On the indentation of an inhomogeneous anisotropic elastic material by multiple straight rigid punches

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Abstract

A generalised plane strain problem concerning the indentation of an inhomogeneous anisotropic elastic material by mutiple straight rigid punches is considered. The problem is reduced to a boundary integral equation with the stresses over the contact regions being represented in terms of Chebyschev polynomials. The boundary integral equation is solved numerically for some particular antiplane contact problems involving one or two contact regions and the stress intensity factors at the ends of the contact regions are calculated. The effect of anisotropy and inhomogeneity on the stress intensity factors is examined through the illustrative examples. The analysis is relevant for a class of geomechanics problems involving inhomogeneous materials.

Key words: Boundary element method, contact problems, anisotropy, inhomogeneous elasticity.

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1 Introduction

Plane contact problems for homogeneous anisotropic materials have been studied quite widely in the past and analytical solutions have been obtained for a variety of problems (see for example Green and Zerna [1], Lekhnitskii [2]). Also a number of generalised plane contact problems for semi-infinite homogeneous anisotropic materials have been solved analytically (see for example Clements [3], [4], [5], [6], Clements and Toy [7] and Fan and Keer [8]) by employing a solution to the equations of elasticity developed by Eshelby, Read and Shockley [9] and Stroh [10].

In some cases where analytical solutions have not been available, numerical methods such as the boundary element method (BEM) have been able to provide numerical values for the stresses and displacements within a material subject to the contact boundary conditions. The standard BEM solution however causes difficulties, or becomes inaccurate in the case when the stress becomes singular at the edges of the contact region.

One procedure for circumventing the problem of stress singularities in the BEM is to introduce a suitable Green's function into the boundary integral equation so as to remove the need to integrate over the segment of the boundary where the singular stress occurs. This procedure has been employed by Clements [11] to consider a certain class of boundary value problems for homogeneous anisotropic materials. However this method of solution does involve boundary integral equations with quite complicated kernels and these lead to

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some difficulties in the implementation of the numerical procedure.

An alternative approach involving the representation of the contact stresses in terms of Chebyschev polynomials is considered in the current study which addresses the problem of contact of an anisotropic inhomogeneous elastic body by a rigid punch. A boundary integral method is formulated which facilitates the computation of the displacements and stresses and, in particular, the stress singularities at the ends of the contact regions for this class of contact problems. Numerical results are obtained for some particular antiplane contact problems in order to illustrate the application of the numerical procedure.

The analysis in the paper holds for a restricted class of inhomogeneous anisotropic materials. In particular the elastic parameters for the materials are required to adopt a multi-parameter form and to satisfy a symmetry condition. These constraints limit the applicability of the analysis but it remains relevant to an important class of materials in geomechanics.

2 Statement of the Problem

Referred to a Cartesian frame $Ox_1x_2x_3$ consider an anisotropic elastic body with a geometry that does not vary in the Ox_3 direction. In the Ox_1x_2 plane let the body occupy the region Ω with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. The material has N non-intersecting continuous contact regions $C^{(\beta)} \in \partial\Omega$ with endpoints at $(a^{(\beta)}, b^{(\beta)})$ and $(c^{(\beta)}, d^{(\beta)})$ which lie on a straight line segment of the boundary (see Figure 1). On each of the contact regions $C^{(\beta)}$, β=1..N mixed boundary conditions corresponding to contact by the rigid punch are specified. On the remainder of the boundary either the displacements or tractions are specified. All of the specified boundary conditions do not vary in the Ox_3 direction.

3 Fundamental Equations

The equilibrium equations governing small generalised plane deformations of an inhomogeneous anisotropic elastic material are

$$\frac{\partial}{\partial x_j}\left[c_{ijkl}(\mathbf{x})\,\frac{\partial u_k(\mathbf{x})}{\partial x_l}\right] = 0, \tag{1}$$

where i, j, k, l = 1,2,3, $\mathbf{x} = (x_1, x_2, x_3)$, u_k denotes the displacement, $c_{ijkl}(\mathbf{x})$ the elastic moduli and the repeated summation convention (summing from 1 to 3) is used for repeated Latin subscripts. The stress displacement relations are given by

$$\sigma_{ij}(\mathbf{x}) = c_{ijkl}\,\frac{\partial u_k}{\partial x_l} \tag{2}$$

and the traction vector P_i on the boundary $\partial\Omega$ is defined as

$$P_i(\mathbf{x}) = \sigma_{ij}\,n_j = c_{ijkl}\,\frac{\partial u_k}{\partial x_l}\,n_j, \tag{3}$$

where $\mathbf{n} = (n_1, n_2)$ denotes the outward pointing normal to the boundary $\partial\Omega$.

For all points in Ω the coefficients $c_{ijkl}(\mathbf{x})$ are required to satisfy the usual symmetry condition

$$c_{ijkl} = c_{ijlk} = c_{jikl} = c_{klij} \tag{4}$$

and also sufficient conditions for the strain energy density to be positive. This requirement ensures that the system of partial differential equations is elliptic throughout Ω.

A solution to (1) is sought which is valid in the region Ω and satisfies the boundary conditions specified in the previous section.

4 Boundary integral equation

The coefficients in (1) are required to take the form

$$c_{ijkl}(\mathbf{x}) = c^{(0)}_{ijkl}\, g(\mathbf{x}), \tag{5}$$

where the $c^{(0)}_{ijkl}$ are constants and $g(x_1, x_2)$ is a twice differentiable function of the variables x_1 and x_2. Also in addition to the symmetry condition (4) the $c^{(0)}_{ijkl}$ are required to satisfy the additional condition

$$c^{(0)}_{ijkl} = c^{(0)}_{ilkj}. \tag{6}$$

Equation (1) may now be written in the form

$$c^{(0)}_{ijkl}\, \frac{\partial}{\partial x_j}\left(g\, \frac{\partial u_k}{\partial x_l}\right) = 0. \tag{7}$$

