Boundary Integral Equation for Axisymmetric Potential Problem
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The boundary integral equation for the three-dimensional potential problem is given in Chapter 6 of the book "*A Beginner's Course in Boundary Element Methods*" as

$$\begin{aligned}\lambda(\xi,\eta,\zeta)\phi(\xi,\eta,\zeta) &= \iint\limits_S (\phi(x,y,z)\frac{\partial}{\partial n}[\Phi_{3\mathrm{D}}(x,y,z;\xi,\eta,\zeta)] \\ &\quad -\Phi_{3\mathrm{D}}(x,y,z;\xi,\eta,\zeta)\frac{\partial}{\partial n}[\phi(x,y,z)])ds(x,y,z), \quad (1)\end{aligned}$$

where ϕ satisfies the three-dimensional Laplace's equation in the region R bounded by a closed surface S, $\lambda(\xi,\eta,\zeta)$ is defined by

$$\lambda(\xi,\eta,\zeta) = \begin{cases} 0 & \text{if } (\xi,\eta,\zeta)\notin R\cup S, \\ 1/2 & \text{if } (\xi,\eta,\zeta) \text{ lies on a smooth part of } S, \\ 1 & \text{if } (\xi,\eta,\zeta)\in R, \end{cases} \quad (2)$$

and $\Phi_{3\mathrm{D}}$ is the fundamental solution given by

$$\Phi_{3\mathrm{D}}(x,y,z;\xi,\eta,\zeta) = -\frac{1}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}. \quad (3)$$

Let us now consider the axisymmetric case in which the surface S of the solution domain can be generated by rotating a curve Γ about the z-axis by an angle of 360°. For example, if S is the sphere $x^2+y^2+(z-2)^2=1$ (sphere of center $(0,0,2)$ and radius 1) then we can generate the surface S by rotating the semi-circle $x^2+(z-2)^2=1$, $x\geq 0$, on the Oxz plane by an angle of 360° about the z-axis. For such a surface S and for ϕ which does not change with the polar coordinate θ but only with r and z, that is, for an axisymmetric problem, the boundary integral over S in (1) can be reduced to an integral over the curve Γ as explained below. (In cylindrical polar coordinates, points can be described using (r,θ,z) instead of (x,y,z), where $x=r\cos\theta$ and $y=r\sin\theta$.)

Firstly, let us define

$$\begin{aligned}\phi^*(r,\theta,z) &= \phi(r\cos\theta, y\sin\theta, z) \\ p^*(r,\theta,z) &= \frac{\partial}{\partial n}[\phi(x,y,z)]\Bigg|_{\substack{(x,y,z)=(r\cos\theta,y\sin\theta,z) \\ (n_x,n_y)=(n_r\cos\theta-n_\theta\sin\theta,n_r\sin\theta+n_\theta\cos\theta)}},\end{aligned}$$

where n_r and n_θ are respectively the r and θ components of the outward unit normal vector to the surface S as explained below. In Cartesian coordinates, the normal vector is given by $[n_x, n_y, n_z]$.

For an axisymmetric problem, ϕ^* is independent of θ and we can write $\phi^*(r, z)$. We will show now that, for axisymmetric problem, p^* also depends only on r and z. We have:

$$\begin{aligned}
\frac{\partial}{\partial n}[\phi(x,y,z)] &= n_x \frac{\partial \phi}{\partial x} + n_y \frac{\partial \phi}{\partial y} + n_z \frac{\partial \phi}{\partial z} \\
&= n_x(\frac{\partial \phi^*}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \phi^*}{\partial \theta}\frac{\partial \theta}{\partial x}) \\
&\quad + n_y(\frac{\partial \phi^*}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \phi^*}{\partial \theta}\frac{\partial \theta}{\partial y}) + n_z \frac{\partial \phi^*}{\partial z}.
\end{aligned}$$

If we introduce a local (polar) coordinate system with base vectors $\underline{\mathbf{e}}_r$, $\underline{\mathbf{e}}_\theta$ and $\underline{\mathbf{e}}_z = \underline{\mathbf{k}}$, then the unit normal vector is given by $n_r\underline{\mathbf{e}}_r + n_\theta\underline{\mathbf{e}}_\theta + n_z\underline{\mathbf{e}}_z$. On a fixed plane $z = c$ (constant), if the body is axisymmetric, the components n_r, n_θ and n_z do not change with θ, but n_x and n_y change with θ. It may be shown that

$$\begin{aligned}
n_x &= n_r \cos\theta - n_\theta \sin\theta \\
n_y &= n_r \sin\theta + n_\theta \cos\theta.
\end{aligned}$$

