CVBEM for a System of Second Order Elliptic Partial Differential Equations

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Abstract

A boundary element method based on the Cauchy's integral formulae and the theory of complex hypersingular integrals is devised for the numerical solution of boundary value problems governed by a system of second order elliptic partial differential equations. The elliptic system has applications in physical problems involving anisotropic media.

Key words: complex variable boundary element method, elliptic partial differential equations, anisotropic media.

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1 INTRODUCTION

Consider the system of second order elliptic partial differential equations given by

$$\sum_{j=1}^{2}\sum_{p=1}^{2}\sum_{k=1}^{N} a_{ijkp}\frac{\partial^2 \phi_k}{\partial x_j \partial x_p} = 0 \quad (i = 1, 2, \cdots, N), \tag{1}$$

where ϕ_k $(k = 1, 2, \cdots, N)$ are functions of x_1 and x_2 and a_{ijkp} $(j, p = 1, 2$ and $i, k = 1, 2, \cdots, N)$ are real constant coefficients which satisfy the symmetry conditions $a_{ijkp} = a_{kpij}$ and are such that

$$\sum_{j=1}^{2}\sum_{p=1}^{2}\sum_{i=1}^{N}\sum_{k=1}^{N} a_{ijkp}\lambda_{ij}\lambda_{kp} \;>\; 0 \text{ for every non-zero } N \times 2 \text{ real matrix } [\lambda_{ij}]. \tag{2}$$

We are interested in solving (1) in a region $\mathcal{R}$ bounded by a simple closed curve $\mathcal{C}$ (on the $0x_1x_2$ plane) subject to

$$\left.\begin{array}{lll} \phi_k(x_1, x_2) = \mu_k(x_1, x_2) & \text{for} & (x_1, x_2) \in \mathcal{C}_1 \\ P_i(x_1, x_2) = Q_i(x_1, x_2) & \text{for} & (x_1, x_2) \in \mathcal{C}_2 \end{array}\right\} \tag{3}$$

where μ_k and Q_i are suitably prescribed functions of x_1 and x_2, $\mathcal{C}_1$ and $\mathcal{C}_2$ are non-intersecting curves such that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and

$$P_i = \sum_{j=1}^{2}\sum_{p=1}^{2}\sum_{k=1}^{N} a_{ijkp}\frac{\partial \phi_k}{\partial x_p} n_j \quad (i = 1, 2, \cdots, N) \tag{4}$$

with n_j $(j = 1, 2)$ being components of the unit outer normal vector to $\mathcal{R}$ on $\mathcal{C}$.

The boundary value problem defined by (1) and (3) has important applications in engineering. As an example, the steady-state temperature distribution in a flat plate which is thermally anisotropic and homogeneous obeys

(1) with $N = 1$. The temperature and heat flux are given by ϕ_1 and (P_1, P_2) respectively, and a_{1j1p} are the heat conduction coefficients.

The plane static deformation of a homogeneous anisotropic elastic solid is governed by (1) with $N = 2$ and x_1 and x_2 as the Cartesian coordinates. The Cartesian displacement and traction are given by (ϕ_1, ϕ_2) and (P_1, P_2) respectively. The coefficients a_{ijkp} are the elastic moduli of the material occupying the solid. For a specific case, the elastostatic behaviour of a transversely-isotropic material which has transverse planes perpendicular to the $0x_1x_2$ plane and which undergoes plane deformation is governed by

$$\begin{aligned} C\frac{\partial^2\phi_1}{\partial x_1^2} + L\frac{\partial^2\phi_1}{\partial x_2^2} + (F+L)\frac{\partial^2\phi_2}{\partial x_1 \partial x_2} &= 0, \\ A\frac{\partial^2\phi_2}{\partial x_2^2} + L\frac{\partial^2\phi_2}{\partial x_1^2} + (F+L)\frac{\partial^2\phi_1}{\partial x_1 \partial x_2} &= 0, \end{aligned} \tag{5}$$

a special case which can be recovered from (1) if we let $N = 2$ and $a_{2222} = A$, $a_{1111} = C$, $a_{1122} = a_{2211} = F$, $a_{1212} = a_{2121} = a_{1221} = a_{2112} = L$ and the remaining a_{ijkl} be zero.