If $g(\mathbf{x})$ satisfies the equation

$$c^{(0)}_{ijkl}\, \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = 0, \tag{8}$$

then a boundary integral equation which provides a solution to (7) has been derived by Azis and Clements [12] in the form

$$\eta\, g^{1/2}(\mathbf{x}_0)\, u_m(\mathbf{x}_0) = -\int_{\partial\Omega} \Big\{ P_i(\mathbf{x}) \left[g^{-1/2}(\mathbf{x})\, \Phi_{im}(\mathbf{x}, \mathbf{x}_0) \right]$$

$$-u_i(\mathbf{x}) \left[g^{1/2}(\mathbf{x})\, \Gamma_{im}(\mathbf{x}, \mathbf{x}_0) - \mathcal{P}^{[g]}_{ki}(\mathbf{x})\, \Phi_{km}(\mathbf{x}, \mathbf{x}_0) \right] \Big\}\, ds(\mathbf{x}). \tag{9}$$

for $m = 1, 2, 3$, where $\mathbf{x}_0$ is the source point, $\eta = 0$ if $\mathbf{x}_0 \notin \Omega$, $\eta = 1$ if $\mathbf{x}_0 \in \Omega$ and $\eta = \frac{1}{2}$ if $\mathbf{x}_0 \in \partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at $\mathbf{x}_0$.

As a result of the symmetry property $c_{ijkl} = c_{klij}$ equation (8) consists of a system of six constant coefficients partial differential equations in the one dependent variable $g^{1/2}$. In general this system will be satisfied by a linear function of the two independent variables x_1, x_2. Thus $g(\mathbf{x})$ may be taken in the form

$$g(\mathbf{x}) = (e x_1 + f x_2 + g)^2\,, \tag{10}$$

where e, f and g are constants which may be used to fit the elastic moduli $c_{ijkl}(\mathbf{x}) = c^{(0)}_{ijkl}\, g(\mathbf{x})$ to given numerical data.

The $\mathcal{P}^{[g]}_{ik}(\mathbf{x})$ in (9) are defined by

$$\mathcal{P}^{[g]}_{ik}(\mathbf{x}) = c^{(0)}_{ijkl} \frac{\partial g^{1/2}}{\partial x_l}\, n_j. \tag{11}$$

Also for generalised plane problems with $\mathbf{x}_0 = (\xi_1, \xi_2)$, $\mathbf{x} = (x_1, x_2)$, the Φ_{im} and Γ_{im} in equation (9) are given by (see for example Clements and Jones [13])

$$\Phi_{im}(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \Re \left[\sum_{\alpha=1}^{3} A_{i\alpha} N_{\alpha k} \log(z_\alpha - c_\alpha) \right] d_{km}, \tag{12}$$

$$\Gamma_{im}(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \Re \left[\sum_{\alpha=1}^{3} L_{ij\alpha} N_{\alpha k} (z_\alpha - c_\alpha)^{-1} \right] n_j d_{km}, \tag{13}$$

where $\Re$ denotes the real part of a complex number, $z_\alpha = x_1 + \tau_\alpha x_2$ and $c_\alpha = \xi_1 + \tau_\alpha \xi_2$, where τ_α are the three roots with positive imaginary part of the sextic in τ

$$|c^{(0)}_{i1k1} + c^{(0)}_{i2k1}\tau + c^{(0)}_{i1k2}\tau + c^{(0)}_{i2k2}\tau^2| = 0. \tag{14}$$

The $A_{i\alpha}$ occurring in (12) are the solutions of the system

$$\left(c^{(0)}_{i1k1} + c^{(0)}_{i2k1}\tau_\alpha + c^{(0)}_{i1k2}\tau_\alpha + c^{(0)}_{i2k2}\tau_\alpha^2\right) A_{k\alpha} = 0. \tag{15}$$

Also the $N_{\alpha k}$, $L_{ij\alpha}$ and d_{km} are defined by

$$\delta_{ik} = \sum_{\alpha=1}^{3} A_{i\alpha} N_{\alpha k}, \tag{16}$$

$$L_{ij\alpha} = (c^{(0)}_{ijk1} + \tau_\alpha c^{(0)}_{ijk2}) A_{k\alpha}, \tag{17}$$

$$\delta_{im} = -\frac{1}{2}\imath \sum_{\alpha=1}^{3} \left\{ L_{i2\alpha} N_{\alpha k} - \overline{L}_{i2\alpha} \overline{N}_{\alpha k} \right\} d_{km}, \tag{18}$$

where the bar denotes the complex conjugate and $\imath$ denotes the square root of minus one.

5 Boundary Element Method

The boundary integral equation (9) for this problem may be written in the form

$$\begin{aligned} \eta\, g^{1/2}(\mathbf{x}_0)\, u_m(\mathbf{x}_0) = -\int_{C\cup D} & \left\{ P_i(\mathbf{x}) \left[g^{-1/2}(\mathbf{x})\, \Phi_{im}(\mathbf{x}, \mathbf{x}_0) \right] \right. \\ & \left. - u_i(\mathbf{x}) \left[g^{1/2}(\mathbf{x})\, \Gamma_{im}(\mathbf{x}, \mathbf{x}_0) - \mathcal{P}^{[g]}_{ki}(\mathbf{x})\, \Phi_{km}(\mathbf{x}, \mathbf{x}_0) \right] \right\} ds(\mathbf{x}). \end{aligned} \tag{19}$$

where $C = C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(N)}$ is the part of the boundary of the body subjected to contact boundary conditions and $D = \partial\Omega \setminus C$ denotes the remainder of the outer boundary

On the βth contact region lying between $(a^{(\beta)}, b^{(\beta)})$ and $(c^{(\beta)}, d^{(\beta)})$. the coordinates x_1 and x_2 may be written in terms of a single parameter t in the form

$$x_1 = X_1^{(\beta)}(t) = [(c^{(\beta)} - a^{(\beta)})t + (c^{(\beta)} + a^{(\beta)})]/2 \;\text{ for }\; t \in [-1, 1], \tag{20}$$

$$x_2 = X_2^{(\beta)}(t) = [(d^{(\beta)} - b^{(\beta)})t + (b^{(\beta)} + d^{(\beta)})]/2 \;\text{ for }\; t \in [-1, 1]. \tag{21}$$

