Complex Variable Boundary Element Method (CVBEM)
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Revised 14 September 2010: Below Eqn (12) $-\pi < \mathrm{Arg}(z) \leq \pi$

Introduction

In the book "A Beginner's Course in Boundary Element Methods", we have considered solving the two-dimensional Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \text{ for } (x, y) \in R, \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} \phi &= f_1(x, y) \text{ for } (x, y) \in C_1, \\ \frac{\partial\phi}{\partial n} &= f_2(x, y) \text{ for } (x, y) \in C_2, \end{aligned} \tag{2}$$

where f_1 and f_2 are suitably prescribed functions and C_1 and C_2 are non-intersecting curves such that $C_1 \cup C_2 = C$ and C is the simple closed curve bounding the two-dimensional region R. We have shown how a boundary integral solution can be derived for (1) and applied (Chapters 1 and 2) to obtain boundary element procedures for the numerical solution of (1) and (2).

An alternative boundary element procedure based on the theory of complex variables can be derived for the numerical solution of (1) and (2) as explained below.

Complex Formulation

According to the theory of complex variables, if $F(z)$ ($z = x + iy$, $i = \sqrt{-1}$) is analytic (holomorphic) in R, then both the real and the imaginary parts of $F(z)$ satisfy the two-dimensional Laplace's equation for $(x, y) \in R$. Thus, we may write

$$\phi(x, y) = \mathrm{Re}\{F(z)\}. \tag{3}$$

We may think of (3) as the general solution of (1).

The boundary value problem defined by (1) and (2) may then be re-stated as:

"Construct a complex function $F(z)$ which is holomorphic in R and such that

$$\begin{aligned}\mathrm{Re}\{F(z)\} &= f_1(x,y) \text{ for } (x,y) \in C_1,\\ \mathrm{Re}\{(n_1+in_2)F'(z)\} &= f_2(x,y) \text{ for } (x,y) \in C_2,\end{aligned}$$

where $[n_1, n_2]$ is the unit normal vector to the curve C pointing away from R."

Numerical Construction of $F(z)$

According to the Cauchy integral formulae, if $F(z)$ is holomorphic in the region R bounded by the simple closed curve C then

$$2\pi i F(z_0) = \oint\limits_C \frac{F(z)}{z-z_0} dz \quad \text{for } z_0 = x_0 + iy_0 \in R, \tag{4}$$

and

$$2\pi i F'(z_0) = \oint\limits_C \frac{F(z)}{(z-z_0)^2} dz \text{ for } z_0 \in R. \tag{5}$$

In the Cauchy integral formulae above, C is assigned an anticlockwise direction.

We will use (4) and (5) to construct numerically[1] the required holomorphic function $F(z)$ whose real part gives the solution of the boundary value problem defined by (1) and (2). As pointed out above, the required function $F(z)$ must satisfy

$$\begin{aligned}\mathrm{Re}\{F(z)\} &= f_1(x,y) \text{ for } (x,y) \in C_1,\\ \mathrm{Re}\{(n_1+in_2)F'(z)\} &= f_2(x,y) \text{ for } (x,y) \in C_2.\end{aligned} \tag{6}$$

[1]The idea of using (4) to construct $F(z)$ numerically was apparently initiated by Hromadka and his co-researchers in the 1980s (see, for example, T. V. Hromadka and C. Lai, *The Complex Variable Boundary Element Method in Engineering Analysis*, Springer-Verlag, 1987). The use of (5) for treating the boundary condition given on the second line of (6) is a later development as found in works such as Linkov and Mogilevskaya [A. M. Linkov and S. G. Mogilevskaya, Complex hypersingular integrals and integral equations in plane elasticity, *Acta Mechanica* **105** (1994) 189-205] and Ang and Park [W. T. Ang and Y. S. Park, A complex variable boundary element method for an exterior boundary value problem governed by an elliptic partial differential equation, *SEA Bulletin of Mathematics* **23** (1999) 541-549].