It follows that (for axisymmetric problem)

$$\begin{aligned}
p^* &= (n_r \cos\theta - n_\theta \sin\theta)[\cos\theta \frac{\partial \phi^*}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \phi^*}{\partial \theta}] \\
&\quad + (n_r \sin\theta + n_\theta \cos\theta)[\sin\theta \frac{\partial \phi^*}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \phi^*}{\partial \theta}] + n_z \frac{\partial \phi^*}{\partial z} \\
&= n_r \frac{\partial \phi^*}{\partial r} + \frac{1}{r} n_\theta \frac{\partial \phi^*}{\partial \theta} + n_z \frac{\partial \phi^*}{\partial z}.
\end{aligned}$$

Since ϕ^*, n_r and n_z are independent of θ, we find that $p^* = n_r\partial\phi^*/\partial r + n_z\partial\phi^*/\partial z$ is also independent of θ.

For convenience, we will now drop the asterik $*$ and write $\phi^*(r, z)$ as merely $\phi(r, z)$ and $p^*(r, z)$ as $p(r, z)$.

For S which is symmetrical about the z-axis, the infinitesimal area $ds(x, y, z)$ in (1) can be written as

$$ds(x, y, z) = r\, d\ell\, d\theta. \tag{4}$$

where $d\ell$ is the length of an infinitesimal portion of the curve Γ.

Consider now (1) for axisymmetric potential problem. For a point $(\xi, \eta, \zeta) = (r_0 \cos\theta_0, r_0 \sin\theta_0, z_0)$ on the Oxz plane (where $y = 0$ or $\theta_0 = 0$), we can rewrite (1) as

$$\begin{aligned} &\lambda(r_0, z_0)\phi(r_0, z_0) \\ = &\iint\limits_S (\phi(r, z)\frac{\partial}{\partial n}[\Phi_{3D}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)] \\ &-\Phi_{3D}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)p(r, z))r\, d\ell\, d\theta, \end{aligned} \tag{5}$$

where $\lambda(r_0, z_0) = 1/2$ if (r_0, z_0) lies on a smooth part of Γ and $\lambda(r_0, z_0) = 1$ if (r_0, z_0) lies in the interior of the solution domain on the Oxz plane.

We need to integrate with respect to θ from 0 to 2π as the complete surface S is obtained by rotating Γ by an angle of 360°. Note that θ appears only in the function Φ_{3D} and not in $\phi(r, z)$. The integration involving the coordinates r and z (that is, with respect to ℓ) is over the curve Γ. Thus, we can rewrite (5) as

$$\begin{aligned} &\lambda(r_0, z_0)T(r_0, z_0) \\ = &\int\limits_\Gamma (T(r, z)\Psi_{\text{axis}}(r, z; r_0, z_0; n_r, n_z) - \Phi_{\text{axis}}(r, z; r_0, z_0)p(r, z))r d\ell(r, z), \end{aligned} \tag{6}$$

where

$$\begin{aligned} &\Phi_{\text{axis}}(r, z; r_0, z_0) \\ = &\int\limits_0^{2\pi} \Phi_{3D}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)d\theta \\ = &-\frac{1}{4\pi}\int\limits_0^{2\pi} \frac{1}{\sqrt{(r\cos\theta - r_0)^2 + r^2\sin^2\theta + (z - z_0)^2}}d\theta \\ = &-\frac{1}{4\pi}\int\limits_0^{2\pi} \frac{1}{\sqrt{r^2 + r_0^2 + (z - z_0)^2 - 2rr_0\cos\theta}}d\theta \\ = &-\frac{1}{\pi\sqrt{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0}}\int\limits_0^{2\pi} \frac{1}{4\sqrt{1 - \frac{2rr_0(1+\cos\theta)}{r^2+r_0^2+(z-z_0)^2+2rr_0}}}d\theta \end{aligned}$$

$$= -\frac{1}{\pi\sqrt{r^2+r_0^2+(z-z_0)^2+2rr_0}}\int_0^\pi \frac{1}{2\sqrt{1-\frac{2rr_0(1+\cos(2t))}{r^2+r_0^2+(z-z_0)^2+2rr_0}}}dt$$

