is N, then the integer P in Subsection 3.2 is given by $M + N$.

We make the approximations

$$C \simeq C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(M-1)} \cup C^{(M)}, \quad (26)$$

and

$$\left.\begin{aligned} u(x_1,x_2) &\simeq u^{(m)} \\ v(x_1,x_2) &\simeq v^{(m)} \\ p(x_1,x_2) &\simeq p^{(m)} \\ t(x_1,x_2) &\simeq t^{(m)} \end{aligned}\right\} \text{ for } (x_1,x_2) \in C^{(m)}, \quad (27)$$

where $p^{(m)} = p(\xi_1^{(m)}, \xi_2^{(m)})$ and $t^{(m)} = t(\xi_1^{(m)}, \xi_2^{(m)})$ for $m = 1, 2, \cdots , M$. Note that $u^{(n)}$ and $v^{(n)}$ are respectively the values of $u(x_1, x_2)$ and $v(x_1, x_2)$ at the n-th collocation point as defined in Subsection 3.2.

If we substitute (26) and (27) into (24) and collocate the resulting equation at $(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1, 2, \cdots , M + N$, we obtain the linear algebraic equations

$$\begin{aligned}
&\lambda(\xi_1^{(n)},\xi_2^{(n)})(u^{(n)}-v^{(n)})\\
&=\sum_{m=1}^{M}(u^{(m)}-v^{(m)})\int\limits_{C^{(m)}}\Gamma(x_1,x_2,\xi_1^{(n)},\xi_2^{(n)})ds(x_1,x_2)\\
&-\sum_{m=1}^{M}(p^{(m)}-t^{(m)})\int\limits_{C^{(m)}}\Phi(x_1,x_2,\xi_1^{(n)},\xi_2^{(n)})ds(x_1,x_2)\\
&\text{for } n=1,2,\cdots,M+N. \qquad (28)
\end{aligned}$$

Note that $\lambda(\xi_1^{(n)},\xi_2^{(n)}) = 1/2$ for $n = 1, 2, \cdots, M$ and $\lambda(\xi_1^{(n)},\xi_2^{(n)}) = 1$ for $n = M + 1, M + 2, \cdots, M + N$.

An account on how the integrals over $C^{(m)}$ in (28) may be evaluated analytically or numerically is given in the Appendix. One may also refer to Ang [3] and Clements [12] for details.

The system (28) contains $M + N$ linear algebraic equations in $4M + 2N$ unknowns. The unknowns are given by $u^{(n)}$, $v^{(n)}$, $p^{(m)}$ and $t^{(m)}$ for $m = 1, 2, \cdots, M$ and $n = 1, 2, \cdots, M + N$. We may regard (28) as a boundary integral approximation of the partial differential equation in (8).

3.4 Numerical procedure

For the numerical solution of the boundary value problem in Section 2, we approximate (7) and (8) by (18) (with $P = M+N$) and (28) respectively. The total number of equations in (7) and (8) is given by $2M+2N$. It is less than the number of unknowns involved. The number of unknowns is $4M + 2N$. Thus, another $2M$ equations are required to complete the numerical formulation.

The boundary conditions in (4) can be expressed in terms of M linear algebraic equations given by

$$u^{(m)} = r(\xi_1^{(m)},\xi_2^{(m)}) \text{ if } u \text{ is specified on } C^{(m)}, \qquad (29)$$

or

$$p^{(m)} + \sum_{q=1}^{M+N} y^{(mq)} u^{(q)} = s(\xi_1^{(m)}, \xi_2^{(m)})$$

$$\text{if } \sum_{i=1}^{2}\sum_{j=1}^{2} k_{ij}(x_1, x_2) n_i(x_1, x_2) \frac{\partial u}{\partial x_j} \text{ is specified on } C^{(m)}, \tag{30}$$