Clements and Rizzo [1] provided a real boundary integral equation method (based on fundamental solutions) for the numerical solution of the boundary value problem defined by (1) and (3). In the present paper, a complex variable boundary element method (CVBEM) is proposed as a useful alternative numerical technique for solving the problem. The method is implemented using constant elements and used to solve some specific problems.

Based on the Cauchy's integral formula, the CVBEM was originally introduced by Hromadka II and Lai [2] for the special case $a_{ijkp} = \delta_{jp}\delta_{i1}\delta_{k1}$ [δ_{ij} is the kronecker-delta], i.e. where the system (1) reduces to the two-dimensional Laplace's equation. Further development and refinement of the method were carried out by Hromadka II and his co-researchers (e.g. Hromadka and Yen [3], Whitley and Hromadka II [6], Hromadka II and Whitley

[4], [7], and Hromadka II [5]).

Introducing the theory of complex Hadamard finite-part (hypersingular) integrals, Linkov and Mogilevskaya [8] formulated a CVBEM to solve certain elastostatic problems governed by a particular system of elliptic partial differential equations. The theory of complex hypersingular integrals is also successfully applied here to develop a CVBEM for solving (1) numerically subject to (3).

2 BASIC EQUATIONS

The system (1) admits solutions of the form (Clements and Rizzo [1])

$$\phi_k(x_1, x_2) = \mathrm{Re}\left\{\sum_{\alpha=1}^{N} A_{k\alpha} f_\alpha(z_\alpha)\right\}, \tag{6}$$

where f_α are holomorphic functions of $z_\alpha = x_1 + p_\alpha x_2$ in $\mathcal{R}$, p_α are the solutions, with positive imaginary parts, of the $(2N)$-th order polynomial (characteristic) equation

$$\det\left(\left[a_{i1k1} + pa_{i2k1} + pa_{i1k2} + p^2 a_{i2k2}\right]\right) = 0, \tag{7}$$

and $A_{k\alpha}$ are solutions of the homogeneous system of equations

$$\sum_{k=1}^{N}\left[a_{i1k1} + p_\alpha a_{i2k1} + p_\alpha a_{i1k2} + p_\alpha^2 a_{i2k2}\right] A_{k\alpha} = 0. \tag{8}$$

Notice that, because of (2), equation (7) admits only non-real solutions which occur in complex conjugate pairs.

From (4) and (6), we find that

$$P_i(x_1, x_2) = \mathrm{Re}\left\{\sum_{j=1}^{2}\sum_{\alpha=1}^{N} L_{ij\alpha} f'_\alpha(z_\alpha) n_j\right\}, \tag{9}$$

where the prime denotes differentiation with respect to the relevant argument and

$$L_{ij\alpha}=\sum_{k=1}^{N}\left(a_{ijk1}+p_{\alpha}a_{ijk2}\right)A_{k\alpha}. \tag{10}$$

Since f_{α} are holomorphic functions of z_{α} in $\mathcal{R}$, Cauchy's integral formulae give

$$\begin{aligned}
2\pi i f_{\alpha}(\xi_1+p_{\alpha}\xi_2) &= \oint_{(x_1,x_2)\in\mathcal{C}}\frac{f_{\alpha}(x_1+p_{\alpha}x_2)d\left(x_1+p_{\alpha}x_2\right)}{\left(x_1+p_{\alpha}x_2-\xi_1-p_{\alpha}\xi_2\right)} &(11)\\
2\pi i f'_{\alpha}(\xi_1+p_{\alpha}\xi_2) &= \oint_{(x_1,x_2)\in\mathcal{C}}\frac{f_{\alpha}(x_1+p_{\alpha}x_2)d\left(x_1+p_{\alpha}x_2\right)}{\left(x_1+p_{\alpha}x_2-\xi_1-p_{\alpha}\xi_2\right)^2} &(12)
\end{aligned}$$

for $(\xi_1,\xi_2)\in\mathcal{R}$. Notice that $f'(z)$ denotes the first order derivative of f with respect to z. We assume $\mathcal{C}$ is assigned an anticlockwise direction.