We begin by noting that if the value of $F(z)$ is known at all points on the boundary C then we have solved the problem as we can then (at least theoretically) use (4) to calculate F at any point z_0 inside the solution domain R. From (6), the value of the function $F(z)$ is only partially known. For example, at any point z on C_1, the value of only the real part of $F(z)$ is known. We will outline here a simple boundary element method (that is, no discretisation of the solution domain R into elements is needed in the numerical procedure) for finding $F(z)$ numerically at all points on C.

We place N consecutive points $z^{(1)}$, $z^{(2)}$, $\cdots$, $z^{(N-1)}$ and $z^{(N)}$ in anticlockwise on the boundary C. Here $z^{(n)} = x^{(n)} + iy^{(n)}$. The m-th element $C^{(m)}$ is the *directed* line from $z^{(m)}$ to $z^{(m+1)}$. (Note that we take $z^{(N+1)} = z^{(1)}$.) So, we make the approximation

$$C \simeq C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(N-1)} \cup C^{(N)}. \tag{7}$$

With (7), (4) can be approximately written as

$$2\pi i F(z_0) = \sum_{n=1}^{N} \int\limits_{C^{(n)}} \frac{F(z)}{z - z_0} dz \;\; \text{for } z_0 \in R. \tag{8}$$

If we write $F(z)$ as its Taylor-Maclaurin series about $z = z_0$ and retains only the first term (in the series), we find that

$$\int\limits_{C^{(n)}} \frac{F(z)}{z - z_0} dz = F(z_0) \int\limits_{C^{(n)}} \frac{1}{z - z_0} dz + O(|z^{(n+1)} - z^{(n)}|). \tag{9}$$

Formally, we can evaluate the line integral on the right hand side of (9) as

$$\begin{aligned} \int\limits_{C^{(n)}} \frac{1}{z - z_0} dz &= \int\limits_{z^{(n)}}^{z^{(n+1)}} \frac{1}{z - z_0} dz \\ &= \ln(z^{(n+1)} - z_0) - \ln(z^{(n)} - z_0). \end{aligned} \tag{10}$$

However, since $\ln(z)$ is a complex function such that its imaginary part may have multiple values for a given z, we have to interpret the imaginary part of $\ln(z^{(n+1)} - z_0) - \ln(z^{(n)} - z_0)$ carefully. Whatever branch we choose to give to the complex logarithmic function, that is, no matter how we define $\arg(z)$ in $\ln(z) = \ln|z| + i\arg(z)$, $\ln(z^{(n+1)} - z_0) - \ln(z^{(n)} - z_0)$ must be the angle $\theta^{(n)}$ shown in Figure 1. As sketched in Figure 1, $\theta^{(n)}$ is positive because $\arg(z^{(n+1)} - z_0)$ and $\arg(z^{(n)} - z_0)$ are both positive angles (by convention) and $\arg(z^{(n+1)} - z_0) > \arg(z^{(n)} - z_0)$. Nevertheless, depending on where z_0 is or the direction of the element $C^{(n)}$, it is possible for $\theta^{(n)}$ to be negative. For example, if the element is orientated as sketched in Figure 2, then $\arg(z^{(n+1)} - z_0) < \arg(z^{(n)} - z_0)$ and hence $\theta^{(n)} < 0$.Whatever the case may be, it is clear that $|\theta^{(n)}|$ cannot be greater than π,that is the magnitude of the angle $\theta^{(n)}$ cannot be greater than 180°. Thus, $-\pi \leq \theta^{(n)} \leq \pi$. In Figure 1, if we push z_0 to tend to the midpoint (say) of the element, then we find that $\theta^{(n)}$ tends to π.