For the class of problems under consideration the unknown tractions become singular at the edges of the contact region. For the purposes of numerical evaluation of the integrals it is convenient to write equation (19) in the form

$$\begin{aligned}
\eta\, g^{1/2}(\mathbf{x}_0)\, u_m(\mathbf{x}_0) = &- \int_D \Big\{ P_i(\mathbf{x}) \left[g^{-1/2}(\mathbf{x})\, \Phi_{im}(\mathbf{x}, \mathbf{x}_0) \right] \\
&- u_i(\mathbf{x}) \left[g^{1/2}(\mathbf{x})\, \Gamma_{im}(\mathbf{x}, \mathbf{x}_0) - \mathcal{P}_{ki}^{[g]}(\mathbf{x})\, \Phi_{km}(\mathbf{x}, \mathbf{x}_0) \right] \Big\}\, ds(\mathbf{x}). \\
&+ \sum_{\beta=1}^{N} \frac{L^{(\beta)}}{2} \int_{-1}^{1} u_i(\mathbf{X}^{(\beta)}(t)) \left[g^{1/2}(\mathbf{X}^{(\beta)}(t))\, \Gamma_{im}(\mathbf{X}^{(\beta)}(t), \mathbf{x}_0) \right. \\
&\qquad\qquad \left. - \mathcal{P}_{ki}^{[g]}(\mathbf{X}^{(\beta)}(t))\, \Phi_{km}(\mathbf{X}^{(\beta)}(t), \mathbf{x}_0) \right] dt \\
&- \sum_{\beta=1}^{N} \frac{L^{(\beta)}}{2} \int_{-1}^{1} P_i(\mathbf{X}^{(\beta)}(t)) \left[g^{-1/2}(\mathbf{X}^{(\beta)}(t))\, \Phi_{im}(\mathbf{X}^{(\beta)}(t), \mathbf{x}_0) \right] dt,
\end{aligned} \tag{22}$$

where $L^{(\beta)}$ is the length of the βth contact region, $(n_1^{(\beta)}, n_2^{(\beta)}) = ([d^{(\beta)} - b^{(\beta)}]/L^{(\beta)}, -[c^{(\beta)} - a^{(\beta)}]/L^{(\beta)})$ and $\mathbf{X}^{(\beta)}(t) = (X_1^{(\beta)}(t), X_2^{(\beta)}(t))$.

To accommodate the singularities at the end of the contact regions the following approximation is made for the purposes of evaluating the last integral in (22) for the cases when the component P_i of traction is unknown

$$g^{-1/2}(\mathbf{X}^{(\beta)}(t))\, P_i(\mathbf{X}^{(\beta)}(t)) \simeq \frac{1}{\sqrt{1-t^2}} \sum_{j=1}^{J} \alpha_{ij}^{(\beta)} T_{j-1}(t), \tag{23}$$

where $T_j(t)$ denotes the Chebyshev polynomial of the first kind and the $\alpha_{ij}^{(\beta)}$ are constants to be determined. To evaluate the other integrals in (22) the part of the outer boundary D is discretised by a series of M line segments

$$D \simeq D_1 \cup D_2 \cup \cdots \cup D_M,$$

and the integrals involving the displacement and the given tractions over the interval [-1,1] are divided into J line segments $L_k^{(\beta)}$ of equal length. Over each of these line segments the displacement and tractions are taken to be constant so that on D, $u_i \simeq u_i^{(k)}$ (constant) and $P_i \simeq P_i^{(k)}$ (constant) for $k = 1, 2, \ldots M$, while on the segments $L_k^{(\beta)}$, $u_i \simeq u_{i\beta}^{(k)}$ (constant) and $P_i \simeq P_{i\beta}^{(k)}$ (constant) for $k = 1, 2, \ldots J$ and $\beta = 1, 2, \ldots N$. Use of these approximations in equation (22) allows it to be approximated by

$$\begin{aligned}
\eta\, g^{1/2}(\mathbf{x}_0)\, u_m(\mathbf{x}_0) \simeq & -\sum_{k=1}^{M} P_i^{(k)} \int_{D_k} \left[g^{-1/2}(\mathbf{x})\, \Phi_{im}(\mathbf{x}, \mathbf{x}_0) \right] ds(\mathbf{x}) \\
& + \sum_{k=1}^{M} u_i^{(k)} \int_{D_k} \left[g^{1/2}(\mathbf{x})\, \Gamma_{im}(\mathbf{x}, \mathbf{x}_0) - \mathcal{P}_{ki}^{[g]}(\mathbf{x})\, \Phi_{km}(\mathbf{x}, \mathbf{x}_0) \right] ds(\mathbf{x}) \\
& + \sum_{\beta=1}^{N} \frac{L^{(\beta)}}{2} \sum_{k=1}^{J} u_{i\beta}^{(k)} \int_{L_k^{(\beta)}} \left[g^{1/2}(\mathbf{X}^{(\beta)}(t))\, \Gamma_{im}(\mathbf{X}^{(\beta)}(t), \mathbf{x}_0) \right. \\
& \qquad \left. - \mathcal{P}_{ki}^{[g]}(\mathbf{X}^{(\beta)}(t))\, \Phi_{km}(\mathbf{X}^{(\beta)}(t), \mathbf{x}_0) \right] dt \\
& - \sum_{\beta=1}^{N} \frac{L^{(\beta)}}{2} \sum_{k=1}^{J} \Lambda_{km}^{(\beta)}(\mathbf{x}_0),
\end{aligned} \tag{24}$$

where if P_i is given on $C^{(\beta)}$ then $\Lambda_{km}^{(\beta)}(\mathbf{x}_0)$ is given by

$$\Lambda_{km}^{(\beta)}(\mathbf{x}_0) = P_{r\beta}^{(k)} \int_{L_k^{(\beta)}} \left[g^{-1/2}(\mathbf{X}^{(\beta)}(t))\, \Phi_{rm}(\mathbf{X}^{(\beta)}(t), \mathbf{x}_0) \right] dt, \tag{25}$$

while if P_i is unknown on $C^{(\beta)}$ then $\Lambda_{km}^{(\beta)}(\mathbf{x}_0)$ is given by

$$\Lambda_{km}^{(\beta)}(\mathbf{x}_0) = \frac{1}{2\pi}\alpha_{rk}^{(\beta)}\Re\left[\sum_{\alpha=1}^{3} A_{r\alpha}N_{\alpha l}d_{lm}\int_{-1}^{1}\frac{T_{k-1}(t)}{\sqrt{1-t^2}}\log(z_\alpha^{(\beta)}(t)-c_\alpha)dt\right], \quad (26)$$

where

$$z_\alpha^{(\beta)}(t) = X_1^{(\beta)}(t) + \tau_\alpha X_2^{(\beta)}(t). \quad (27)$$