For the case where (ξ_1,ξ_2) lies on a smooth part of $\mathcal{C}$, the formulae (11) and (12) can be modified to become

$$\begin{aligned}
\pi i f_{\alpha}(\xi_1+p_{\alpha}\xi_2) &= \mathcal{P}\oint_{(x_1,x_2)\in\mathcal{C}}\frac{f_{\alpha}(x_1+p_{\alpha}x_2)d\left(x_1+p_{\alpha}x_2\right)}{\left(x_1+p_{\alpha}x_2-\xi_1-p_{\alpha}\xi_2\right)} &(13)\\
\pi i f'_{\alpha}(\xi_1+p_{\alpha}\xi_2) &= \mathcal{H}\oint_{(x_1,x_2)\in\mathcal{C}}\frac{f_{\alpha}(x_1+p_{\alpha}x_2)d\left(x_1+p_{\alpha}x_2\right)}{\left(x_1+p_{\alpha}x_2-\xi_1-p_{\alpha}\xi_2\right)^2} &(14)
\end{aligned}$$

where $\mathcal{P}$ and $\mathcal{H}$ denote that the integral is to be interpreted in the Cauchy principal and Hadamard finite-part sense, respectively. For further details on the theory of complex Cauchy principal and Hadamard finite-part integrals, refer to Linkov and Mogilevskaya [8].

3 CVBEM

A CVBEM for the approximate solution of the boundary value problem defined by (1) and (3) may be devised using (6), (9) and (11)-(14) as follows.

The boundary $\mathcal{C}$ is discretised by putting M closely-packed well-spaced out points $(x_1^{(1)},x_2^{(1)})$, $(x_1^{(2)},x_2^{(2)}),\cdots$, $(x_1^{(k)},x_2^{(k)})$, $\cdots$, and $(x_1^{(M)},x_2^{(M)})$ (in anticlockwise direction) on it. If we denote the straight line segment from $(x_1^{(k)}, x_2^{(k)})$ to $(x_1^{(k+1)}, x_2^{(k+1)})$ by $\mathcal{C}^{(k)}$ $[k = 1, 2, \cdots, M]$, then we make the approximation $\mathcal{C} \simeq \mathcal{C}^{(1)} \cup \mathcal{C}^{(2)} \cup \cdots \cup \mathcal{C}^{(M)}$. [Note that we take $(x_1^{(M+1)}, x_2^{(M+1)}) = (x_1^{(1)}, x_2^{(1)})$.]

Let us write

$$u_k(x_1, x_2) + iv_k(x_1, x_2) = \sum_{\alpha=1}^{N} A_{k\alpha} f_\alpha(z_\alpha), \tag{15}$$

where u_k and v_k are real functions $[u_k = \phi_k]$.

Inverting (15), we obtain

$$f_\alpha(z_\alpha) = \sum_{k=1}^{N} N_{\alpha k} \left[u_k(x_1, x_2) + iv_k(x_1, x_2) \right], \tag{16}$$

where $[N_{\alpha k}]$ is the inverse of $[A_{k\alpha}]$. The existence of $[N_{k\alpha}]$ is guaranteed if p_1, $p_2, \cdots, p_{N-1}$ and p_N are all distinct (Clements and Rizzo [1]).

Approximating u_k and v_k by

$$\left.\begin{array}{l} u_k(x_1, x_2) \simeq u_k^{(q)} \\ v_k(x_1, x_2) \simeq v_k^{(q)} \end{array}\right\} \text{ for } (x_1, x_2) \in \mathcal{C}^{(q)}, \tag{17}$$

then for $(\xi_1, \xi_2) \in \mathcal{R} \cup \mathcal{C}$ the imaginary parts of both (11) and (13) give rise to the approximation

$$v_p(\xi_1, \xi_2) \simeq \frac{1}{2\pi} \sum_{m=1}^{M} \sum_{k=1}^{N} \left\{ -u_k^{(m)} \Gamma_{pk}^{(m)}(\xi_1, \xi_2) + v_k^{(m)} \Theta_{pk}^{(m)}(\xi_1, \xi_2) \right\}, \tag{18}$$