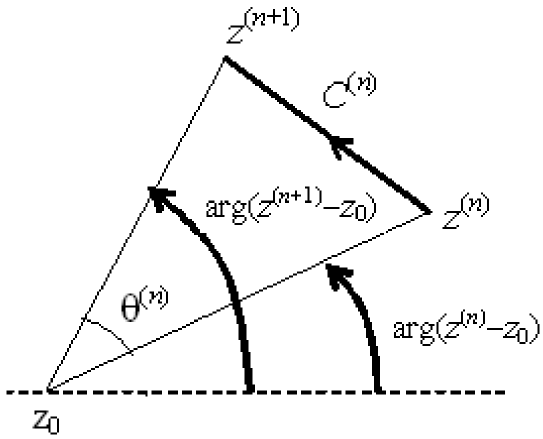

Figure 1

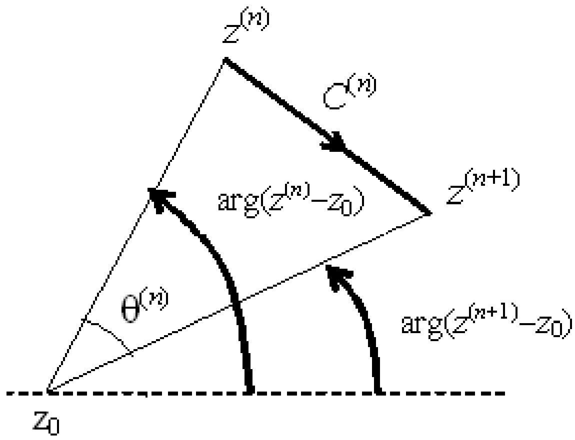

Figure 2

We can now write

$$
\begin{aligned}
\int\limits_{C^{(n)}} \frac{1}{z - z_0} dz &= \ln(z^{(n+1)} - z_0) - \ln(z^{(n)} - z_0) \\
&= \gamma^{(n)}(z_0) + i\theta^{(n)}(z_0),
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
\gamma^{(n)}(z_0) &= \ln \left| \frac{z^{(n+1)} - z_0}{z^{(n)} - z_0} \right|, \\
\theta^{(n)}(z_0) &= \begin{cases} \Omega^{(n)}(z_0) & \text{if} \quad -\pi < \Omega^{(n)}(z_0) \leq \pi \\ \Omega^{(n)}(z_0) + 2\pi & \text{if} \quad -2\pi \leq \Omega^{(n)}(z_0) \leq -\pi \\ \Omega^{(n)}(z_0) - 2\pi & \text{if} \quad \pi < \Omega^{(n)}(z_0) \leq 2\pi \end{cases}, \\
\Omega^{(n)}(z_0) &= \mathrm{Arg}(z^{(n+1)} - z_0) - \mathrm{Arg}(z^{(n)} - z_0).
\end{aligned} \tag{12}
$$

Note that $\mathrm{Arg}(z)$ is the principal argument of z such that $-\pi < \mathrm{Arg}(z) \leq \pi$ and $\theta^{(n)}(z_0)$ as defined above is for $z_0 \in R \cup C$ and is such that $-\pi < \theta^{(n)}(z_0) \leq \pi$.

For convex region R, the angle $\theta^{(n)}(z_0)$ is always positive if z_0 lies in R. Thus, from the cosine rule for the sides of a triangle, we can write

$$
\theta^{(n)}(z_0) = \cos^{-1}\left(\frac{\left|z^{(n+1)} - z_0\right|^2 + \left|z^{(n)} - z_0\right|^2 - \left|z^{(n+1)} - z^{(n)}\right|^2}{2\left|z^{(n+1)} - z_0\right|\left|z^{(n)} - z_0\right|} \right) \quad \text{for } z_0 \text{ in } R \text{ which is convex.} \tag{13}
$$

Note that (9), (11), (12) and (13) are still all valid even if we let z_0 tends towards the midpoint of any of the boundary elements, that is, all those equations still hold if we let z_0 be the midpoint of any of the boundary elements. If we let $z_0 = \widehat{z}^{(p)}$ (midpoint of $C^{(p)}$) in (8), use (9) and (11) and take the real parts of both sides of the equation, we obtain

$$
v^{(p)} = \frac{1}{2\pi} \sum_{n=1}^{N} \{\theta^{(n)}(\widehat{z}^{(p)}) v^{(n)} - \gamma^{(n)}(\widehat{z}^{(p)}) u^{(n)}\} \text{ for } p = 1, 2, \cdots, N, \tag{14}
$$

where $u^{(p)} = \mathrm{Re}\{F(\widehat{z}^{(p)})\}$and $v^{(p)} = \mathrm{Im}\{F(\widehat{z}^{(p)})\}$.

