A Complex Variable Boundary Element Method for a Class of Boundary Value Problems in Anisotropic Thermoelasticity

W. T. Ang, D. L. Clements and T. Cooke

Abstract

A boundary element method based on the Cauchy's integral formulae, called the complex variable boundary element method (CVBEM), is proposed for the numerical solution of boundary value problems governing plane thermoelastic deformations of anisotropic elastic bodies. The method is applicable for a wide class of problems which do not involve inertia or coupling effects and can be easily and efficiently implemented on the computer. It is applied to solve specific test problems.

Key words: complex variable boundary element method, plane thermoelasticity, anisotropic media.

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1 INTRODUCTION

The present paper proposes a boundary element method based on the Cauchy's integral formulae, known as the complex variable boundary element method (CVBEM), for the numerical solution of boundary value problems arising in thermoelastic deformations of anisotropic bodies. Many authors have considered problems of this type, e.g. Dargush and Banerjee [1], Sladek and Sladek [2], Deb [3] and Ang, Clements and Cooke [4]. In all the papers just cited, the authors used various approaches to devise real boundary element methods which involve only boundary integrals to solve the thermoelastic problems.

For plane thermoelastic deformations where inertia and coupling effects may be ignored, the solutions of the governing equations can be expressed in terms of complex functions which are holomorphic in the domain of interest. The approach adopted in the present work is to apply the Cauchy's integral formulae to construct approximately holomorphic functions which satisfy the boundary conditions of the problem under consideration. The proposed procedure requires only the boundary of the domain to be discretised, hence the term CVBEM.

The CVBEM was originally introduced by Hromadka II and Lai [5] for solving boundary value problems governed by the two-dimensional Laplace's equation. More details of the method for the Laplace's equation may be found in Hromadka and Yen [6], Whitley and Hromadka II [9], Hromadka II and Whitley [7], [10], and Hromadka II [8].

More recently, introducing the theory of complex Hadamard finite-part (hypersingular) integrals, Linkov and Mogilevskaya [11], [14] extended the CVBEM to solve certain boundary value problems in plane elastostatics. The theory of complex hypersingular integrals is also successfully applied in the present paper to devise the CVBEM for the thermoelastic problems.

2 THE PROBLEM

Referring to a $0x_1x_2x_3$ Cartesian frame, consider a homogeneous anisotropic elastic body whose geometry does not vary along the x_3-direction. In the

$0x_1x_2$ plane, it occupies the region $\mathcal{R}$ which is bounded by a simple closed curve $\mathcal{C}$. At each and every point on the boundary $\mathcal{C}$, either the temperature or heat flux and either the displacements or tractions are suitably specified. The prescribed boundary conditions are assumed to be independent of time and the spatial coordinate x_3. (Hence, the temperature and displacement fields vary with x_1 and x_2 only.) It is required to determine the temperature and displacement fields throughout the body.

3 GOVERNING EQUATIONS

The temperature distribution T at any general point (x_1, x_2) in the body $\mathcal{R}$ satisfies the partial differential equation (Nowacki [12])

$$\lambda_{kj} \frac{\partial^2 T}{\partial x_k \partial x_j} = 0, \tag{1}$$

where $\lambda_{ij} = \lambda_{ji}$ $(i, j = 1, 2, 3)$ are the constant heat conduction coefficients satisfying the strict inequality $\lambda_{11}\lambda_{22} - \lambda_{12}^2 > 0$. The convention of summing over a repeated index is adopted only for latin subscripts which run from 1 to 3.

Since either temperature or heat flux is specified at each and every point on $\mathcal{C}$, we are required to solve (1) subject to

$$\left.\begin{array}{lll} T(x_1, x_2) = \Phi(x_1, x_2) & \text{for} & (x_1, x_2) \in \mathcal{C}_1 \\ P(x_1, x_2) = \Psi(x_1, x_2) & \text{for} & (x_1, x_2) \in \mathcal{C}_2 \end{array}\right\} \tag{2}$$

where Φ and Ψ are suitably prescribed functions of x_1 and x_2, $\mathcal{C}_1$ and $\mathcal{C}_2$ are non-intersecting curves such that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and P is the heat flux defined by $P = \lambda_{kj} n_k \partial T/\partial x_j$,with n_k being components of the unit outer normal vector to $\mathcal{R}$ on $\mathcal{C}$.

Now if $\mathcal{C}_2 = \mathcal{C}$, i.e. if the heat flux P is specified over the whole boundary $\mathcal{C}$, then the function Ψ in (2) is required to satisfy

$$\oint_{\mathcal{C}} \Psi(x_1, x_2) dS(x_1, x_2) = 0.$$

If the displacement and the stress are denoted by u_k and σ_{pj} respectively then

$$\sigma_{pj}(x_1, x_2) = c_{pjkr} \frac{\partial u_k}{\partial x_r} - \beta_{pj} T(x_1, x_2), \tag{3}$$

where c_{pjkr} and $\beta_{pj} = \beta_{jp}$ are, respectively, the constant elastic moduli and stress-temperature coefficients of the anisotropic medium.

Substituting (3) into the equilibrium equations $\partial\sigma_{pj}/\partial x_j = 0$, we find that

$$c_{pjkr}\frac{\partial^2 u_k}{\partial x_j \partial x_r} = \beta_{pj}\frac{\partial T}{\partial x_j}. \quad (4)$$

Notice that the temperature $T(x_1, x_2)$ is regarded as known after solving (1) subject to (2). Hence, (4) is an inhomogeneous system of partial differential equations.