with the constant coefficients $y^{(mq)}$ defined by

$$y^{(mq)} = \sum_{i=1}^{2}\sum_{j=1}^{2} k_{ij}^{(1)}(\xi_1^{(m)}, \xi_2^{(m)}) n_i^{(m)} \sum_{r=1}^{M+N} \varphi^{(rq)} \left.\frac{\partial}{\partial x_j}(\rho^{(r)}(x_1, x_2))\right|_{(x_1,x_2)=(\xi_1^{(m)},\xi_2^{(m)})}, \tag{31}$$

where $n_i^{(m)}$ is the x_i component of the unit vector that is normal to $C^{(m)}$ and that points out of the solution domain. Note that (30) is derived from the second line in (4) by taking $k_{ij} = k_{ij}^{(0)} + k_{ij}^{(1)}$ and using (12) and the first equation in (25).

Another M linear algebraic equations may be derived by using (12) and the second equation in (25). They are given by

$$t^{(m)} - \sum_{q=1}^{M+N} z^{(mq)} v^{(q)} = 0 \text{ for } m = 1, 2, \cdots, M, \tag{32}$$

where

$$z^{(mq)} = \sum_{i=1}^{2}\sum_{j=1}^{2} k_{ij}^{(0)} n_i^{(m)} \sum_{r=1}^{M+N} \varphi^{(rq)} \left.\frac{\partial}{\partial x_j}(\rho^{(r)}(x_1, x_2))\right|_{(x_1,x_2)=(\xi_1^{(m)},\xi_2^{(m)})}. \tag{33}$$

For the numerical solution of the boundary value problem in Section 2, we solve the system of $4M + 2N$ linear algebraic equations given by (18) (with $P = M + N$), (28), (29) or (30), and (32). If one wishes to solve a smaller system of linear algebraic equations, one may substitute (29) or (30) together with (32) directly into (28). This will reduce the number of unknowns and also the number of equations in the formulation to $2M + 2N$. The computer coding of the smaller system of equations is, however, more involved. Once the unknowns are determined, the required solution $u(x_1, x_2)$ can be computed approximately in an explicit manner at any general point (x_1, x_2) in the solution domain by using (12).

4 Specific problems

The numerical procedure described above is applied to solve the following specific problems. The radial basis function given in (21) is used in the numerical computation.

Problem 1. Take the coefficients q_i and k_{ij} to be given by

$$q_1 = q_2 = 0, \ \frac{1}{2}k_{11} = k_{22} = k_{12} = k_{21} = (x_1^2 - 2x_1x_2 + 2)^2. \tag{34}$$

The problem is to solve the partial differential equation given by (3) together with (34) in the solution domain $0 < x_1 < 1$, $0 < x_2 < 1$, subject to the boundary conditions

$$\left.\begin{array}{lcl} u(x_1, 0) & = & \dfrac{1 + x_1}{x_1^2 + 2} \\ u(x_1, 1) & = & \dfrac{x_1}{x_1^2 - 2x_1 + 2} \end{array}\right\} \text{for } 0 < x_1 < 1,$$

$$\left.\begin{array}{lcl} u(0, x_2) & = & \dfrac{1 - x_2}{2} \\ u(1, x_2) & = & \dfrac{2 - x_2}{3 - 2x_2} \end{array}\right\} \text{for } 0 < x_2 < 1. \tag{35}$$

It may be verified by direct substitution that the analytical solution of the problem is given by

$$u(x_1, x_2) = \frac{1 + x_1 - x_2}{x_1^2 - 2x_1x_2 + 2}. \tag{36}$$

For the numerical solution of the problem, each sides of the square solution domain is discretized into M_0 equal length elements and the interior collocation points are chosen to be well spaced out points given by $(i/(N_0 + 1), j/(N_0 + 1))$ for $i, j = 1, 2, \cdots, N_0$ (so that $M = 4M_0$ and $N = N_0^2$). We take $k_{ij}^{(0)}$ to be the average value of k_{ij} over all the interior collocation points, that is,

$$k_{ij}^{(0)} = \frac{1}{N_0^2} \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} k_{ij}\left(\frac{i}{N_0 + 1}, \frac{j}{N_0 + 1}\right). \tag{37}$$

Table 1. Comparison of numerical and analytical values of u at selected interior collocation points. The average absolute error (AAE) is given for each set of the numerical values.