where $u_k^{(p)}$ and $v_k^{(p)}$ are constants, $\Gamma_{pk}^{(m)}(\xi_1, \xi_2)$ and $\Theta_{pk}^{(m)}(\xi_1, \xi_2)$ are real parameters defined by

$$\begin{aligned} & \Gamma_{pk}^{(m)}(\xi_1, \xi_2) + i\Theta_{pk}^{(m)}(\xi_1, \xi_2) \\ = & \sum_{\alpha=1}^{N} A_{p\alpha} N_{\alpha k} \left[\gamma_\alpha^{(m)}(\xi_1, \xi_2) + i\theta_\alpha^{(m)}(\xi_1, \xi_2) \right], \end{aligned}$$

$\gamma_\alpha^{(m)}(\xi_1,\xi_2) = \ln\left|z_\alpha^{(m+1)} - c_\alpha\right| - \ln\left|z_\alpha^{(m)} - c_\alpha\right|$ and

$$\theta_\alpha^{(m)}(\xi_1,\xi_2) = \begin{cases} \Omega_\alpha^{(m)}(\xi_1,\xi_2) & \text{if} \quad -\pi \le \Omega_\alpha^{(m)}(\xi_1,\xi_2) \le \pi \\ \Omega_\alpha^{(m)}(\xi_1,\xi_2) + 2\pi & \text{if} \quad -2\pi \le \Omega_\alpha^{(m)}(\xi_1,\xi_2) < -\pi \\ \Omega_\alpha^{(m)}(\xi_1,\xi_2) - 2\pi & \text{if} \quad \pi < \Omega_\alpha^{(m)}(\xi_1,\xi_2) \le 2\pi \end{cases}$$

where $\Omega_\alpha^{(m)}(\xi_1,\xi_2) = \mathrm{Arg}(z_\alpha^{(m+1)} - c_\alpha) - \mathrm{Arg}(z_\alpha^{(m)} - c_\alpha)$, $\mathrm{Arg}(z)$ denotes the principal value of the argument of the complex number z, $z_\alpha^{(m)} = x_1^{(m)} + p_\alpha x_2^{(m)}$ and $c_\alpha = \xi_1 + p_\alpha \xi_2$.

If the region $\mathcal{R}$ is convex, then $\theta_\alpha^{(m)}(\xi_1,\xi_2)$ can be calculated directly from

$$\theta_\alpha^{(m)}(\xi_1,\xi_2) = \cos^{-1}\left(\frac{\left|z_\alpha^{(m+1)} - c_\alpha\right|^2 + \left|z_\alpha^{(m)} - c_\alpha\right|^2 - \left|z_\alpha^{(m+1)} - z_\alpha^{(m)}\right|^2}{2\left|z_\alpha^{(m+1)} - c_\alpha\right|\left|z_\alpha^{(m)} - c_\alpha\right|}\right).$$

By letting $(\xi_1,\xi_2) = (y_1^{(p)}, y_2^{(p)}) \equiv (x_1^{(p+1)} + x_1^{(p)}, x_2^{(p+1)} + x_2^{(p)})/2$ (the mid-point of $\mathcal{C}^{(p)}$) in (18), we obtain the approximate system

$$v_p^{(q)} = \frac{1}{2\pi}\sum_{m=1}^{M}\sum_{k=1}^{N}\left\{-u_k^{(m)}\Gamma_{pk}^{(m)}(y_1^{(q)}, y_2^{(q)}) + v_k^{(m)}\Theta_{pk}^{(m)}(y_1^{(q)}, y_2^{(q)})\right\} \tag{19}$$

for $p = 1, 2, \cdots, N$ and $q = 1, 2, \cdots, M$. Equations (19) constitute a system of MN linear algebraic equations in $2MN$ unknowns $u_k^{(m)}$ and $v_k^{(m)}$ ($k = 1, 2, \cdots, N$; $m = 1, 2, \cdots, M$).