For further details of equations (3) and (4), refer to Clements [13].

Once the temperature T is determined from (1) and (2), our task is to solve (4) subject to

$$\left.\begin{array}{rl} u_k(x_1, x_2) = \mu_k(x_1, x_2) & \text{for} \quad (x_1, x_2) \in \mathcal{C}_3 \\ t_k(x_1, x_2) = \rho_k(x_1, x_2) & \text{for} \quad (x_1, x_2) \in \mathcal{C}_4 \end{array}\right\} \quad (5)$$

where μ_k and ρ_k are suitably prescribed functions of x_1 and x_2, $\mathcal{C}_3$ and $\mathcal{C}_4$ are non-intersecting curves such that $\mathcal{C} = \mathcal{C}_3 \cup \mathcal{C}_4$ and t_k are the tractions defined by $t_k = \sigma_{kj} n_j$.

4 CVBEM

We shall now outline a simple CVBEM for solving (1) and (2) for the temperature field and also (4) and (5) for the displacement field.

4.1 Temperature field

The system (1) admits general solutions of the form

$$T(x_1 + x_2) = \mathrm{Re}\left\{f(x_1 + \tau x_2)\right\}, \quad (6)$$

where $\tau = \left(-\lambda_{12} + i\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}\right)/\lambda_{22}$, $i = \sqrt{-1}$ and f is a holomorphic function of $x_1 + \tau x_2$ for $(x_1, x_2) \in \mathcal{R} \cup \mathcal{C}$.

Since f is a holomorphic function of $x_1 + \tau x_2$ for $(x_1, x_2) \in \mathcal{R} \cup \mathcal{C}$, the Cauchy's integral formulae give

$$2\pi i f(\xi_1 + \tau\xi_2) = \oint_{(x_1,x_2)\in\mathcal{C}} \frac{f(x_1 + \tau x_2)d\left(x_1 + \tau x_2\right)}{\left(x_1 + \tau x_2 - \xi_1 - \tau\xi_2\right)} \quad (7)$$

$$2\pi i f'(\xi_1 + \tau\xi_2) = \oint_{(x_1,x_2)\in\mathcal{C}} \frac{f(x_1 + \tau x_2)d\left(x_1 + \tau x_2\right)}{\left(x_1 + \tau x_2 - \xi_1 - \tau\xi_2\right)^2} \quad (8)$$

for $(\xi_1,\xi_2)\in\mathcal{R}$. Notice that $f'(z)$ denotes the first order derivative of f with respect to z. We take the direction of the simple closed curve $\mathcal{C}$ to be anticlockwise.

For the case where (ξ_1,ξ_2) lies on a smooth part of the curve $\mathcal{C}$, the above formulae can be modified to become

$$\begin{aligned}\pi i f(\xi_1+\tau\xi_2) &= \mathcal{P}\oint_{(x_1,x_2)\in\mathcal{C}}\frac{f(x_1+\tau x_2)d\,(x_1+\tau x_2)}{(x_1+\tau x_2-\xi_1-\tau\xi_2)} && (9)\\ \pi i f'(\xi_1+\tau\xi_2) &= \mathcal{H}\oint_{(x_1,x_2)\in\mathcal{C}}\frac{f(x_1+\tau x_2)d\,(x_1+\tau x_2)}{(x_1+\tau x_2-\xi_1-\tau\xi_2)^2}, && (10)\end{aligned}$$

where $\mathcal{P}$ and $\mathcal{H}$ denotes that the integral is to be interpreted in the Cauchy principal and Hadamard finite-part sense, respectively. For further details on the theory of complex Cauchy principal and Hadamard finite-part integrals, refer to the recent work of Linkov and Mogilevskaya [11], [14].

Equations (6) together with (7)-(10) are used to determine $f(x_1+\tau x_2)$ as follows.

The boundary $\mathcal{C}$ is discretised by putting M closely-packed well-spaced out (nodal) points $(x_1^{(1)},x_2^{(1)})$, $(x_1^{(2)},x_2^{(2)})$, $\cdots$, $(x_1^{(k)},x_2^{(k)})$, $\cdots$, and $(x_1^{(M)},x_2^{(M)})$ (in anticlockwise direction) on it. If we denote the straight line segment from $(x_1^{(k)},x_2^{(k)})$ to $(x_1^{(k+1)},x_2^{(k+1)})$ by $\mathcal{C}^{(k)}$ $[k=1,2,\cdots,M$ and $(x_1^{(M+1)},x_2^{(M+1)})=(x_1^{(1)},x_2^{(1)})]$ then we make the approximation $\mathcal{C}\simeq\mathcal{C}^{(1)}\cup\mathcal{C}^{(2)}\cup\cdots\cup\mathcal{C}^{(M)}$.