Point (x_1, x_2)	$M_0 = 10$ $N_0 = 4$	$M_0 = 40$ $N_0 = 19$	Analytical
$(0.2, 0.2)$	0.5096	0.5102	0.5102
$(0.2, 0.4)$	0.4242	0.4254	0.4255
$(0.2, 0.6)$	0.3300	0.3329	0.3333
$(0.2, 0.8)$	0.2292	0.2325	0.2326
$(0.4, 0.2)$	0.6003	0.5999	0.6000
$(0.4, 0.4)$	0.5462	0.5434	0.5435
$(0.4, 0.6)$	0.4770	0.4761	0.4762
$(0.4, 0.8)$	0.3985	0.3946	0.3947
$(0.6, 0.2)$	0.6598	0.6603	0.6604
$(0.6, 0.4)$	0.6387	0.6382	0.6383
$(0.6, 0.6)$	0.6076	0.6097	0.6098
$(0.6, 0.8)$	0.5697	0.5712	0.5714
$(0.8, 0.2)$	0.6910	0.6896	0.6897
$(0.8, 0.4)$	0.7039	0.6999	0.7000
$(0.8, 0.6)$	0.7188	0.7142	0.7143
$(0.8, 0.8)$	0.7414	0.7352	0.7353
AAE	2.3×10^{-3}	1.2×10^{-4}	–

In Table 1, the numerical values of u obtained using $(M_0, N_0) = (10, 4)$ and $(M_0, N_0) = (40, 19)$ are compared with the analytical solution u in (36) at selected interior collocation points. There is an obvious improvement in the accuracy of the numerical solution when the calculation is refined by increasing the number of boundary elements and interior collocation points.

For the calculation of the flux, the approximate formula for u in (12) may be partially differentiated with respect to x_i to compute the first order partial derivatives of u with respect to x_1 and x_2 denoted by $u_{,1}$ and $u_{,2}$ repectively. The numerical values of u from the calculation using $(M_0, N_0) = (40, 19)$ are used in (12) to compute the approximate values of $u_{,1}$ and $u_{,2}$ at selected interior points. In Table 2, the approximate values obtained are compared with the values calculated from the analytical solution in (36). As may be expected, for the same set of boundary elements and interior collocation points, the accuracy of the numerical values of $u_{,1}$ and $u_{,2}$ is less than that of u, since $u_{,1}$ and $u_{,2}$ are secondary quantities obtained by post-processing the numerical values of u at the collocation points. Specifically, the average absolute errors of the numerical values of $u_{,1}$ and $u_{,2}$ in Table 2 are about 15 times larger than that of the numerical values of u in the third column of Table 1.

Table 2. Comparison of numerical and analytical values of $u_{,1}$ and $u_{,2}$ at selected interior points. The average absolute error (AAE) is given for each set of the numerical values.

Point (x_1, x_2)	Numerical $u_{,1}$	Analytical $u_{,1}$	Numerical $u_{,2}$	Analytical $u_{,2}$
$(0.25, 0.25)$	0.5164	0.5161	−0.3830	−0.3829
$(0.25, 0.50)$	0.6656	0.6659	−0.4374	−0.4376
$(0.25, 0.75)$	0.7683	0.768 2	−0.5049	−0.5048
$(0.50, 0.25)$	0.3437	0.3438	−0.1877	−0.1875
$(0.50, 0.50)$	0.5714	0.5714	−0.2448	−0.2449
$(0.50, 0.75)$	0.8336	0.8333	−0.3337	−0.3333
$(0.75, 0.25)$	0.1435	0.1437	0.01314	0.01306
$(0.75, 0.50)$	0.3617	0.3615	0.01912	0.01902
$(0.75, 0.75)$	0.6954	0.6956	0.02987	0.03025
AAE	2.0×10^{-3}	–	2.1×10^{-3}	–

The numerical procedure is executed on a desktop personal computer with an entry level Intel Core i3-4150 processor (3M cache, 3.5 GHz). For $(M_0, N_0) = (10, 4)$ which involves 192 unknowns, the CPU time used to set up and solve the equations is around 0.01 seconds. For $(M_0, N_0) = (40, 19)$ with the number of unknowns increased to 1362, the CPU time used increases to about 4 seconds.

