A numerical method based on boundary integral equations and radial basis functions for plane anisotropic thermoelastostatic equations with general variable coefficients

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Abstract

A boundary integral method with radial basis function approximation is proposed for solving numerically an important class of boundary value problems governed by a system of thermoelastostatic equations with variable coefficients. The equations describe the thermoelastic behaviors of nonhomogeneous anisotropic materials with properties that vary smoothly from point to point in space. No restriction is imposed on the spatial variations of the thermoelastic coefficients as long as all the requirements of the laws of physics are satisfied. To check the validity and accuracy of the proposed numerical method, some specific test problems with known solutions are solved.

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1 Introduction

The plane themoelastostatic deformation of an anisotropic material is governed by the elliptic systems of partial differential equations

$$\frac{\partial}{\partial x_i}(k_{ij}\frac{\partial T}{\partial x_j}) + Q = 0, \tag{1}$$

and

$$\frac{\partial}{\partial x_j}(c_{ijkr}\frac{\partial u_k}{\partial x_r}) = \frac{\partial}{\partial x_j}(\beta_{ij}T) - f_i. \tag{2}$$

where Latin subscripts take the values of 1 and 2, the Einsteinian convention of summing over a repeated subscript is assumed here, x_1 and x_2 are the Cartesian coordinates of points in the material, T and u_k which are functions of x_1 and x_2 are the temperature and the x_k component of the Cartesian displacement, k_{ij}, c_{ijkr} and β_{ij} are respectively the heat conduction coefficients, the elastic moduli and the stress-temperature coefficients of the material, and Q and f_i are given heat source and body force terms respectively. For more details on the equations in (1) and (2), one may refer to Clements [11, 12] and Nowacki [24].

The material properties k_{ij}, c_{ijkr} and β_{ij} are constants for homogeneous materials. For nonhomogeneous materials such as functionally graded materials, they are given by functions that vary smoothly with x_1 and x_2. Experimental and theoretical works on thermal and mechanical behaviors of functionally graded materials have been a subject of considerable interest among many researchers (see, for example, Batra [9], Kapuria, Bhattacharyya and Kumar [20], Kawasaki and Watanabe [21], Pei, Ocelik and de Hosson [26] and Vel and Batra [35]), as the materials play an important role in many engineering applications. Partial differential equations of the form (1) and (2) with variable coefficients are also of interest in geotechnical engineering such as in the modeling of the properties of soils that vary with depth (Gibson [19] and Ter-Martirosyan and Mirnyi [34]).

In general, partial differential equations with variable coefficients are inherently difficult to solve. Thus, to simplify thermal and elastic analyses involving functionally graded materials, many researchers assume that the heat conduction coefficients k_{ij} and the elastic moduli c_{ijkr} are of the form

$$k_{ij} = \kappa_{ij} g(x_1, x_2) \text{ and } c_{ijkr} = \gamma_{ijkr} h(x_1, x_2), \tag{3}$$

where κ_{ij} and γ_{ijkr} are constants and $g(x_1, x_2)$ and $h(x_1, x_2)$ are grading functions given by specific elementary functions. For example, the grading functions are given by exponential functions of x_1 and x_2 in Kuo and Chen [22], Petrova and Sadowski [27] and Sladek, Sladek and Chang [32], while they are taken as quadratic functions in Wang and Qin [36] and Yuan and Yin [37].

For homogeneous materials, boundary integral methods for the generalized heat equation (1) with $Q = 0$ (that is, for steady state heat anisotropic heat conduction problem with zero heat source) and for the elastic equations in (2) with $\beta_{ij} = 0$ and $f_i = 0$ (anisotropic elastostatic problem without body force) are well established (see Clements [12]). For functionally graded materials with thermal conductivity of the form given in (3), a dual-reciprocity boundary element method for solving the anisotropic heat conduction problem is presented in Ang, Clements and Vahdati [7]. In Azis and Clements [8], the elastic moduli of functionally graded materials are taken to be of the form in (3) with some extra conditions imposed on the constants γ_{ijkr} and the grading function $g(x_1, x_2)$ in order to derive a boundary integral method for solving the anisotropic elastostatic problem. Recently, a boundary integral approach with radial basis function approximations is proposed in Ang [3] for solving the anisotropic elastostatic problem with the elastic moduli given by (3) but without the extra conditions on γ_{ijkr} and $g(x_1, x_2)$ imposed in [8]. Radial basis functions are widely used in the development of meshless methods for approximating partial differential equations in engineering science (see, for example, Chu, Wang, Zhong and He [10], Fasshauer [16] and Sarler and Vertnik [28]).

Boundary element methods for solving partial differential equations of thermoelasticity for homogeneous materials may be found in, for example, Ang, Clements and Cooke [5, 6], Dargush and Banerjee [13], Deb [14], Shiah and Tan [30] and Sladek and Sladek [31]. In Gao [18], plane thermoelastostatic problems involving isotropic functionally graded materials are formulated in terms of boundary-domain integral equations with the domain integrals treated by a radial integration method.

In the current paper, we employ a numerical method based on boundary integral equations and radial basis function approximations for solving boundary value problems governed by the equations of thermoelasticity in (1) and (2) for nonhomogeneous materials with spatially varying properties. No restrictive form (such as the one in (3)) is imposed here on the properties k_{ij}, β_{ij} and c_{ijkr}. The coefficients k_{ij}, β_{ij} and c_{ijkr} may be individually given by any smoothly varying functions as long as they satisfy the symmetric relations and the positive definiteness conditions required by the laws of physics. In the approach here, the governing equations (1) and (2) are reduced to linear algebraic equations by using radial basis functions and the standard boundary integral equations for anisotropic heat conduction and elastostatics. No domain integral is involved in the formulation. The numerical procedure does not require the solution domain to be discretized into elements. Some specific test problems are solved to check the validity and accuracy of the numerical method presented here.