$$= -\frac{1}{\pi\sqrt{r^2+r_0^2+(z-z_0)^2+2rr_0}}\int_0^\pi \frac{1}{2\sqrt{1-\frac{4rr_0\cos^2(t)}{r^2+r_0^2+(z-z_0)^2+2rr_0}}}dt$$

$$= -\frac{1}{\pi\sqrt{r^2+r_0^2+(z-z_0)^2+2rr_0}}\int_0^{\pi/2} \frac{1}{\sqrt{1-\frac{4rr_0\cos^2(t)}{r^2+r_0^2+(z-z_0)^2+2rr_0}}}dt$$

$$= -\frac{1}{\pi\sqrt{r^2+r_0^2+(z-z_0)^2+2rr_0}}\int_0^{\pi/2} \frac{1}{\sqrt{1-\frac{4rr_0\sin^2(t)}{r^2+r_0^2+(z-z_0)^2+2rr_0}}}dt.$$

If we define the function $K(m)$ as

$$K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1-m\sin^2(t)}}dt, \tag{7}$$

then we can write

$$\Phi_{\text{axis}}(r,z;r_0,z_0) = -\frac{1}{\pi\sqrt{r^2+r_0^2+(z-z_0)^2+2rr_0}}K\left(\frac{4rr_0}{r^2+r_0^2+(z-z_0)^2+2rr_0}\right). \tag{8}$$

In mathematics, K is a special function and is called the *complete elliptic integral of the first kind*. There is a simple approximate but accurate formula in Abramowitz and Stegun's *Handbook of Mathematical Functions* for evaluating $K(m)$. Some mathematical softwares may have inbuilt functions for calculating $K(m)$. Note that $0 \le \frac{4rr_0}{r^2+r_0^2+(z-z_0)^2+2rr_0} \le 1$ and $K(m)$ is undefined for $m = 1$.

Also, for axisymmetric body, we have:

$$
\begin{aligned}
& \Psi_{\text{axis}}(r,z;r_0,z_0;n_r,n_z) \\
= & \int_0^{2\pi} \frac{\partial}{\partial n}[\Phi_{\text{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)]d\theta \\
= & \int_0^{2\pi} (n_r \frac{\partial}{\partial r}[\Phi_{\text{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)] \\
& + \frac{1}{r} n_\theta \frac{\partial}{\partial \theta}[\Phi_{\text{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)] \\
& + n_z \frac{\partial}{\partial z}[\Phi_{\text{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)])d\theta \qquad (9) \\
= & \; n_r \frac{\partial}{\partial r}[\Phi_{\text{axis}}(r,z;r_0,z_0)] + n_z \frac{\partial}{\partial z}[\Phi_{\text{axis}}(r,z;r_0,z_0)]
\end{aligned}
$$

For the axisymmetric body, can you see why $n_\theta = 0$?

There is this relationship:

$$
\frac{d}{dm}(K(m)) = \frac{1}{2m}(\frac{E(m)}{1-m} - K(m)), \qquad (10)
$$

where

$$
E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2 t}\, dt. \qquad (11)
$$

Note that $E(m)$ is known as the *complete elliptic integral of the second kind.* Details on computing $E(m)$ are available in Abramowitz and Stegun.

From (9) and (10), it can be shown that

$$
\begin{aligned}
& \Psi_{\text{axis}}(r,z;r_0,z_0;n_r,n_z) \\
= & -\frac{1}{\pi\sqrt{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0}} \\
& \times\{\frac{n_r}{2r}[\frac{r_0^2 - r^2 + (z-z_0)^2}{r^2 + r_0^2 + (z-z_0)^2 - 2rr_0} E(\frac{4rr_0}{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0}) \\
& \quad -K(\frac{4rr_0}{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0})] \\
& + n_z \frac{z_0 - z}{r^2 + r_0^2 + (z-z_0)^2 - 2rr_0} E(\frac{4rr_0}{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0})\}. \qquad (12)
\end{aligned}
$$

If we use (6) to devise a boundary element method for solving the axisymmetric potential problem, we have to discretize the curve Γ on the rz space (that is, the Oxz plane). The curve Γ can be discretized into straight line elements. The k-th typical element which has endpoints $(r^{(k)}, z^{(k)})$ and $(r^{(k+1)}, z^{(k+1)})$