Let $f(x_1+\tau x_2)=w(x_1,x_2)+iv(x_1,x_2)$. We assume that w and v are constant over a given straight line segment $\mathcal{C}^{(k)}$ or more precisely

$$\left.\begin{aligned}w(x_1,x_2)\simeq w^{(k)}\\ v(x_1,x_2)\simeq v^{(k)}\end{aligned}\right\}\text{ for }(x_1,x_2)\in\mathcal{C}^{(k)},\qquad(11)$$

where $w^{(k)}$ and $v^{(k)}$ are constants, then for $(\xi_1,\xi_2)\in\mathcal{R}$ the real and imaginary parts of (7) gives rise to the approximation

$$\begin{aligned}2\pi w(\xi_1,\xi_2) &\simeq \sum_{m=1}^{M}\left\{v^{(m)}\gamma^{(m)}(\xi_1,\xi_2)+w^{(m)}\theta^{(m)}(\xi_1,\xi_2)\right\},\\ 2\pi v(\xi_1,\xi_2) &\simeq \sum_{m=1}^{M}\left\{v^{(m)}\theta^{(m)}(\xi_1,\xi_2)-w^{(m)}\gamma^{(m)}(\xi_1,\xi_2)\right\}, && (12)\end{aligned}$$

where $\gamma^{(m)}(\xi_1,\xi_2)=\ln\left|z^{(m+1)}-c\right|-\ln\left|z^{(m)}-c\right|$, $z^{(m)}=x_1^{(m)}+\tau x_2^{(m)}$, $c=$

$\xi_1 + \tau\xi_2$ and

$$\theta^{(m)}(\xi_1, \xi_2) = \begin{cases} \Omega^{(m)}(\xi_1, \xi_2) & \text{if} \quad -\pi \leq \Omega^{(m)}(\xi_1, \xi_2) \leq \pi \\ \Omega^{(m)}(\xi_1, \xi_2) + 2\pi & \text{if} \quad -2\pi \leq \Omega^{(m)}(\xi_1, \xi_2) < -\pi \\ \Omega^{(m)}(\xi_1, \xi_2) - 2\pi & \text{if} \quad \pi < \Omega^{(m)}(\xi_1, \xi_2) \leq 2\pi \end{cases} \qquad (13)$$

where $\Omega^{(m)}(\xi_1, \xi_2) = \text{Arg}(z^{(m+1)} - c) - \text{Arg}(z^{(m)} - c)$ and $\text{Arg}(z)$ denotes the principal value of the argument of the complex number z.

If the curve $\mathcal{C}$ has certain shape, e.g. if it is such that $\mathcal{R}$ is a convex region, then, for $(\xi_1, \xi_2) \in \mathcal{R} \cup \mathcal{C}$, $\theta^{(m)}(\xi_1, \xi_2)$ can be calculated directly from

$$\theta^{(m)}(\xi_1, \xi_2) = \cos^{-1}\left(\frac{\left|z^{(m+1)} - c\right|^2 + \left|z^{(m)} - c\right|^2 - \left|z^{(m+1)} - z^{(m)}\right|^2}{2\left|z^{(m+1)} - c\right|\left|z^{(m)} - c\right|}\right). \qquad (14)$$

Equation (12) is valid not only for $(\xi_1, \xi_2) \in \mathcal{R}$. From (9) and (11), we can show that it is also applicable if (ξ_1, ξ_2) is the midpoint of $\mathcal{C}^{(k)}$.

Thus, we may let $(\xi_1, \xi_2) = (y_1^{(p)}, y_2^{(p)}) \equiv \frac{1}{2}(x_1^{(p+1)} + x_1^{(p)}, x_2^{(p+1)} + x_2^{(p)})$ (the midpoint of $\mathcal{C}^{(p)}$) in the second equation of (12) to obtain the approximate system

$$\sum_{m=1}^{M} \left\{ v^{(m)}\theta^{(m)}(y_1^{(p)}, y_2^{(p)}) - w^{(m)}\gamma^{(m)}(y_1^{(p)}, y_2^{(p)}) \right\} = 2\pi v^{(p)} \qquad (15)$$

for $p = 1, 2, \cdots, M$.

The system (15) contains M linear algebraic equations in $2M$ unknowns $w^{(m)}$ and $v^{(m)}$ $(m = 1, 2, \cdots, M)$.

Over each of the line segments $\mathcal{C}^{(m)}$ $(m = 1, 2, \cdots, M)$, either T or P are specified according to (2).

Thus, from (2) and (6), we obtain

$$w^{(p)} = \Phi(y_1^{(p)}, y_2^{(p)}) \text{ if } T \text{ is specified over } \mathcal{C}^{(p)}. \qquad (16)$$

Over the line segment $\mathcal{C}^{(p)}$, the heat flux is related to the holomorphic function f by

$$P(x_1, x_2) = \text{Re}\left\{(J_s + iK_s)f'(y_1^{(p)} + \tau y_2^{(p)})\right\} n_s^{(p)} \text{ for } (x_1, x_2) \in \mathcal{C}^{(p)}, \qquad (17)$$

where J_s and K_s are real constants such that $J_s + iK_s = \lambda_{s1} + \tau\lambda_{s2}$ and $n_s^{(m)}$ are the components of the unit normal vector to $\mathcal{R}$ on $\mathcal{C}^{(m)}$.

Using (2), (10) and (17) and letting $(\xi_1, \xi_2) = (y_1^{(p)}, y_2^{(p)})$, we find that

$$\begin{aligned}
&\sum_{m=1}^{M} \left\{ \left(J_s q^{(pm)} - K_s r^{(pm)} \right) v^{(m)} \right. \\
&\left. + \left(J_s r^{(pm)} + K_s q^{(pm)} \right) w^{(m)} \right\} n_s^{(p)} \\
= \ & \pi \Psi(y_1^{(p)}, y_2^{(p)}) \text{ if } P \text{ are specified over } \mathcal{C}^{(p)}, \qquad (18)
\end{aligned}$$

where $q^{(pm)}$ and $r^{(pm)}$ are real parameters such that

$$q^{(pm)} + i r^{(pm)} = \frac{1}{z^{(m)} - c^{(p)}} - \frac{1}{z^{(m+1)} - c^{(p)}}, \qquad (19)$$

where $c^{(p)} = (z^{(p)} + z^{(p+1)})/2$.