Problem 2. Take the coefficients q_i and k_{ij} as

$$\begin{aligned} q_1 &= -e^{2x_1+x_2} - 6e^{-x_1} - 2e^{-x_2} - 4,\ q_2 = 1, \\ k_{11} &= 1 + e^{-x_1},\ k_{22} = e^{-x_2},\ k_{12} = k_{21} = 0, \end{aligned} \tag{38}$$

and solve (3) in the solution domain $0 < x_1 < 1$, $0 < x_2 < 1$ subject to the boundary conditions

$$\left.\begin{aligned} \sum_{j=1}^{2} k_{2j} \frac{\partial u}{\partial x_j}\bigg|_{x_2=0} &= -e^{-2x_1} \\ \sum_{j=1}^{2} k_{2j} \frac{\partial u}{\partial x_j}\bigg|_{x_2=1} &= -e^{-2(1+x_1)} \end{aligned}\right\} \text{ for } 0 < x_1 < 1,$$

$$\left.\begin{aligned} \sum_{j=1}^{2} k_{1j} \frac{\partial u}{\partial x_j}\bigg|_{x_1=0} &= -4e^{-x_2} \\ u(1, x_2) &= e^{-2-x_2} \end{aligned}\right\} \text{ for } 0 < x_2 < 1. \tag{39}$$

As in Problem 1, the boundary of the square solution domain is discretized into $4M_0$ equal length elements and N_0^2 interior collocation points are selected for the numerical calculation. Also, the values of $k_{ij}^{(0)}$ are computed using the formula in (37).

It is easy to check that the analytical solution of the problem here is given by

$$u(x_1, x_2) = e^{-2x_1-x_2}. \tag{40}$$

Once the numerical values of u at all the collocation points, that is, the values of $u^{(m)}$ $(m = 1, 2, \cdots, 4M_0 + N_0^2)$ are known, the formula for u in (12) can be used to compute approximately the solution at any point of interest in the solution domain. In Table 3, approximate values of u thus computed using $(M_0, N_0) = (10, 5)$ and $(M_0, N_0) = (20, 10)$ are compared with the analytical solution at various points on the line $x_2 = 0.50$ in the solution domain. The numerical values agree well with the values from the analytical solution and convergence in the numerical solution is observed when the values of M_0 and N_0 are doubled.

Table 3. Comparison of numerical and analytical values of $u(x_1, 0.50)$ for various values of x_1. The average absolute error (AAE) is given for each set of the numerical values.

x_1	$M_0 = 10$ $N_0 = 5$	$M_0 = 20$ $N_0 = 10$	Analytical
0.1	0.4944	0.4967	0.4966
0.2	0.4045	0.4050	0.4066
0.3	0.3384	0.3330	0.3329
0.4	0.2695	0.2718	0.2725
0.5	0.2229	0.2230	0.2231
0.6	0.1817	0.1826	0.1827
0.7	0.1475	0.1485	0.1496
0.8	0.1219	0.1226	0.1225
0.9	0.1016	0.09826	0.1003
AAE	2.0×10^{-3}	6.6×10^{-4}	–

Problem 3. Solve (3) together with

$$q_1 = q_2 = 0,\ k_{11} = \frac{1}{2}k_{22} = e^{c(x_1^2+x_2^2)},\ \ k_{12} = k_{21} = 0, \tag{41}$$

in the quarter circular domain $x_1^2 + x_2^2 < 1$, $x_1 > 0$, $x_2 > 0$, subject to the boundary conditions

$$\begin{aligned} u(x_1, 0) &= 0 \text{ for } 0 < x_1 < 1, \\ k_{11}x_1\frac{\partial u}{\partial x_1} + k_{22}x_2\frac{\partial u}{\partial x_2} &= 1 \text{ on } x_1^2 + x_2^2 = 1,\ \ x_1 > 0,\ \ x_2 > 0, \\ u(0, x_2) &= 0 \text{ for } 0 < x_2 < 1. \end{aligned} \tag{42}$$

Note that c in (41) is a real constant.