Equation (24) generates $3M$ linear equations when $\mathbf{x}_0$ corresponds to the midpoints of the boundary segments D_k. Since there are $3(M+NJ)$ unknowns, the remaining $3NJ$ equations may be obtained by evaluating equation (24) at $\mathbf{x}_0 = \mathbf{X}^{(\beta)}(s_p)$, setting $s_p = \cos([2p-1]\pi/[2J])$, $(p = 1, 2, \cdots, J)$ for each contact region $C^{(\beta)}$.

The integrals in (24) and (25) may be evaluated using standard boundary element procedures (see Clements and Jones [13]).

As $\mathbf{x}_0 = (\xi_1, \xi_2)$ approaches the βth contact region, the integral (26) over the βth contact region with $k \neq i$ may be conveniently evaluated by making the substitution t=sinθ and using a standard numerical integration rule on the resulting integral. Also as $\mathbf{x}_0 = (\xi_1, \xi_2)$ approaches the βth contact region, the integral (26) over the βth contact region contains a logarithmic singularity.

Now if $\mathbf{x}_0$ is on the βth contact region $C^{(\beta)}$ then the coordinates ξ_1 and ξ_2 may be written in terms of a single parameter s in the form

$$\xi_1 = X_1^{(\beta)}(s) = [(c^{(\beta)} - a^{(\beta)})s + (c^{(\beta)} + a^{(\beta)})]/2 \text{ for } s \in [-1, 1], \quad (28)$$

$$\xi_2 = X_2^{(\beta)}(s) = [(d^{(\beta)} - b^{(\beta)})s + (b^{(\beta)} + d^{(\beta)})]/2 \text{ for } s \in [-1, 1]. \quad (29)$$

Hence if $\mathbf{x}_0 \in C^{(\beta)}$ and $\mathbf{x} \in C^{(\beta)}$ then

$$
\begin{aligned}
z_\alpha^{(\beta)}(t) - c_\alpha &= X_1^{(\beta)}(t) + \tau_\alpha X_2^{(\beta)}(t) - X_1^{(\beta)}(s) - \tau_\alpha X_2^{(\beta)}(s) \\
&= \frac{1}{2}\left[(c^{(\beta)} - a^{(\beta)}) + \tau_\alpha(d^{(\beta)} - b^{(\beta)})\right](t-s).
\end{aligned} \tag{30}
$$

Equation (30) may be used together with (16) in (26) to provide

$$
\begin{aligned}
\Lambda_{km}^{(\beta)}(\mathbf{x}_0) &= \alpha_{rk}^{(\beta)}\left[f_{rm}^{(\beta)}\int_{-1}^{1}\frac{T_{k-1}(t)}{\sqrt{1-t^2}}dt + d_{rm}\int_{-1}^{1}\frac{T_{k-1}(t)}{\sqrt{1-t^2}}\log|t-s|dt\right] \\
&= \pi\alpha_{r1}^{(\beta)}f_{rm}^{(\beta)} + \alpha_{rk}^{(\beta)}d_{rm}\begin{cases} -\frac{1}{2(k-1)}T_{k-1}(s) & \text{for } k>1 \\ -\frac{1}{2}\log 2 & \text{for } k=1. \end{cases}
\end{aligned} \tag{31}
$$

where the orthogonality property of Chebyshev polynomials has been employed to evaluate the first integral and the $f_{rm}^{(\beta)}$ is given by

$$
f_{rm}^{(\beta)} = \frac{1}{2\pi}\Re\sum_{\alpha=1}^{3}A_{r\alpha}N_{\alpha l}\,\log\left\{\left[(c^{(\beta)} - a^{(\beta)}) + \tau_\alpha(d^{(\beta)} - b^{(\beta)})\right]/2\right\}\,d_{lm}. \tag{32}
$$

Since the total number of linear equations is now equal to the number of unknowns, the entire system may be solved. The stress singularities are calculated from the $\alpha_{kj}^{(\beta)}$s and the displacements at any point in the material may be calculated from (24).

6 Numerical Results

For the purposes of illustrating the procedure outlined in the previous sections some particular antiplane contact problems will be considered in this section. Specifically consider an anisotropic elastic material for which the plane $x_3 = 0$ is a plane of elastic symmetry. For such a material the generalised plane problem uncouples into separate plane and antiplane problems (see for example Clements [14]). For the antiplane problem the elastic moduli of interest are $c_{1313}, c_{1323}, c_{2323}$. Then from the analysis in previous sections it follows that the relevant constants are given by $A_{31} = 1$, $L_{321} = c^{(0)}_{3231} + \tau_1 c^{(0)}_{3232}$, $N_{13} = 1$ where τ_1 is the root with positive imaginary part of the equation $c^{(0)}_{1313} + 2\tau c^{(0)}_{1323} + \tau^2 c^{(0)}_{2323} = 0$. The only non zero displacement and traction for the antiplane problem are, respectively, u_3 and P_3 while the non zero stresses are σ_{13} and σ_{23}.