We find that (16) and/or (18) give rise to an additional M equations in $w^{(m)}$ and $v^{(m)}$.

From numerical experiments, we find that, depending on the discretisation of $\mathcal{C}$, the system of $2M$ linear algebraic equations in the unknowns $w^{(m)}$ and $v^{(m)}$ $(m = 1, 2, \cdots, M)$, as given by (15) with $p = 1, 2, \cdots, M$ together with (16) and/or (18), may be poorly conditioned and may give rise to numerical values of v which are extremely large in magnitude, ruining subsequent calculations. Perhaps this is not surprising due to the fact that the function v is not uniquely determined by the boundary value problem. A possible way (successfully used in section 5) to overcome this difficulty is to fix the value of v across the segment $\mathcal{C}^{(M)}$, e.g. set $v^{(M)} = 0$, and solve (15) for only $p = 1, 2, \cdots, M-1$ together with (16) and/or (18) for the other remaining boundary values of w and v.

Once the unknowns $w^{(m)}$ and $v^{(m)}$ are completely known, we can compute f approximately via (12) and hence $T(\xi_1, \xi_2)$ [using (6)] at any point (ξ_1, ξ_2) in the interior of $\mathcal{R}$.

4.2 Displacement field

If the temperature is given by (6), for the solution of (4) subject to (5), we take

$$u_k = \mathrm{Re}\left\{ D_k g(x_1 + \tau x_2) + \sum_{\alpha=1}^{3} A_{k\alpha} h_\alpha(z_\alpha) \right\}, \qquad (20)$$

where $h_\alpha(z_\alpha)$ are holomorphic functions of $z_\alpha = x_1 + p_\alpha x_2$ for $(x_1, x_2) \in \mathcal{R}$, p_α and $A_{k\alpha}$ are constants related to the elastic moduli c_{kjpl} as defined in

Clements [13], $g(\zeta)$ is a holomorphic function of $\zeta = x_1 + \tau x_2$ for $(x_1, x_2) \in \mathcal{R}$ such that $g'(\zeta) = f(\zeta)$ and D_k are constants satisfying

$$\left(c_{i1k1} + \tau c_{i1k2} + \tau c_{i2k1} + \tau^2 c_{i2k2}\right) D_k = \beta_{i1} + \tau\beta_{i2}. \tag{21}$$

The boundary $\mathcal{C}$ is discretised as before. If $g(x_1 + \tau x_2) = s(x_1, x_2) + ik(x_1, x_2)$ and the real functions s and k are assumed to be constants $s^{(m)}$ and $k^{(m)}$ respectively over the segment $\mathcal{C}^{(m)}$, then from $g'(\zeta) = f(\zeta)$ and the formula (10) [with f replaced by g], we obtain the approximation

$$\pi i\left[w^{(p)} + iv^{(p)}\right] = \sum_{m=1}^{M}\left(q^{(pm)} + ir^{(pm)}\right)\left(s^{(m)} + ik^{(m)}\right) \tag{22}$$

for $p = 1, 2, \cdots, M$.

The function $f(x_1 + \tau x_2) = w + iv$ is assumed to be known at least numerically from solving (15) together with (16) and/or (18). The function $g(x_1 + \tau x_2)$ can be determined numerically from (22). To do this properly, we fix the value of g across $\mathcal{C}^{(M)}$, e.g. set $s^{(M)} + ik^{(M)} = 0$, and solve (22) for $p = 1, 2, \cdots, M-1$. Once this is done, the approximate value of g at any point (ξ_1, ξ_2) in the interior of $\mathcal{R}$ can be calculated using (12) with $w^{(m)}$ and $v^{(m)}$ replaced by $s^{(m)}$ and $k^{(m)}$ $(m = 1, 2, \cdots, M)$ respectively.

With $g(x_1 + \tau x_2)$ determined, the remaining task is to find the holomorphic functions $h_\alpha(z_\alpha)$.

If we write

$$w_k(x_1, x_2) + iv_k(x_1, x_2) = \sum_{\alpha=1}^{3} A_{k\alpha} h_\alpha(z_\alpha), \tag{23}$$

where w_k and v_k are real functions ($w_k = u_k$), then

$$h_\alpha(z_\alpha) = N_{\alpha k}\left[w_k(x_1, x_2) + iv_k(x_1, x_2)\right], \tag{24}$$

where $[N_{\alpha k}]$ is the inverse of $[A_{k\alpha}]$. The existence of $[N_{\alpha k}]$ is guaranteed if p_1, p_2 and p_3 have distinct values.