We may regard (14) as a system of N linear algebraic equations in $2N$ unknowns $u^{(n)}$ and $v^{(n)}$ ($n = 1, 2, \cdots, N$). Another N equations are needed to complete the system. This comes from the boundary conditions on the N boundary elements.

If the boundary condition on an element is given by the second line of (6), we have to calculate the first order derivative of the holomorphic function F at the midpoint of the element. To do this, we use the second Cauchy integral formula as given in (5).

From (5), we can write

$$2\pi i F'(z_0) = \sum_{n=1}^{N} \int_{C^{(n)}} \frac{F(z)}{(z-z_0)^2} dz \quad \text{for } z_0 \in R. \tag{15}$$

If we write $F(z)$ as its Taylor-Maclaurin series about $z = z_0$, we find that

$$\begin{aligned} \int_{C^{(n)}} \frac{F(z)}{(z-z_0)^2} dz &= F(z_0) \int_{C^{(n)}} \frac{1}{(z-z_0)^2} dz + F'(z_0) \int_{C^{(n)}} \frac{1}{(z-z_0)} dz \\ &\quad + \cdots + \frac{F^{(k)}(z_0)}{k!} \int_{C^{(n)}} \frac{1}{(z-z_0)^{2-k}} dz + \cdots . \end{aligned} \tag{16}$$

Now so long as z_0 is a point inside R, none of the integrals on the right hand side of (16) is improper and the second and subsequent terms can all be shown to be $O(|z^{(n+1)} - z^{(n)}|)$. However, if we let z_0 to tend to the midpoint of $C^{(p)}$,as we will have to if we are interested in using (15) to approximate $F'(\widehat{z}^{(p)})$, we have to bear in mind that for $n = p$ the second term on the right hand side of (16) tends to $\pi i F'(\widehat{z}^{(p)})$ as $z_0 \to \widehat{z}^{(p)}$. Hence, we make the approximation

$$\begin{aligned} & \int_{C^{(n)}} \frac{F(z)}{(z-\widehat{z}^{(p)})^2} dz \\ \simeq\ & F(\widehat{z}^{(p)}) \int_{C^{(n)}} \frac{1}{(z-\widehat{z}^{(p)})^2} dz \\ =\ & F(\widehat{z}^{(p)})[-\frac{1}{(z^{(n+1)} - \widehat{z}^{(p)})} + \frac{1}{(z^{(n)} - \widehat{z}^{(p)})}] \text{ for } n \neq p, \end{aligned} \tag{17}$$

and

$$
\begin{aligned}
&\int\limits_{C^{(p)}} \frac{F(z)}{(z-\widehat{z}^{(p)})^2}dz \\
\simeq\ & F(\widehat{z}^{(p)}) \int\limits_{C^{(p)}} \frac{1}{(z-\widehat{z}^{(p)})^2}dz + \pi i F'(\widehat{z}^{(p)}) \\
=\ & F(\widehat{z}^{(p)})[-\frac{1}{\left(z^{(p+1)}-\widehat{z}^{(p)}\right)} + \frac{1}{\left(z^{(p)}-\widehat{z}^{(p)}\right)}] + \pi i F'(\widehat{z}^{(p)}).
\end{aligned} \tag{18}
$$

It follows that (15) can be approximately written as

$$
\pi i F'(\widehat{z}^{(p)}) = \sum_{n=1}^{N} \{-\frac{1}{\left(z^{(n+1)}-\widehat{z}^{(p)}\right)} + \frac{1}{\left(z^{(n)}-\widehat{z}^{(p)}\right)}\} F(\widehat{z}^{(n)}). \tag{19}
$$