Over each of the line segments $\mathcal{C}^{(m)}$ ($m = 1, 2, \cdots, M$), either ϕ_k or P_i are specified according to (3). Thus, from (3), (6), (9) and (14), we obtain either

$$u_k^{(p)} = \mu_k(y_1^{(p)}, y_2^{(p)}) \text{ if } \phi_k \text{ are specified over } \mathcal{C}^{(p)}, \tag{20}$$

or

$$\begin{aligned} &\frac{1}{\pi}\sum_{m=1}^{M}\sum_{\alpha=1}^{N}\sum_{j=1}^{2}\sum_{k=1}^{N}\left\{\left(J_{ij\alpha k}q_\alpha^{(pm)} - K_{ij\alpha k}w_\alpha^{(pm)}\right)v_k^{(m)}\right. \\ &\left.+\left(J_{ij\alpha k}w_\alpha^{(pm)} + K_{ij\alpha k}q_\alpha^{(pm)}\right)u_\alpha^{(m)}\right\}n_j^{(p)} \\ = \; & Q_i(y_1^{(p)}, y_2^{(p)}) \text{ if } P_i \text{ are specified over } \mathcal{C}^{(p)}, \end{aligned} \tag{21}$$

where $J_{ij\alpha k} = \mathrm{Re}\{L_{ij\alpha}N_{\alpha k}\}$, $K_{ij\alpha k} = \mathrm{Im}\{L_{ij\alpha}N_{\alpha k}\}$, $n_j^{(m)}$ are the components of the unit normal vector to $\mathcal{R}$ on $\mathcal{C}^{(m)}$, $q_\alpha^{(pm)}$ and $w_\alpha^{(pm)}$ are real constants such that $q_\alpha^{(pm)} + iw_\alpha^{(pm)} = (z_\alpha^{(m)} - c_\alpha^{(p)})^{-1} - (z_\alpha^{(m+1)} - c_\alpha^{(p)})^{-1}$ and $c_\alpha^{(p)} = (z_\alpha^{(p)} + z_\alpha^{(p+1)})/2$. We find that (20) and/or (21) gives rise to an additional MN equations in $u_\alpha^{(m)}$ and $v_\alpha^{(m)}$.

We may solve (19) [for $p = 1, 2, \cdots, N$ and $q = 1, 2, \cdots, M$] together with (20) and/or (21) as a system of $2MN$ linear algebraic equations for the $2MN$ unknowns $u_k^{(m)}$ and $v_k^{(m)}$ ($k = 1, 2, \cdots, N$; $m = 1, 2, \cdots, M$). However, from numerical experiments, we find that, depending on the discretisation of the boundary $\mathcal{C}$, (19) [for $p = 1, 2, \cdots, N$ and $q = 1, 2, \cdots, M$] together with (20) and/or (21) may be poorly conditioned and may give rise to numerical values of v_k that are extremely large in magnitude, ruining subsequent calculations of ϕ_k in the interior of $\mathcal{R}$. Perhaps this observation is not surprising at all due to the fact that v_k are not uniquely determined by the boundary value problem under consideration (but $u_k = \phi_k$ are).

One method which is used successfully to overcome the difficulty associated with the poorly conditioned system of linear algebraic equations is to fix the value of v_k across the segment $\mathcal{C}^{(M)}$, e.g set $v_k^{(M)} = 0$, and solve (19), with $p = 1, 2, \cdots, N$ and $q = 1, 2, \cdots, M-1$, together with (20) and/or (21) as a well conditioned system of $(2M-1)N$ linear algebraic equations.

Once $u_k^{(m)}$ and $v_k^{(m)}$ ($k = 1, 2, \cdots, N$; $m = 1, 2, \cdots, M$) are all known, we can compute f_α approximately via

$$f_\alpha(\xi_1 + p_\alpha \xi_2) \simeq \frac{1}{2\pi i} \sum_{m=1}^{M} \sum_{k=1}^{N} N_{\alpha k} \left[u_k^{(m)}(\xi_1, \xi_2) + i v_k^{(m)}(\xi_1, \xi_2) \right] \times \left(\gamma_\alpha^{(m)}(\xi_1, \xi_2) + i\theta_\alpha^{(m)}(\xi_1, \xi_2) \right) \quad (22)$$

and hence $\phi_k(\xi_1, \xi_2)$ [using (6)] at any point (ξ_1, ξ_2) in the interior of $\mathcal{R}$.