Making the approximation

$$\left.\begin{array}{r} w_k(x_1, x_2) \simeq w_k^{(m)} \\ v_k(x_1, x_2) \simeq v_k^{(m)} \end{array}\right\} \text{ for } (x_1, x_2) \in \mathcal{C}^{(m)}, \tag{25}$$

and proceeding as in the previous subsection using the Cauchy's integral formulae, we obtain (for $q = 1, 2, \cdots, M$)

$$\sum_{m=1}^{M} \left\{ v_k^{(m)} \Theta_{pk}^{(m)}(y_1^{(q)}, y_2^{(q)}) - w_k^{(m)} \Gamma_{pk}^{(m)}(y_1^{(q)}, y_2^{(q)}) \right\} = 2\pi v_p^{(q)}, \tag{26}$$

where $\Gamma_{pk}^{(m)}(\xi_1, \xi_2)$ and $\Theta_{pk}^{(m)}(\xi_1, \xi_2)$ are real parameters defined by

$$\begin{aligned} & \Gamma_{pk}^{(m)}(\xi_1, \xi_2) + i\Theta_{pk}^{(m)}(\xi_1, \xi_2) \\ = & \sum_{\alpha=1}^{3} A_{p\alpha} N_{\alpha k} \left[\gamma_\alpha^{(m)}(\xi_1, \xi_2) + i\theta_\alpha^{(m)}(\xi_1, \xi_2) \right], \end{aligned} \tag{27}$$

where $\gamma_\alpha^{(m)}(\xi_1, \xi_2) = \ln \left| z_\alpha^{(m+1)} - c_\alpha \right| - \ln \left| z_\alpha^{(m)} - c_\alpha \right|$, $z_\alpha^{(m)} = x_1^{(m)} + p_\alpha x_2^{(m)}$, $c_\alpha = \xi_1 + p_\alpha \xi_2$ and

$$\theta_\alpha^{(m)}(\xi_1, \xi_2) = \begin{cases} \Omega_\alpha^{(m)}(\xi_1, \xi_2) & \text{if} \quad -\pi \leq \Omega_\alpha^{(m)}(\xi_1, \xi_2) \leq \pi \\ \Omega_\alpha^{(m)}(\xi_1, \xi_2) + 2\pi & \text{if} \quad -2\pi \leq \Omega_\alpha^{(m)}(\xi_1, \xi_2) < -\pi \\ \Omega_\alpha^{(m)}(\xi_1, \xi_2) - 2\pi & \text{if} \quad \pi < \Omega_\alpha^{(m)}(\xi_1, \xi_2) \leq 2\pi \end{cases} \tag{28}$$

where $\Omega_\alpha^{(m)}(\xi_1, \xi_2) = \text{Arg}(z_\alpha^{(m+1)} - c_\alpha) - \text{Arg}(z_\alpha^{(m)} - c_\alpha)$. As before, if $\mathcal{R}$ is a convex region, we may compute $\theta_\alpha^{(m)}(\xi_1, \xi_2)$ directly using (14) with $\theta^{(m)}(\xi_1, \xi_2)$, $z^{(m)}$ and c superceded by $\theta_\alpha^{(m)}(\xi_1, \xi_2)$, $z_\alpha^{(m)}$ and c_α respectively.

From conditions (5), we obtain

$$\begin{aligned} w_k^{(p)} = & -\text{Re}\left\{ D_k \left[s^{(p)} + ik^{(p)} \right] \right\} \\ & +\mu_k(y_1^{(p)}, y_2^{(p)}) \text{ if } u_k \text{ are specified over } \mathcal{C}^{(p)}, \end{aligned} \tag{29}$$

or

$$\begin{aligned} & \frac{1}{\pi} \sum_{m=1}^{M} \sum_{\alpha=1}^{3} \left\{ \left(J_{ij\alpha k} q_\alpha^{(pm)} - K_{ij\alpha k} r_\alpha^{(pm)} \right) v_k^{(m)} \right. \\ & \left. + \left(J_{ij\alpha k} r_\alpha^{(pm)} + K_{ij\alpha k} q_\alpha^{(pm)} \right) w_k^{(m)} \right\} n_j^{(p)} \\ = & -\text{Re}\left\{ B_{ij} \left[w^{(p)} + iv^{(p)} \right] \right\} n_j^{(p)} \\ & +\rho_i(y_1^{(p)}, y_2^{(p)}) \text{ if } t_i \text{ are specified over } \mathcal{C}^{(p)}, \end{aligned} \tag{30}$$

where $J_{ij\alpha k} = \text{Re}\{L_{ij\alpha} N_{\alpha k}\}$, $K_{ij\alpha k} = \text{Im}\{L_{ij\alpha} N_{\alpha k}\}$, $L_{ij\alpha} = (c_{ijk1} + p_\alpha c_{ijk2}) A_{k\alpha}$, $B_{ij} = (c_{ijk1} + \tau c_{ijk2}) D_k$, $q_\alpha^{(pm)}$ and $r_\alpha^{(pm)}$ are real constants defined by

$$q_\alpha^{(pm)} + ir_\alpha^{(pm)} = \frac{1}{z_\alpha^{(m)} - c_\alpha^{(p)}} - \frac{1}{z_\alpha^{(m+1)} - c_\alpha^{(p)}}, \tag{31}$$

where $c_\alpha^{(p)} = (z_\alpha^{(p)} + z_\alpha^{(p+1)})/2$.

Equations (26) together with (29) and/or (30) constitute a system of $6M$ linear algebraic equations in $6M$ unknowns $w_k^{(m)}$ and $v_k^{(m)}$ ($k = 1, 2, 3$; $m = 1, 2, \cdots, M$). As before, to avoid poorly conditioned systems of linear algebraic equations, we fix the values of v_k over the segment $\mathcal{C}^{(M)}$, e.g. $v_k = 0$, and use (26) with $q = 1, 2, \cdots, M - 1$.