From (23) with $J = 4$ the antiplane traction over the contact region is given by

$$
\begin{aligned}
\frac{P_3(\mathbf{X}^{(\beta)}(t))}{\mu_0} &\simeq \frac{1}{\sqrt{1-t^2}} \left[g^{1/2}(\mathbf{X}^{(\beta)}(t)) \sum_{j=1}^{4} \overline{\alpha}^{(\beta)}_{ij} T_{j-1}(t) \right] \\
&= \frac{K^{(\beta)}(t)}{\sqrt{1-t^2}},
\end{aligned} \tag{33}
$$

where μ_0 is a reference shear stress, $\overline{\alpha}^{(\beta)}_{ij} = \alpha^{(\beta)}_{ij}/\mu_0$ and $K^{(\beta)}(t)$ is given by

$$
\begin{aligned}
K^{(\beta)}(t) &= g^{1/2}(\mathbf{X}^{(\beta)}(t)) \sum_{j=1}^{4} \overline{\alpha}^{(\beta)}_{ij} T_{j-1}(t) \\
&= g^{1/2}(\mathbf{X}^{(\beta)}(t)) \, [\overline{\alpha}^{(\beta)}_{31} + \overline{\alpha}^{(\beta)}_{32} t + \overline{\alpha}^{(\beta)}_{33}(2t^2-1) + \overline{\alpha}^{(\beta)}_{34}(4t^3-3t)].
\end{aligned} \tag{34}
$$

Once the coefficients $\overline{\alpha}^{(\beta)}_{ij}$ are determined this formula permits the tractions over the contact regions to be calculated.

6.1 Case A: Homogeneous materials - the effect of anisotropy

In this section numerical results are presented for some antiplane contact problems for a homogeneous anisotropic elastic material with a geometry shown in Figure 2. For the problems considered the bottom surface of the material ($x_2/l = -2$ where l is a reference length) is held fixed ($u_3 = 0$) and the contact regions on the top surface of the material are subjected to a constant anti-plane displacement. The remainder of the boundary is traction free ($P_3 = 0$). For each contact problem considered numerical values for the unknown coefficients $\overline{\alpha}_{3k}^{(\beta)}$ and the unknown boundary displacements and forces outside the contact region are obtained by solving the system of linear equations (24). In the calculation of the numerical results, convergence to three decimal places was obtained by setting $J = 4$ for each contact region, and by dividing the outer boundary $C \setminus D$ into equally spaced segments of size 0.1.

Table 1 shows calculated values of the coefficients $\overline{\alpha}_{3k}^{(1)}$ for the instance of a single contact region running from $x_1/l = -0.5$ to $x_1/l = 0.5$ along the upper surface $x_2/l = 2$ over which a constant anti-plane displacement of $u_3/u_0 = 1$ is specified where u_0 is a reference displacement. These values have been listed first for the special case of an isotropic material for which the elastic constants are given by

$$c_{1331} = \mu_0 \ , c_{1332} = 0 \ , \ c_{2332} = \mu_0 .$$

The table also contains values for an anisotropic material with elastic moduli

$$c_{1331} = \mu_0 \ , c_{1332} = 0 \ , \ c_{2332} = 2\mu_0 .$$

The values obtained have been compared against known analytical solutions for a similar problem, described in Clements [14]. This analytical solution required knowledge of the total load, which was obtained by numerical integration of the stress σ_{32}/μ_0 over the bottom surface $x_2/l = -2$. Similarly the displacement profiles for these two cases have been plotted and compared against known analytic solutions in Figures 3 and 4. The analytical solution is for a slightly different problem from that solved numerically, in that the analytical solution is for an infinite half-plane, while the numerical solution is for a problem in a finite material. As a result, the two solutions agree very well close to the contact region, but become distinct further away.

Table 2 gives values for the stress intensity factors (SIFs) $K^{(1)}(1)$ and $K^{(1)}(-1)$ obtained by employing (34) together with the numerical values given in Table 1. The results indicate that for this particular problem the SIFs at the two ends of the contact region are equal with the SIFs for the anisotropic problem greater than the SIFs for the isotropic problem due to the larger value of the shear modulus c_{2323}.

For the two contact problem depicted in Figure 2, Table 3 and Table 4 provide numerical values for the coefficients $\overline{\alpha}_{3k}^{(\beta)}$ and SIFs for an anisotropic material having the same elastic moduli as the second case in Table 1. The first contact region runs from $x_1/l = -2.5$ to $x_1/l = -1.5$, over which the anti-plane displacement is given by $u_3/u_0 = 1$. The second contact region runs from $x_1/l = 1.5$ to $x_1/l = 2.5$ and is given an anti-plane displacement of $u_3/u_0 = \rho$ (ρ constant). The tables list the coefficients $\overline{\alpha}_{3k}^{(\beta)}$ and the SIFs for three different values of ρ.

The results in Tables 3 and 4 when taken together with the formula given by equations (33) and (34) indicate the effect of the interaction of the two contact regions on the stress distribution over the contact regions.

6.2 Case B: Inhomogeneous anisotropic materials

Table 5 gives the coefficients $\overline{\alpha}_{3k}^{(\beta)}$ and Table 6 gives the SIFs for a single contact lying between the points $(-0.5, 2)$ and $(0.5, 2)$ on the surface of an inhomogeneous anisotropic material occupying the region depicted in Figure 2.

The contact region is subjected to a constant antiplane displacement $u/u_0 = 1$ and there is zero displacement on the bottom surface with the remainder of the boundary traction free. The inhomogeneous materials under consideration have elastic coefficients

$$c_{ijkl} = c_{ijkl}^{(0)}(1 + 0.1x_1/l)^2 \quad \text{and} \quad c_{ijkl} = c_{ijkl}^{(0)}(1 + 0.1x_2/l)^2, \tag{35}$$

where

$$c_{1331}^{(0)} = \mu_0 \ , c_{1332}^{(0)} = 0 \ , \ c_{2332}^{(0)} = 2\mu_0. \tag{36}$$

When compared with the relevant values for anisotropy in Tables 1 and 2 the Tables 5 and 6 indicate the effect of the inhomogeneity on the stress distribution in the contact region. In comparison with the corresponding homogeneous case given in Table 1, the case when $c_{ijkl} = c_{ijkl}^{(0)}(1 + 0.1x_1/l)^2$ has the effect of reducing the SIF at the end of the contact region at $(-0.5, 2)$ and increasing the SIF at the end at $(0.5, 2)$. From a qualitative viewpoint this behaviour is in line with expectations since the elastic moduli for this particular inhomogeneous material are smaller in magnitude at $(-0.5, 2)$ and larger in magnitude at $(0.5, 2)$ than the corresponding elastic moduli for the anisotropic homogeneous problem for which results are given in Table 1. For the inhomogeneous material with $c_{ijkl} = c_{ijkl}^{(0)}(1 + 0.1x_2/l)^2$ the the SIFs at the two ends of the contact region are equal and smaller in magnitude than the SIFS for the corresponding anisotropic homogeneous problem in Table 1. For this particular inhomogeneous material the elastic moduli are larger in magnitude for $x_2 > 0$ and smaller in magnitude for $x_2 < 0$ than the corresponding elastic moduli for the anisotropic homogeneous problem in Table 1. The results indicate these smaller elastic moduli from $x_2 = 0$ to the rigid boundary at $x_2/l = -2$ reduce the antiplane force required to impose the applied displacement $u/u_0 = 1$ over the contact region.