4 SPECIFIC PROBLEMS

We shall now apply the CVBEM described above to solve some specific problems.

Problem 1. Take the elliptic partial differential equation

$$5\frac{\partial^2\phi}{\partial x_1^2} + 2\frac{\partial^2\phi}{\partial x_1\partial x_2} + \frac{\partial^2\phi}{\partial x_2^2} = 0 \tag{23}$$

which is a special case of (1) with $N = 1$ and $a_{1j1p} = 5\delta_{j1}\delta_{p1} + \delta_{j1}\delta_{p2} + \delta_{j2}\delta_{p1} + \delta_{j2}\delta_{p2}$.

We apply the CVBEM to solve (23) in the square region $\mathcal{R}$ given by

$$\mathcal{R} = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < 1\}$$

subject to the conditions

$$\begin{aligned}
\left.\frac{\partial\phi}{\partial x_2}\right|_{x_2=1} &= \frac{x_1^2 - 8x_1 - 4}{(x_1^2+4)^2} \text{ for } 0 < x_1 < 1, \\
\phi(0, x_2) &= \frac{-x_2 + 1}{(-x_2+1)^2 + 4x_2^2} \text{ for } 0 < x_2 < 1, \\
\phi(x_1, 0) &= \frac{1}{(x_1+1)} \text{ for } 0 < x_1 < 1, \\
\phi(1, x_2) &= \frac{2 - x_2}{(2-x_2)^2 + 4x_2^2} \text{ for } 0 < x_2 < 1.
\end{aligned} \tag{24}$$

It can be easily verified by substitution that the boundary value problem defined by (23) and (24) has the exact solution

$$\phi(x_1, x_2) = \frac{x_1 - x_2 + 1}{(x_1 - x_2 + 1)^2 + 4x_2^2}. \tag{25}$$

To apply the CVBEM to solve the boundary value problem, we place M equally spaced out nodal points on the square boundary (the nodal points include the four vertices of the square). Once the relevant holomorphic function is completely determined on the square boundary, we compute $\phi(x_1, x_2)$ at various points in the interior of the square region. The numerical results obtained by using $M = 40$ and $M = 120$ are compared with the exact solution (25) in Table 1. It is obvious that the numerical values compare favourably with the exact ones. More importantly, there is a noticeable improvement in the numerical results as the number of nodal points increases from 40 to 120.

Table 1. Comparison of numerical values of ϕ at various interior points (x_1, x_2) with the exact solution.

(x_1, x_2)	$M = 40$	$M = 120$	Exact
(0.10,0.10)	1.0062	0.9693	0.9615
(0.10,0.50)	0.4835	0.4423	0.4412
(0.50,0.10)	0.6983	0.7021	0.7000
(0.90,0.10)	0.5215	0.5486	0.5488
(0.50,0.50)	0.5127	0.5005	0.5000
(0.50,0.90)	1.6765	1.6645	1.6667
(0.70,0.90)	1.9536	2.0601	2.0619

Problem 2. Let the elliptic partial differential equation be given by

$$\frac{\partial^2\phi}{\partial x_1^2} + 4\frac{\partial^2\phi}{\partial x_2^2} = 0 \tag{26}$$

which is a special case of (1) with $N = 1$ and $a_{1j1p} = \delta_{j1}\delta_{p1} + 4\delta_{j2}\delta_{p2}$.

The boundary $\mathcal{C}$ is chosen to be a pentagon with vertices A $(0, 0)$, B $(0, 1)$, C $(1, 1)$, D $(1/2, 1/2)$ and E $(1, 0)$. For this particular case, $\mathcal{R}$ is a concave region.