Once all the unknowns $w_k^{(m)}$ and $v_k^{(m)}$ ($k = 1, 2, 3$; $m = 1, 2, \cdots, M$) are known, we can compute g and h_α approximately as in the previous subsection and hence $u_k(\xi_1, \xi_2)$ [using (20)] at any point (ξ_1, ξ_2) in the interior of $\mathcal{R}$.

5 TEST PROBLEMS

Problem 1. Let us assume that the thermoelastic behaviour of the material under consideration is such that the governing equations (1) and (4) are specifically given by

$$\begin{aligned}
\frac{\partial^2 T}{\partial x_1^2} + 4\frac{\partial^2 T}{\partial x_2^2} &= 0, \\
18\frac{\partial^2 u_1}{\partial x_1^2} + 11\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + 4\frac{\partial^2 u_1}{\partial x_2^2} &= \frac{\partial T}{\partial x_1}, \\
16\frac{\partial^2 u_2}{\partial x_2^2} + 11\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + 4\frac{\partial^2 u_2}{\partial x_1^2} &= 4\frac{\partial T}{\partial x_2}.
\end{aligned} \tag{32}$$

A solution of (32) is

$$\begin{aligned}
T(x_1, x_2) &= \mathrm{Re}\left\{\left[x_1 + \frac{1}{2}i\,(x_2 - 2)\right]^{-2}\right\}, \\
u_k(x_1, x_2) &= -\mathrm{Re}\left\{D_k\left[x_1 + \frac{1}{2}i\,(x_2 - 2)\right]^{-1}\right\} \\
&\quad + \mathrm{Re}\left\{\sum_{\alpha=1}^{2} A_{k\alpha}\left[x_1 + 1 + p_\alpha x_2\right]^2\right\},
\end{aligned} \tag{33}$$

where p_1 and p_2 are the solutions of $64p^4 + 183p^2 + 72 = 0$ such that $\mathrm{Im}\{p_1\} > 0$ and $\mathrm{Im}\{p_2\} > 0$ and

$$[D_k] = \frac{1}{121}\begin{bmatrix} 44 \\ 114i \end{bmatrix},$$

$$
\begin{aligned}
[A_{k\alpha}] &= \begin{bmatrix} -11ip_1(18+4p_1^2)^{-1} & -11ip_2(18+4p_2^2)^{-1} \\ i & i \end{bmatrix}, \\
[N_{k\alpha}] &= \frac{1}{11\nu}\begin{bmatrix} -2i\,(9+2p_1^2)\,(9+2p_2^2) & -11ip_2\,(9+2p_1^2) \\ 2i\,(9+2p_1^2)\,(9+2p_2^2) & 11ip_1\,(9+2p_2^2) \end{bmatrix},
\end{aligned} \tag{34}
$$

where $\nu = (2p_2p_1 - 9)\,(-p_2 + p_1)$. Notice that $p_1 \simeq 1.5454241i$ and $p_2 \simeq 0.68632303i$.

The heat flux and tractions which correspond to (33) are

$$
\begin{aligned}
P &= \operatorname{Re}\left\{-2(n_1 + 2in_2)\left[x_1 + \frac{i}{2}(x_2 - 2)\right]^{-3}\right\}, \\
t_k &= \operatorname{Re}\left\{B_{kj}n_j\left[x_1 + \frac{i}{2}(x_2 - 2)\right]^{-2}\right\} \\
&\quad + \operatorname{Re}\left\{\sum_{\alpha=1}^{2} 2L_{kj\alpha}n_j\,[x_1 + 1 + p_\alpha x_2]\right\},
\end{aligned} \tag{35}
$$

where

$$
\begin{aligned}
[B_{kj}] &= \frac{1}{121}\begin{pmatrix} 393 & 544i \\ 544i & -604 \end{pmatrix}, \\
[L_{k1\alpha}] &= \begin{pmatrix} ip_1\left[7 - 77\,(18+4p_1^2)^{-1}\right] & ip_2\,[7 - 77\,(18+4p_2^2)]^{-1} \\ 4i\,(18-7p_1^2)\,/\,(18+4p_1^2) & 4i\,(18-7p_2^2)\,/\,(18+4p_2^2) \end{pmatrix}, \\
[L_{k2\alpha}] &= \begin{pmatrix} 4i\,(18-7p_1^2)\,/\,(18+4p_1^2) & 4i\,(18-7p_2^2)\,/\,(18+4p_2^2) \\ ip_1\left[16 - 77\,(18+4p_1^2)^{-1}\right] & ip_2\left[16 - 77\,(18+4p_2^2)^{-1}\right] \end{pmatrix}.
\end{aligned} \tag{36}
$$

The solution (33) is valid everywhere in the $0x_1x_2$ plane except at the point $(0, 2)$. Thus, we may choose the region $\mathcal{R}$ to be given by

$$
\mathcal{R} = \{(x_1, x_2) : 0 < x_1 < 1,\ \ 0 < x_2 < 1\}, \tag{37}
$$

i.e. $\mathcal{C}$ is a square whose sides are of unit length.

The heat flux and tractions are prescribed according to (35) on the side $0 < x_1 < 1$, $x_2 = 1$ of the square. On the remaining sides, the temperature and displacements are specified in accordance with (33). By discretising the sides of the square into 80 equal length segments, we apply the CVBEM described in section 4 to solve (32) in $\mathcal{R}$ subject to the generated boundary data on $\mathcal{C}$.