Table 7, Table 8, Table 9 and Table 10 give the coefficients $\overline{\alpha}_{3k}^{(\beta)}$ and the SIFs $K^{(\beta)}$ for the two contact problem considered previously for a homogeneous anisotropic material (for which results are given in Table 3 and Table 4) but with the elastic coefficients now given by (35) and (36).

When compared with the results in Table 4 the results in Table 8 indicate the effect of the inhomogeneity $c_{ijkl} = c_{ijkl}^{(0)}(1 + 0.1x_1/l)^2$ on the SIFs. Specifically, compared with the corresponding homogeneous material the presence of the inhomogenity reduces the SIFs at the ends of the contact region between $(-2.5, 2)$ and $(-1.5, 2)$ and increases the SIFs at the end of the contact region between $(1.5, 2)$ and $(2.5, 2)$. From a qualitative viewpoint this move in SIFs is to be expected since the elastic coefficients are smaller than the corresponding coefficents for the homogeneous material for $x_1 < 0$ and larger for $x_1 > 0$.

Also when compared with the results in Table 3 and Table 4 the results in Table 9 and Table 10 indicate the effect of the inhomogeneity $c_{ijkl} = c_{ijkl}^{(0)}(1+0.1x_2/l)^2$ on the surface traction P_3/μ_0 over the contact regions and the SIFs. In this case the values of $\overline{\alpha}_{3k}^{(1)}$ and $\overline{\alpha}_{3k}^{(2)}$ for $\rho = 0.5$ when taken together with equation (33) indicate that the surface stress P_3/μ_0 over the contact region between $(-2.5, 2)$ and $(-1.5, 2)$ is opposite in sign to the surface stress over the contact region between $(1.5, 2)$ and $(2.5, 2)$. Thus in this case if a surface displacement of $u/u_0 = 1$ is imposed on the contact region between $(-2.5, 2)$ and $(-1.5, 2)$ then a force opposite in sign is required to impose a surface displacement of $u/u_0 = 1/2$ over the contact region between $(1.5, 2)$ and $(2.5, 2)$. As a result the SIFs $K^{(1)}(1)$ and $K^{(1)}(-1)$ for $\rho = 0.5$ are both positive while the SIFs

$K^{(2)}(1)$ and $K^{(2)}(-1)$ for $\rho = 0.5$ are both negative. With the exception of the SIF $K^{(2)}(1)$ for $\rho = 0.5$ all the SIFs for this inhomogeneous material are smaller in magnitude than the corresponding stress intensity factors for the corresponding homogenous problem.

7 Summary and final remarks

A boundary integral equation method has been obtained for the numerical solution of a generalised plane contact problem for inhomogeneous anisotropic elastic media. The problem involves the indentation of the elastic material by multiple straight rigid punches. The tractions over the contact regions between the punches and the elastic material are unknown and these tractions exhibit singularities at the ends of these regions. The unknown tractions are expressed in terms of a series involving Chebychev polynomials and the coefficients in the series are obtained numerically through a boundary integral equation. Numerical values for some particular antiplane problems illustrate how the method may be used to assess the effect of anisotropy and inhomogeneity on the contact stress over one or two contact regions.

The class of inhomogeneous anistropic materials for which the analysis holds is restricted to be of the form given by (5) and (10) with the symmetry condition (6). A consequence of this symmetry condition is that the elastic modulus relating the normal stress $\sigma_{\alpha\alpha}$ for $\alpha = 1, 2, 3$ to the normal strain $\epsilon_{\beta\beta}$ for $\beta = 1, 2, 3$ ($\beta \neq \alpha$) is equal to the elastic modulus relating the shear stress $\sigma_{\alpha\beta}$ to the shear strain $\epsilon_{\alpha\beta}$. For isotropic materials this symmetry condition gives rise to the requirement that the Lamé parameters λ and μ are equal which provides a Poisson's ratio of 0.25.

Although these constraints limit the applicability of the analysis it remains relevant for an important class of materials. In particular a Poisson's ratio of 0.25 is not uncommon for rock materials (see Manolis and Shaw [15] and Turcotte and Schubert [16]). Also the geotechnical analysis of certain subterraean regions by Ward, Burland and Gallois [17] indicates that the elastic parameters of such regions take numerical values which may be closely approximated by a multi-parameter form of the type given by (5), (6) and (10) with appropriate values of the constants $c_{ijkl}^{(0)}$, e, f and g (see Azis and Clements [12]). Thus the present analysis is applicable to relevant classes of geomechanical contact problems.