For a specific boundary value problem, we solve (26) in $\mathcal{R}$ subject to

$$\begin{aligned}
\phi(0,x_2) &= \cos\left(\frac{1}{2}x_2\right) \text{ for } 0<x_2<1 \text{ (on } AB\text{)},\\
\phi(x_1,1) &= \exp(x_1)\cos\left(\frac{1}{2}\right) \text{ for } 0<x_1<1 \text{ (on } BC\text{)},\\
\phi(x_1,x_1) &= \exp(x_1)\cos\left(\frac{1}{2}x_1\right) \text{ for } \frac{1}{2}<x_1<1 \text{ (on } CD\text{)},\\
\phi(x_1,1-x_1) &= \exp(x_1)\cos\left(\frac{1}{2}\left[1-x_1\right]\right) \text{ for } \frac{1}{2}<x_1<1 \text{ (on } DE\text{)},\\
\phi(x_1,0) &= \exp(x_1) \text{ for } 0<x_1<1 \text{ (on } AE\text{)}. \qquad (27)
\end{aligned}$$

It can be easily verified by direct substitution that the exact solution of the boundary value problem is

$$\phi(x_1,x_2)=\exp(x_1)\cos\left(\frac{1}{2}x_2\right). \qquad (28)$$

We discretise the sides AB, BC and AE into 20 equal length boundary elements per side and each of DE and CD into 10 elements per side (so that $M=80$) and apply the CVBEM to solve (26) in the concave region $\mathcal{R}$ subject to (27). Once all the relevant holomorphic functions are constructed approximately by the CVBEM, we compute ϕ at various interior points in the concave region.

The numerical values of ϕ thus obtained at selected interior points are compared with the exact values from (28) in Table 2. The numerical and exact values show reasonable agreement with each other.

Table 2. Comparison of numerical values of ϕ at various interior points (x_1, x_2) in the concave region with the exact solution.

(x_1, x_2)	CVBEM	Exact
$(0.25, 0.25)$	1.282	1.274
$(0.25, 0.50)$	1.253	1.244
$(0.25, 0.75)$	1.203	1.195
$(0.50, 0.25)$	1.639	1.636
$(0.50, 0.75)$	1.538	1.534
$(0.60, 0.25)$	1.798	1.808
$(0.60, 0.75)$	1.686	1.695

Problem 3. Consider the system of partial differential equations (5) with $A = 16$, $C = 18$, $F = 7$ and $L = 4$, i.e.

$$\begin{aligned} 18\frac{\partial^2 \phi_1}{\partial x_1^2} + 11\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} + 4\frac{\partial^2 \phi_1}{\partial x_2^2} &= 0, \\ 16\frac{\partial^2 \phi_2}{\partial x_2^2} + 11\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + 4\frac{\partial^2 \phi_2}{\partial x_1^2} &= 0. \end{aligned} \tag{29}$$

For the system (29), the constants $A_{k\alpha}$, $L_{kj\alpha}$ and $N_{k\alpha}$ required in the CVBEM's calculations are given by

$$\begin{aligned} [A_{k\alpha}] &= \begin{bmatrix} -11ip_1(18+4p_1^2)^{-1} & -11ip_2(18+4p_2^2)^{-1} \\ i & i \end{bmatrix}, \\ [N_{k\alpha}] &= \frac{1}{11\nu}\begin{bmatrix} -2i\,(9+2p_1^2)\,(9+2p_2^2) & -11ip_2\,(9+2p_1^2) \\ 2i\,(9+2p_1^2)\,(9+2p_2^2) & 11ip_1\,(9+2p_2^2) \end{bmatrix}, \\ [L_{k1\alpha}] &= \begin{pmatrix} ip_1\left[7-77\,(18+4p_1^2)^{-1}\right] & ip_2\,[7-77\,(18+4p_2^2)]^{-1} \\ 4i\,(18-7p_1^2)\,/\,(18+4p_1^2) & 4i\,(18-7p_2^2)\,/\,(18+4p_2^2) \end{pmatrix}, \\ [L_{k2\alpha}] &= \begin{pmatrix} 4i\,(18-7p_1^2)\,/\,(18+4p_1^2) & 4i\,(18-7p_2^2)\,/\,(18+4p_2^2) \\ ip_1\left[16-77\,(18+4p_1^2)^{-1}\right] & ip_2\left[16-77\,(18+4p_2^2)^{-1}\right] \end{pmatrix}, \end{aligned} \tag{30}$$

where $\nu = (2p_2p_1 - 9)(-p_2 + p_1)$ and p_1 and p_2 are the solutions of $64p^4 + 183p^2 + 72 = 0$ such that $\mathrm{Im}\{p_1\} > 0$ and $\mathrm{Im}\{p_2\} > 0$, i.e. $p_1 \simeq 1.5454241i$ and $p_2 \simeq 0.68632303i$.