The numerical values of T and u_k calculated at various points in the interior of the square domain are compared with the exact solution in Tables 1 and 2. The numerical and exact values show good agreement.

Table 1. Comparison of numerical values of the temperature T at various interior points (x_1, x_2) with the exact solution (test problem 1).

(x_1, x_2)	CVBEM	Exact
$(0.25, 0.25)$	-1.030	-1.025
$(0.25, 0.50)$	-1.288	-1.280
$(0.25, 0.75)$	-1.608	-1.598
$(0.50, 0.25)$	-0.5090	-0.4999
$(0.50, 0.50)$	-0.4891	-0.4734
$(0.50, 0.75)$	-0.3676	-0.3427
$(0.75, 0.25)$	-0.1254	-0.1152
$(0.75, 0.50)$	-0.0152	0.0000
$(0.75, 0.75)$	0.1679	0.1892

Table 2. Comparison of numerical values of the displacement (u_1, u_2) at various interior points (x_1, x_2) with the exact solution (test problem 1).

(x_1, x_2)	CVBEM	Exact
$(0.25, 0.25)$	$(3.433, -0.3894)$	$(3.453, -0.3994)$
$(0.25, 0.50)$	$(2.418, -1.672)$	$(2.474, -1.659)$
$(0.25, 0.75)$	$(0.8417, -2.907)$	$(0.8481, -2.885)$
$(0.50, 0.25)$	$(5.121, -0.8397)$	$(5.089, -0.8621)$
$(0.50, 0.50)$	$(4.119, -2.478)$	$(4.102, -2.478)$
$(0.50, 0.75)$	$(2.496, -4.101)$	$(2.471, -4.102)$
$(0.75, 0.25)$	$(7.129, -1.326)$	$(7.079, -1.332)$
$(0.75, 0.50)$	$(6.104, -3.292)$	$(6.099, -3.277)$
$(0.75, 0.75)$	$(4.523, -5.258)$	$(4.484, -5.241)$

Problem 2. As in problem 1, the governing partial differential equations are given by (32). The boundary $\mathcal{C}$ is chosen to be a pentagon with vertices

A $(0,0)$, B $(0,1)$, C $(1,1)$, D $(1/2,1/2)$ and E $(1,0)$. For this particular case, $\mathcal{R}$ is a concave region.

For a specific boundary value problem, we use the particular solution

$$\begin{aligned} T(x_1,x_2) &= \mathrm{Re}\left\{\exp\left[x_1+\frac{1}{2}ix_2\right]\right\}, \\ u_k(x_1,x_2) &= \mathrm{Re}\left\{D_k\exp\left[x_1+\frac{1}{2}ix_2\right]\right\} \\ &\quad+\mathrm{Re}\left\{\sum_{\alpha=1}^{2}A_{k\alpha}\exp\left[x_1+p_\alpha x_2\right]\right\} \end{aligned} \tag{38}$$

to generate the temperature and the displacements at all points on the boundary of the polygon.

We discretise the sides AB, BC and AE into 20 equal length boundary elements per side and each of DE and CD into 10 elements per side (so that $M = 80$) and apply the CVBEM to solve (32) in $\mathcal{R}$ subject to the generated boundary data on $\mathcal{C}$. Once all the relevant holomorphic functions are constructed approximately by the CVBEM, we compute the temperature and displacements at various interior points in the concave region.

The numerical values of T and u_k thus obtained at selected interior points are compared with the exact values from (38) in Tables 3 and 4. The numerical and exact values show reasonable agreement with each other.

Table 3. Comparison of numerical values of the temperature T at various interior points (x_1, x_2) with the exact solution (test problem 2).

(x_1, x_2)	CVBEM	Exact
$(0.25, 0.25)$	1.282	1.274
$(0.25, 0.50)$	1.253	1.244
$(0.25, 0.75)$	1.203	1.195
$(0.50, 0.25)$	1.639	1.636
$(0.50, 0.75)$	1.538	1.534
$(0.60, 0.25)$	1.798	1.808
$(0.60, 0.75)$	1.686	1.695

Table 4. Comparison of numerical values of the displacement (u_1, u_2) at various interior points (x_1, x_2) with the exact solution (test problem 2).

(x_1, x_2)	CVBEM	Exact
$(0.25, 0.25)$	$(3.426, -0.8966)$	$(3.450, -0.8540)$
$(0.25, 0.50)$	$(2.815, -1.643)$	$(2.869, -1.628)$
$(0.25, 0.75)$	$(1.953, -2.237)$	$(1.992, -2.252)$
$(0.50, 0.25)$	$(4.349, -1.101)$	$(4.429, -1.096)$
$(0.50, 0.75)$	$(2.438, -2.842)$	$(2.558, -2.892)$
$(0.60, 0.25)$	$(4.773, -1.167)$	$(4.895, -1.212)$
$(0.60, 0.75)$	$(2.674, -3.129)$	$(2.827, -3.196)$