For the numerical solution of the boundary value problem, each of the the three parts of the boundary, namely where $x_1 = 0$, $x_2 = 0$ and $x_1^2 + x_2^2 = 1$, is discretized into M_0 boundary elements (so that $M = 3M_0$) and N_0^2 collocation points in the interior solution domain are taken to be given by

$$(x_1, x_2) = \frac{m}{N_0 + 1}(\cos\frac{n\pi}{2(N_0 + 1)}, \sin\frac{n\pi}{2(N_0 + 1)}) \text{ for } m, n = 1, 2, \cdots, N_0. \tag{43}$$

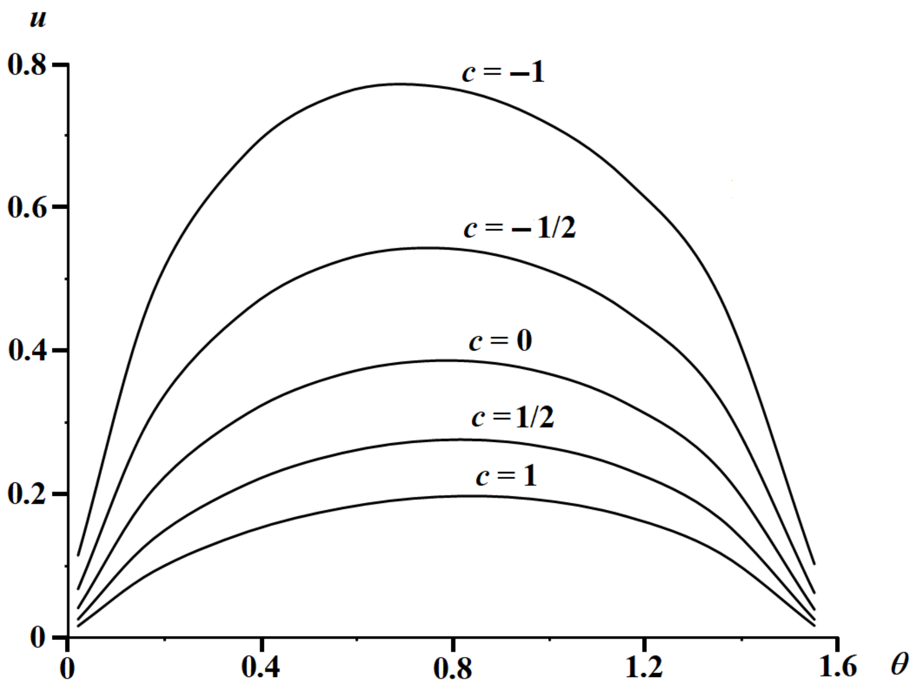

Figure 1. Plots of u on $x_1^2 + x_2^2 = 1$, $x_1 > 0$, $x_2 > 0$, against $\theta = \arctan(x_2/x_1)$.

For $(M_0, N_0) = (40, 6)$, the numerically obtained values of the u on $x_1^2 + x_2^2 = 1$, $x_1 > 0$, $x_2 > 0$ (where $k_{11}x_1\partial u/\partial x_1 + k_{22}x_2\partial u/\partial x_2$ is specified) are plotted against $\theta = \arctan(x_2/x_1)$ for selected values of c in Figure 1. The plots in the figure are very close to those given in Ang et al. [5] where the same problem in the context of a functionally graded material undergoing an antiplane deformation was solved numerically using a different boundary element procedure.