Acknowlegments

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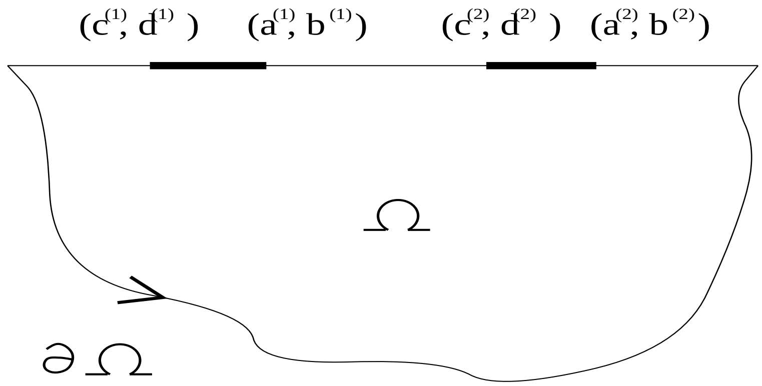

Fig. 1. Geometry of the problem for two contact regions

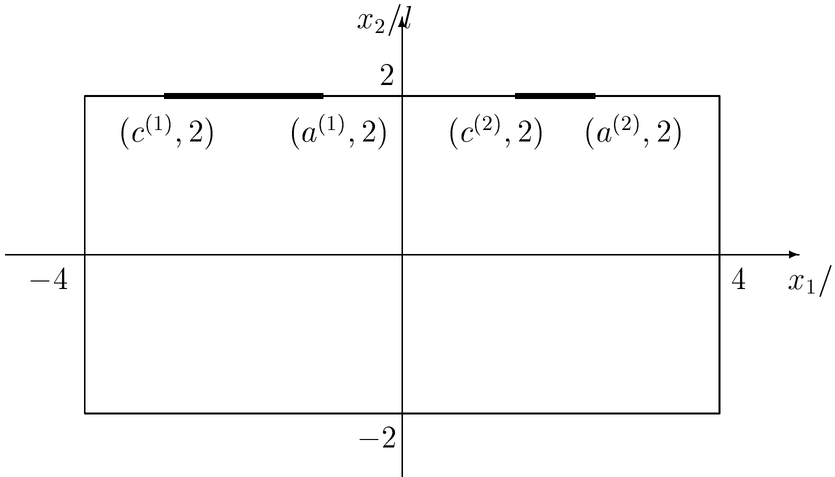

Fig. 2. Geometry for numerical examples.

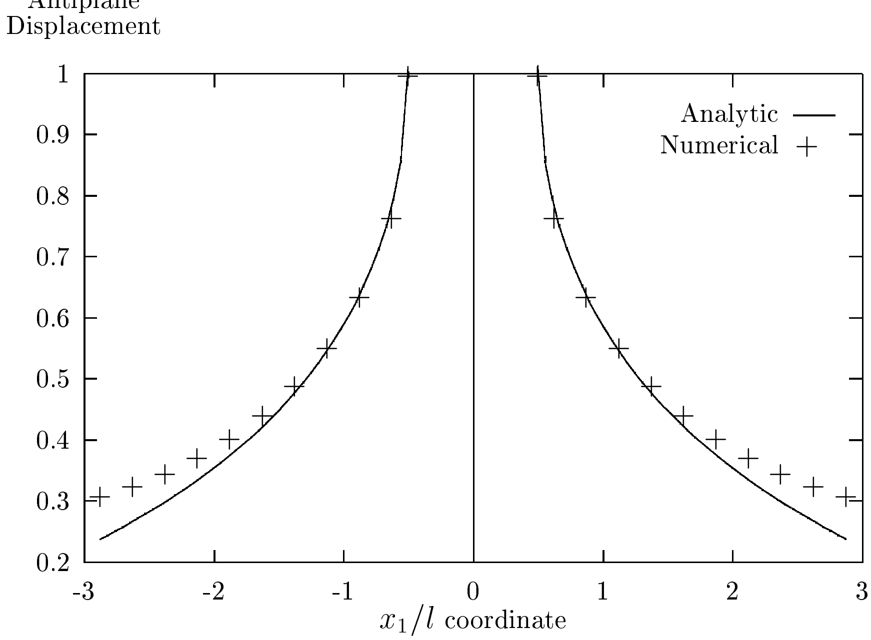

Fig. 3. Comparison of displacements in an isotropic medium.

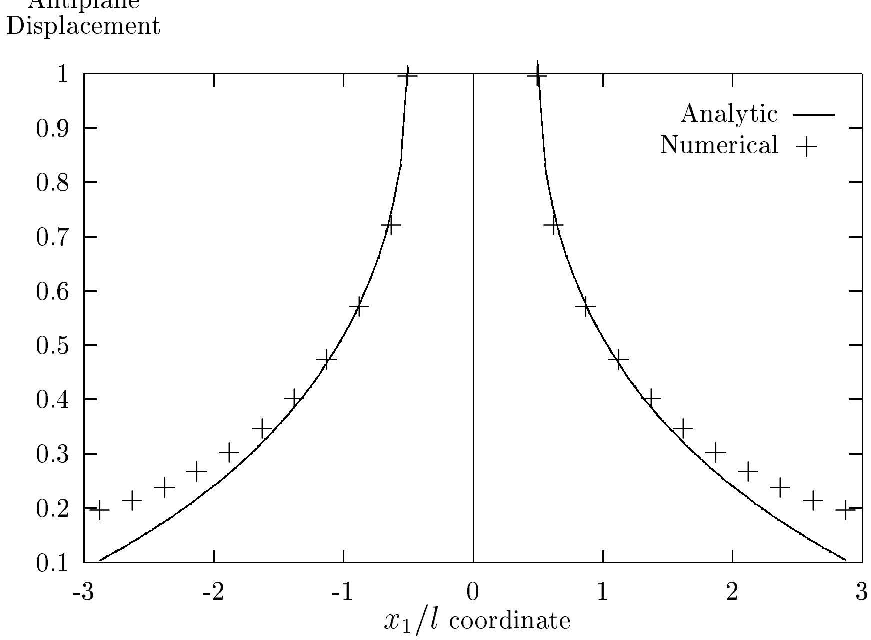

Fig. 4. Comparison of displacements in an anisotropic medium.

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Table 1

Coefficients $\overline{\alpha}_{3k}^{(\beta)}$ for a single contact.

	$\overline{\alpha}_{31}^{(1)}$	$\overline{\alpha}_{32}^{(1)}$	$\overline{\alpha}_{33}^{(1)}$	$\overline{\alpha}_{34}^{(1)}$
Isotropic (B.E.M.)	0.623	0.000	−0.004	0.000
Isotropic (Analytic)	0.625	0	0	0
Anisotropic (B.E.M.)	1.036	0.000	−0.009	0.000
Anisotropic (Analytic)	1.040	0	0	0

Table 2

SIFs $K^{(1)}$ for a single contact.