Problem 3. The governing partial differential equations are given by

$$\begin{aligned}\frac{\partial^2 T}{\partial x_1^2}+\frac{\partial^2 T}{\partial x_2^2} &= 0,\\ 3\frac{\partial^2 u_1}{\partial x_1^2}+2\frac{\partial^2 u_2}{\partial x_1 \partial x_2}+\frac{\partial^2 u_1}{\partial x_2^2} &= \frac{\partial T}{\partial x_1},\\ 3\frac{\partial^2 u_2}{\partial x_2^2}+2\frac{\partial^2 u_1}{\partial x_1 \partial x_2}+\frac{\partial^2 u_2}{\partial x_1^2} &= \frac{\partial T}{\partial x_2}.\end{aligned} \tag{39}$$

The system (39) governs the thermoelastic behaviours of an *isotropic* body with Lamé constants $\lambda = \mu = 1$. It is usually regarded as a degenerate case of (4) as it gives rise to identical value for the constants p_1 and p_2, i.e. $p_1 = p_2 = i$, causing the matrix $[A_{k\alpha}]$ to become singular. Thus, the analysis in the present paper seems to break down for this particular case. Fortunately, we can easily recover it by introducing a very slight perturbation in one of the coefficients of the second equation of the system (39). Specifically, we replace that equation in an approximate sense by

$$(3+\varepsilon)\frac{\partial^2 u_1}{\partial x_1^2}+2\frac{\partial^2 u_2}{\partial x_1 \partial x_2}+\frac{\partial^2 u_1}{\partial x_2^2}=\frac{\partial T}{\partial x_1}, \tag{40}$$

where ε is a positive parameter of extremely small magnitude. (Ideally, we should let $\varepsilon \to 0^+$.) As long as $\varepsilon \neq 0$, the values of p_1 and p_2 are distinct, but close to i if ε is near 0.

Thus, for the case of an isotropic body, relevant constants such as p_1, p_2, $A_{k\alpha}$, D_k and B_{kj} can be computed (approximately) using the elastic moduli

$$\begin{aligned}c_{2222} &= 3,\ c_{1111}=3+\varepsilon,\ c_{1122}=c_{2211}=1,\\ c_{1212} &= c_{2121}=c_{1221}=c_{2112}=1,\end{aligned} \tag{41}$$

where ε is a selected positive parameter of very small magnitude (e.g. $\varepsilon = 0.000001$). Elastic moduli which are not listed above may be taken as zero.

As before, for the problem presently under consideration, we use a particular solution of the partial differential equations (39) to generate boundary data on a chosen boundary $\mathcal{C}$ and apply the CVBEM to solve the equations subject to the generated boundary data.

Specifically, we take the particular solution

$$\begin{aligned} T(x_1, x_2) &= \operatorname{Re}\left\{[x_1 + i\,(x_2 - 2)]^{-2}\right\}, \\ u_k(x_1, x_2) &= -\operatorname{Re}\left\{D_k\,[x_1 + i\,(x_2 - 2)]^{-1}\right\} \\ &\quad + \operatorname{Re}\left\{\sum_{\alpha=1}^{2} A_{k\alpha}\,[x_1 + 1 + ix_2]^2\right\}. \end{aligned} \tag{42}$$

The heat flux and tractions which correspond to (42) are

$$\begin{aligned} P &= \operatorname{Re}\left\{-2(n_1 + 2in_2)\,[x_1 + i(x_2 - 2)]^{-3}\right\}, \\ t_k &= \operatorname{Re}\left\{B_{kj}n_j\,[x_1 + i(x_2 - 2)]^{-2}\right\} \\ &\quad + \operatorname{Re}\left\{\sum_{\alpha=1}^{2} 2L_{kj\alpha}n_j\,[x_1 + 1 + ix_2]\right\}. \end{aligned} \tag{43}$$

As in problem 1, we choose the region $\mathcal{R}$ to be given by

$$\mathcal{R} = \{(x_1, x_2) : 0 < x_1 < 1,\ 0 < x_2 < 1\}, \tag{44}$$

i.e. $\mathcal{C}$ is a square whose sides are of unit length.

Also, the heat flux and tractions are prescribed according to (43) on the side $0 < x_1 < 1$, $x_2 = 1$ of the square. On the remaining sides, the temperature and displacements are specified in accordance with (42). The sides of the square are discretised into 80 equal length segments. The CVBEM is applied to solve (39) in $\mathcal{R}$ subject to the generated boundary data on $\mathcal{C}$.

The numerical values of u_k thus obtained at various points in the interior of the square domain are compared with the exact solution in Table 5. Once again, the numerical and exact values show good agreement.

Table 5. Comparison of numerical values of the displacement (u_1, u_2) at various interior points (x_1, x_2) with the exact solution (test problem 3).

(x_1, x_2)	CVBEM	Exact
$(0.10, 0.10)$	$(2.336, -0.3630)$	$(2.386, -0.4400)$
$(0.50, 0.50)$	$(3.891, -2.989)$	$(3.900, -3.000)$
$(0.90, 0.10)$	$(7.066, -0.8015)$	$(7.098, -0.7600)$
$(0.40, 0.70)$	$(2.855, -3.897)$	$(2.832, -3.920)$

6 SUMMARY

A CVBEM is proposed for the numerical solution of time independent uncoupled plane thermoelastic problems involving anisotropic bodies. The method requires only the boundary of the domain of interest to be discretised. It can be efficiently implemented on the computer as the coefficients of the linear algebraic equations that approximate the problems under consideration are simple in form and easy to compute. The method is used to solve some specific test problems, including one involving a concave domain and an isotropic body. Even for relatively low number of boundary elements, the numerical results obtained by using the CVBEM show reasonably good agreement with the exact solutions.

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