5 Summary and final remarks

A numerical method based on boundary integral equation and radial basis function approximation is derived and successfully implemented in the computer for solving a two-dimensional steady state anisotropic heat or mass diffusion equation with a linear source term and general variable coefficients. The numerical procedure requires only the boundary of the solution domain to be discretized into elements. However, the formulation does not involve only unknowns at boundary collocation points but also at well distributed collocation points in the interior of the solution domain. The validity and accuracy of the boundary element procedure is verified by applying it to solve some specific problems with known solutions. The numerical solutions obtained show convergence and agree well with the known solutions.

In the current paper, the boundary integral equation in the formulation is discretized using only constant elements. With constant elements, the accuracy of the numerical solution is expected to be $O(h)$ (where h is the length of a typical element). This is reflected in the reduction in the errors of the numerical solutions of the specific test problems when the computation is refined by increasing the number of boundary elements and interior collocation points. In spite of the constant elements, we have managed to obtain reasonably accurate numerical solutions for the test problems even with a relatively low number of boundary elements and interior collocation points. Higher order elements such as the discontinuous linear elements may be employed in the boundary integral equation if a more accurate numerical solution is desired such as in the post-processing computation of the flux. The implementation of higher order elements is, however, algebraically more tedious.

The proposed solution approach based on boundary integral and radial basis function approximations provides an interesting and viable alternative to existing numerical methods for analyzing heat and mass transfer in anisotropic media with spatially varying material properties. It may be explored further and extended to solve more complicated problems involving anisotropic media such as those considered in Aksoy and Şenocak [2], Baron [7], Li et al. [26], Marin and Lesnic [27] and Fahmy [20, 21].

Acknowledgment

The author would like to thank the Editor-in-Chief and the anonymous reviewers for their constructive criticisms and for the opportunity to revise and improve on the work presented here.

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Appendix

If the points (x_1, x_2) on $C^{(m)}$ are expressed using the parametric equations

$$\left.\begin{aligned} x_1 &= \frac{1}{2}(x_1^{(m+1)} + x_1^{(m)}) + \frac{1}{2}(x_1^{(m+1)} - x_1^{(m)})t \\ x_2 &= \frac{1}{2}(x_2^{(m+1)} + x_2^{(m)}) + \frac{1}{2}(x_2^{(m+1)} - x_2^{(m)})t \end{aligned}\right\} \text{ for } -1 \leq t \leq 1,$$

where $(x_1^{(m)}, x_2^{(m)})$ and $(x_1^{(m+1)}, x_2^{(m+1)})$ are the endpoints of the element $C^{(m)}$, the integrals over $C^{(m)}$ in (28) can be rewritten as

$$\begin{aligned} &\int\limits_{C^{(m)}} \Phi(x_1, x_2, \xi_1^{(n)}, \xi_2^{(n)}) ds(x_1, x_2) \\ = &\frac{\ell^{(m)}}{2\pi\sqrt{k_{11}^{(0)}k_{22}^{(0)} - (k_{12}^{(0)})^2}} \int\limits_{-1}^{1} \ln(A^{(m)}t^2 + B^{(mn)}t + E^{(mn)})dt, \quad \text{(A1)} \end{aligned}$$

and

$$\int\limits_{C^{(m)}} \Gamma(x_1, x_2, \xi_1^{(n)}, \xi_2^{(n)}) ds(x_1, x_2)$$

$$= \frac{\ell^{(m)}}{4\pi\sqrt{k_{11}^{(0)} k_{22}^{(0)} - (k_{12}^{(0)})^2}} \int\limits_{-1}^{1} \frac{(G^{(m)}t + H^{(mn)})dt}{A^{(mn)}t^2 + B^{(mn)}t + E^{(mn)}}, \tag{A2}$$

where $\ell^{(m)} = \sqrt{(x_1^{(m+1)} - x_1^{(m)})^2 + (x_2^{(m+1)} - x_2^{(m)})^2}$ and $A^{(m)}$, $B^{(mn)}$, $E^{(mn)}$, $G^{(m)}$ and $H^{(mn)}$ are real parameters defined by