Now we are ready to set up N more equations using (6) and (19). We obtain

$$
u^{(p)} = f_1^{(p)} \text{ if } \phi \text{ is specified on } C^{(p)}, \tag{20}
$$

or

$$
\sum_{n=1}^{N} \{q^{(pn)}u^{(n)} - r^{(pn)}v^{(n)}\} = f_2^{(p)} \text{ if } \frac{\partial \phi}{\partial n} \text{ is specified on } C^{(p)}, \tag{21}
$$

where $f_1^{(p)}$ and $f_2^{(p)}$ are the specified boundary values and $q^{(pn)}$and $r^{(pn)}$ are real parameters defined by

$$
q^{(pn)} + ir^{(pn)} = \frac{1}{\pi i}\{-\frac{1}{\left(z^{(n+1)}-\widehat{z}^{(p)}\right)} + \frac{1}{\left(z^{(n)}-\widehat{z}^{(p)}\right)}\}(n_1^{(p)} + in_2^{(p)}), \tag{22}
$$

where $[n_1^{(p)}, n_2^{(p)}]$ is the unit normal vector to $C^{(p)}$ pointing out of R.

We may solve (14) together with (20) or (21) as a system of $2N$ linear algebraic equations in $2N$ unknowns $u^{(n)}$ and $v^{(n)}$ ($n = 1, 2, \cdots, N$). If ϕ is specified at at least one point on C then the constants $u^{(n)}$ are uniquely determined. From a theoretical point of view, $\text{Im}\{F(z)\}$ is determined only up to an arbitrary constant by (6). Thus, if we solve (14) together with (20) or (21) without specifying the value of $\text{Im}\{F(z)\}$ at one point on C, we may end up having $v^{(n)}$ whose values are all extremely large in magnitude. To avoid such a situation, we may set $v^{(N)} = 0$ and solve (14) for $p = 1, 2, \cdots, N-1$ (instead of N) together with (20) or (21).

Once $u^{(n)}$ and $v^{(n)}$ are known, we may calculate F at any interior point $z_0 = x_0 + iy_0$ in R approximately using

$$F(z_0) = \frac{1}{2\pi i}\sum_{n=1}^{N}(u^{(n)} + iv^{(n)})[\gamma^{(n)}(z_0) + i\theta^{(n)}(z_0)] \text{ for } z_0 \in R, \tag{23}$$

and the approximate value of the required solution ϕ at the point (x_0, y_0) is given by the real part of $F(z_0)$.

A Test Problem

Take the test problem in Example 1.2 in the book "A Beginner's Course in Boundary Element Methods". The CVBEM codes (in Fortran 77) for the numerical solution of the test problem are listed in the file `CVBEM.FOR`. As in the book, the boundary is discretised into $12N_0$ elements. In the table below, we compare the numerical solution obtained using $N_0 = 10$ and $N_0 = 40$ at some points in the interior of the solution domain.

(x, y)	$N_0 = 10$	$N_0 = 40$	Exact
$(1.50, 0.50)$	0.3618	0.3602	0.3591
$(0.10, 1.20)$	0.7333	0.7353	0.7330
$(1.08, 0.63)$	-0.3795	-0.3982	-0.4042
$(0.88, 1.52)$	-0.9171	-0.9245	-0.9274
$(1.06, 1.06)$	-1.0702	-1.0867	-1.0923

Cauchy and Hadamard finite-part integrals

We could have started the derivation of the boundary element procedure by using

$$\pi i F(z_0) = \mathcal{C}\oint_C \frac{F(z)}{z - z_0}dz \quad \text{for } z_0 = x_0 + iy_0 \in C, \tag{24}$$

and

$$\pi i F'(z_0) = \mathcal{H}\oint_C \frac{F(z)}{(z - z_0)^2}dz \text{ for } z_0 \in C, \tag{25}$$

where $\mathcal{C}$ and $\mathcal{H}$ denote that the complex integral over C is to be interpreted in the Cauchy principal and Hadamard finite-part sense. If we do so, we would have to define clearly what Cauchy principal and Hadamard finite-part integrals are in complex variables.

In the derivation given in this document, we have avoided going explicitly into the Cauchy principal and Hadamard finite-part integrals. This is done by discretising the curve C first into elements and then letting z_0 tend to the midpoint of an element. The point z_0 tends to the midpoint of an element, but it is always within R, so we can use (4) and (5) instead of having to start from (24) and (25).