Coordinates	$(-0.5, 2)$	$(0.5, 2)$
Isotropic	$K^{(1)}(1) = 0.619$	$K^{(1)}(-1) = 0.619$
Anisotropic	$K^{(1)}(1) = 1.027$	$K^{(1)}(-1) = 1.027$

Table 3

Coefficients $\overline{\alpha}_{3k}^{(\beta)}$ for two contacts.

ρ	$\overline{\alpha}_{31}^{(1)}$	$\overline{\alpha}_{32}^{(1)}$	$\overline{\alpha}_{33}^{(1)}$	$\overline{\alpha}_{34}^{(1)}$	$\overline{\alpha}_{31}^{(2)}$	$\overline{\alpha}_{32}^{(2)}$	$\overline{\alpha}_{33}^{(2)}$	$\overline{\alpha}_{34}^{(2)}$
0.5	0.630	-0.046	-0.010	0.000	0.151	-0.045	-0.011	0.000
1.0	0.521	0.000	-0.014	0.000	0.521	0.000	-0.014	0.000
1.5	0.411	0.045	-0.018	0.000	0.891	0.046	-0.017	0.000

Table 4

SIFs $K^{(\beta)}$ for two contacts.

Coordinates	$(-2.5, 2)$	$(-1.5, 2)$	$(1.5, 2)$	$(2.5, 2)$
$K^{(\beta)}$	$K^{(1)}(1)$	$K^{(1)}(-1)$	$K^{(2)}(1)$	$K^{(2)}(-1)$
$\rho = 0.5$	0.574	0.666	0.095	0.185
$\rho = 1$	0.507	0.507	0.507	0.507
$\rho = 1.5$	0.438	0.348	0.920	0.828

Table 5

Coefficients $\overline{\alpha}_{3k}^{(\beta)}$ for a single contact.

	$\overline{\alpha}_{31}^{(1)}$	$\overline{\alpha}_{32}^{(1)}$	$\overline{\alpha}_{33}^{(1)}$	$\overline{\alpha}_{34}^{(1)}$
$c_{ijkl} = c_{ijkl}^{(0)}$	1.036	0.000	-0.009	0.000
$c_{ijkl} = c_{ijkl}^{(0)}(1 + 0.1x_1/l)^2$	1.038	-0.137	-0.008	0.000
$c_{ijkl} = c_{ijkl}^{(0)}(1 + 0.1x_2/l)^2$	0.999	0	-0.022	0

Table 6

SIFs $K^{(1)}$ for a single contact.

Coordinates	$(-0.5, 2)$	$(0.5, 2)$
$c_{ijkl} = c^{(0)}_{ijkl}(1 + 0.1x_1/l)^2$	$K^{(1)}(1) = 0.850$	$K^{(1)}(-1) = 1.225$
$c_{ijkl} = c^{(0)}_{ijkl}(1 + 0.1x_2/l)^2$	$K^{(1)}(1) = 0.977$	$K^{(1)}(-1) = 0.977$

Table 7

Coefficients $\overline{\alpha}_{3k}^{(\beta)}$ for two contacts with $c_{ijkl} = c_{ijkl}^{(0)}(1 + 0.1x_1/l)^2$.

ρ	$\overline{\alpha}_{31}^{(1)}$	$\overline{\alpha}_{32}^{(1)}$	$\overline{\alpha}_{33}^{(1)}$	$\overline{\alpha}_{34}^{(1)}$	$\overline{\alpha}_{31}^{(2)}$	$\overline{\alpha}_{32}^{(2)}$	$\overline{\alpha}_{33}^{(2)}$	$\overline{\alpha}_{34}^{(2)}$
0.5	0.540	-0.116	-0.007	0.000	0.230	-0.062	-0.011	0.000
1.0	0.412	-0.056	-0.011	0.000	0.634	-0.057	-0.016	0.000
1.5	0.285	-0.005	-0.016	0.001	1.038	-0.052	-0.021	0.001

Table 8

SIFs $K^{(\beta)}$ for two contacts with $c_{ijkl} = c_{ijkl}^{(0)}(1+0.1x_1/l)^2$.

Coordinates	$(-2.5, 2)$	$(-1.5, 2)$	$(1.5, 2)$	$(2.5, 2)$
$K^{(\beta)}$	$K^{(1)}(1)$	$K^{(1)}(-1)$	$K^{(2)}(1)$	$K^{(2)}(-1)$
$\rho = 0.5$	0.313	0.552	0.181	0.351
$\rho = 1$	0.266	0.388	0.645	0.844
$\rho = 1.5$	0.206	0.232	1.111	1.335

Table 9

Coefficients $\overline{\alpha}_{3k}^{(\beta)}$ for two contacts with $c_{ijkl} = c_{ijkl}^{(0)}(1 + 0.1x_2/l)^2$.

ρ	$\overline{\alpha}_{31}^{(1)}$	$\overline{\alpha}_{32}^{(1)}$	$\overline{\alpha}_{33}^{(1)}$	$\overline{\alpha}_{34}^{(1)}$	$\overline{\alpha}_{31}^{(2)}$	$\overline{\alpha}_{32}^{(2)}$	$\overline{\alpha}_{33}^{(2)}$	$\overline{\alpha}_{34}^{(2)}$
0.5	0.543	-0.079	-0.035	-0.001	-0.145	-0.077	-0.033	-0.001
1.0	0.353	-0.001	-0.045	0.000	0.353	0.001	-0.045	0.000
1.5	0.162	0.077	-0.055	0.001	0.720	0.079	-0.057	0.001

Table 10

SIFs $K^{(\beta)}$ for two contacts with $c_{ijkl} = c^{(0)}_{ijkl}(1 + 0.1x_2/l)^2$.

Coordinates	$(-2.5, 2)$	$(-1.5, 2)$	$(1.5, 2)$	$(2.5, 2)$
$K^{(\beta)}$	$K^{(1)}(1)$	$K^{(1)}(-1)$	$K^{(2)}(1)$	$K^{(2)}(-1)$
$\rho = 0.5$	0.428	0.588	-0.256	-0.100
$\rho = 1$	0.307	0.309	0.309	0.307
$\rho = 1.5$	0.185	0.029	0.743	0.583