$$\begin{aligned}
A^{(m)} &= \frac{1}{4}(x_1^{(m+1)} - x_1^{(m)} + \mathrm{Re}\{\tau\}(x_2^{(m+1)} - x_2^{(m)}))^2 \\
&\quad + \frac{1}{4}(\mathrm{Im}\{\tau\})^2(x_2^{(m+1)} - x_2^{(m)})^2, \\
B^{(mn)} &= \frac{1}{2}(x_1^{(m+1)} - x_1^{(m)} + \mathrm{Re}\{\tau\}(x_2^{(m+1)} - x_2^{(m)})) \\
&\quad \times (x_1^{(m+1)} + x_1^{(m)} - 2\xi_1^{(n)} + \mathrm{Re}\{\tau\}(x_2^{(m+1)} + x_2^{(m)} - 2\xi_2^{(n)})) \\
&\quad + \frac{1}{2}(\mathrm{Im}\{\tau\})^2(x_2^{(m+1)} - x_2^{(m)})(x_2^{(m+1)} + x_2^{(m)} - 2\xi_2^{(n)}), \\
E^{(mn)} &= \frac{1}{4}(x_1^{(m+1)} + x_1^{(m)} - 2\xi_1^{(n)} + \mathrm{Re}\{\tau\}(x_2^{(m+1)} + x_2^{(m)} - 2\xi_2^{(n)}))^2 \\
&\quad + \frac{1}{2}(\mathrm{Im}\{\tau\})^2(x_2^{(m+1)} - x_2^{(m)})(x_2^{(m+1)} + x_2^{(m)} - 2\xi_2^{(n)})^2, \\
G^{(m)} &= \mathrm{Re}\{L^{(m)}(x_1^{(m+1)} - x_1^{(m)} + \mathrm{Re}\{\tau\}(x_2^{(m+1)} - x_2^{(m)}) \\
&\quad - i\,\mathrm{Im}\{\tau\}(x_2^{(m+1)} - x_2^{(m)}))\} \\
H^{(mn)} &= \mathrm{Re}\{L^{(m)}(x_1^{(m+1)} + x_1^{(m)} - 2\xi_1^{(n)} + \mathrm{Re}\{\tau\}(x_2^{(m+1)} + x_2^{(m)} - 2\xi_2^{(n)}) \\
&\quad - i\,\mathrm{Im}\{\tau\}(x_2^{(m+1)} + x_2^{(m)} - 2\xi_2^{(n)}))\}, \\
L^{(m)} &= (k_{11}^{(0)} + \tau k_{12}^{(0)})n_1^{(m)} + (k_{21}^{(0)} + \tau k_{22}^{(0)})n_2^{(m)}.
\end{aligned}$$

Note that $n_i^{(m)}$ is the x_i component of the unit vector that is normal to $C^{(m)}$ and that points out of the solution domain.

For $m \neq n$, the expression $4A^{(mn)}E^{(m)} - (B^{(mn)})^2$ is strictly greater than zero, and hence the integrals in (A1) and (A2) are proper. The proper integrals may be evaluated either approximately by using a numerical integration formula or by using the analytical formulae

$$\begin{aligned}
\int \ln(at^2+bt+c)dt &= t(\ln(a)-2)+(t+\frac{b}{2a})\ln(t^2+\frac{b}{a}t+\frac{c}{a}) \\
&\quad +\frac{1}{a}\sqrt{4ac-b^2}\arctan(\frac{2at+b}{\sqrt{4ac-b^2}}), \\
\int \frac{(gt+h)dt}{at^2+bt+c} &= \frac{g}{2a}\ln(at^2+bt+c) \\
&\quad ++\frac{(2ah-gb)}{a\sqrt{4ac-b^2}}\arctan(\frac{2at+b}{\sqrt{4ac-b^2}}).
\end{aligned}$$

which are valid for $4ac-b^2>0$.

For $n=m$, the integrals on the right hand side of (A1) and (A2) are improper to be interpreted in the Cauchy principal sense (with an integrable singularity at $t=0$), since $B^{(mm)}=0$, $E^{(mm)}=0$ and $H^{(mm)}=0$. The resulting Cauchy principal integrals are, however, straightforward to evaluate analytically since their integrands are in considerably simple forms.