For a particular boundary value problem, we choose the region $\mathcal{R}$ to be given by

$$\mathcal{R} = \{(x_1, x_2) : 0 < x_1 < 1,\ \ 0 < x_2 < 1\}.$$

We are interested in solving (29) in $\mathcal{R}$ subject to

$$\begin{aligned}
P_k(x_1, 1) &= \mathrm{Re}\left\{\sum_{\alpha=1}^{2} 2L_{k2\alpha}\left[x_1 + 1 + p_\alpha\right]\right\} \text{ for } 0 < x_1 < 1, \\
\phi_k(0, x_2) &= \mathrm{Re}\left\{\sum_{\alpha=1}^{2} A_{k\alpha}\left[1 + p_\alpha x_2\right]^2\right\} \text{ for } 0 < x_2 < 1, \\
\phi_k(x_1, 0) &= \mathrm{Re}\left\{\sum_{\alpha=1}^{2} A_{k\alpha}\left[x_1 + 1\right]^2\right\} \text{ for } 0 < x_1 < 1, \\
\phi_k(1, x_2) &= \mathrm{Re}\left\{\sum_{\alpha=1}^{2} A_{k\alpha}\left[2 + p_\alpha x_2\right]^2\right\} \text{ for } 0 < x_2 < 1.
\end{aligned} \tag{31}$$

The exact solution of the boundary value problem defined by (29) and (31) is given by

$$\phi_k(x_1, x_2) = \mathrm{Re}\left\{\sum_{\alpha=1}^{2} A_{k\alpha}\left[x_1 + 1 + p_\alpha x_2\right]^2\right\}. \tag{32}$$

By discretising the sides of the square into 80 equal length segments, we apply the CVBEM to solve the boundary value problem and calculate ϕ_k at various points in the interior of the square domain. The numerical values of ϕ_k are compared with the exact values from (32) at various points in the interior of the square in Table 3. It is obvious that a reasonably good accuracy in the numerical results is achieved by the CVBEM.

Table 3. Comparison of numerical values of (ϕ_1, ϕ_2) at various interior points (x_1, x_2) in the square with the exact solution.

(x_1, x_2)	CVBEM	Exact
$(0.25, 0.25)$	$(3.553, -1.418)$	$(3.562, -1.395)$
$(0.25, 0.50)$	$(2.614, -2.816)$	$(2.620, -2.790)$
$(0.25, 0.75)$	$(1.064, -4.219)$	$(1.050, -4.185)$
$(0.50, 0.25)$	$(5.252, -1.695)$	$(5.268, -1.674)$
$(0.50, 0.50)$	$(4.312, -3.370)$	$(4.325, -3.348)$
$(0.50, 0.75)$	$(2.755, -5.043)$	$(2.754, -5.021)$
$(0.75, 0.25)$	$(7.254, -1.971)$	$(7.284, -1.953)$
$(0.75, 0.50)$	$(6.314, -3.920)$	$(6.341, -3.906)$
$(0.75, 0.75)$	$(4.752, -5.868)$	$(4.770, -5.858)$

5 SUMMARY

A simple CVBEM is presented for the numerical solution of boundary value problems involving a rather general system of second order elliptic partial differential equations. It is applied to solve some specific problems, including one involving a concave region. For each of the problems, the numerical results obtained compare favourably with the known (exact) solution.

The CVBEM can be efficiently implemented on the computer. Like other boundary element methods, it requires only the boundary of the domain of interest to be discretised. Furthermore, the coefficients of the linear algebraic equations which approximate the boundary value problems [equations (19), (20) and/or (21)] are easy to compute.

The proposed CVBEM can be refined further for better accuracy by using higher order elements in the approximation of the holomorphic functions as in Hromadka II and Lai [2], or by following recent developments in CVBEM, e.g. Hromadka II and Whitley [4]-[7].

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