2 Boundary value problems

With reference to an $Ox_1x_2x_3$ Cartesian frame, consider an anisotropic body with geometry and properties that do not vary along the x_3 direction. At each and every point on the boundary of the body, either the temperature or heat flux and either the displacement or the traction are suitably specified. The prescribed boundary conditions are independent of time and the spatial coordinate x_3. The problem of interest here is to determine the temperature and the displacement fields throughout the body.

If the body occupies the region $\mathcal{R}$ on the Ox_1x_2 plane, where $\mathcal{R}$ is bounded by a simple closed curve $\mathcal{C}$, then the boundary value problems to solve in $\mathcal{R}$ in order to determine the temperature and displacement are as described below.

At any general point (x_1, x_2) in $\mathcal{R}$, the temperature T, which is a function of x_1 and x_2 only, is required to satisfy the elliptic partial differential equation (1) where the heat conduction coefficients k_{ij} $(i, j = 1, 2)$ and the heat source term Q are smoothly varying functions of x_1 and x_2.

Since either the temperature or the heat flux is specified at each and every point on $\mathcal{C}$, the thermal boundary conditions may be written as

$$\begin{array}{ll} T(x_1, x_2) = \Omega(x_1, x_2) & \text{for } (x_1, x_2) \text{ on } \mathcal{C}_1, \\ \digamma(x_1, x_2) = \Psi(x_1, x_2) & \text{for } (x_1, x_2) \text{ on } \mathcal{C}_2, \end{array} \tag{4}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are non-intersecting curves such that $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{C}$, $\Omega(x_1, x_2)$ and $\Psi(x_1, x_2)$ are suitably prescribed functions on $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively and $\digamma$ is the thermal heat flux function on $\mathcal{C}$ defined by

$$\digamma = k_{ij} n_i \frac{\partial T}{\partial x_j}, \tag{5}$$

where $n_i(x_1, x_2)$ is the x_i component of the unit outer normal vector to $\mathcal{R}$ at the point (x_1, x_2) on $\mathcal{C}$.

To determine the temperature T in the solid, the boundary value problem to solve is defined by (1) and (4).

If the displacement and the stress, which are also functions of x_1 and x_2 only, are denoted by u_k and σ_{ij} respectively then (Clements [11, 12] and Nowacki [24])

$$\sigma_{ij}(x_1, x_2) = c_{ijkr}\frac{\partial u_k}{\partial x_r} - \beta_{ij}T(x_1, x_2), \tag{6}$$

where the elastic moduli c_{ijkr} and the stress-temperature coefficients β_{ij} are smoothly varying functions of x_1 and x_2.

Substituting (6) into the equilibrium equations $\partial\sigma_{ij}/\partial x_j = -f_i$, where the body force terms f_i are smoothly varying functions of x_1 and x_2, we obtain the elliptic system of partial differential equations in (2).

Either the displacement or the traction is suitably specified at each and every point $\mathcal{C}$. This may be written as

$$\begin{aligned} u_k(x_1, x_2) &= F_k(x_1, x_2) \quad \text{for } (x_1, x_2) \text{ on } \mathcal{C}_3, \\ t_k(x_1, x_2) &= G_k(x_1, x_2) \quad \text{for } (x_1, x_2) \text{ on } \mathcal{C}_4, \end{aligned} \tag{7}$$

where $\mathcal{C}_3$ and $\mathcal{C}_4$ are non-intersecting curves such that $\mathcal{C}_3 \cup \mathcal{C}_4 = \mathcal{C}$, $F_k(x_1, x_2)$ and $G_k(x_1, x_2)$ are suitably prescribed functions on $\mathcal{C}_3$ and $\mathcal{C}_4$ respectively and t_k is the x_k component of the traction as defined by

$$t_k = \sigma_{kj}n_j. \tag{8}$$

The boundary value problem to solve in order to determine the displacements u_k is governed by (2) and (7). Note that the temperature $T(x_1, x_2)$ is known in (2) after (1) is solved subject to (4).

3 Overview of solution approach

We rewrite the elliptic partial differential equation (1) and (2) respectively as

$$\frac{\partial}{\partial x_i}(k_{ij}^{(0)}\frac{\partial T}{\partial x_j} + k_{ij}^{(1)}\frac{\partial T}{\partial x_j}) = -Q, \tag{9}$$

and

$$\frac{\partial}{\partial x_j}(c_{ijkr}^{(0)}\frac{\partial u_k}{\partial x_r} + c_{ijkr}^{(1)}\frac{\partial u_k}{\partial x_r} - \beta_{ij}T) = -f_i, \tag{10}$$

where $k_{ij}^{(0)}$ and $c_{ijkr}^{(0)}$ are constants which may be obtained by averaging respectively k_{ij} and c_{ijkr} over $\mathcal{R}$ and $k_{ij}^{(1)} = k_{ij} - k_{ij}^{(0)}$ and $c_{ijkr}^{(1)} = c_{ijkr} - c_{ijkr}^{(0)}$ are, in general, functions that vary smoothly with x_1 and x_2 in $\mathcal{R}$.

By letting

$$T(x_1, x_2) = V(x_1, x_2) + W(x_1, x_2). \tag{11}$$

we replace the partial differential equation in (9) by

$$\frac{\partial}{\partial x_i}(k_{ij}^{(0)}\frac{\partial V}{\partial x_j} + k_{ij}^{(1)}\frac{\partial T}{\partial x_j}) = -Q, \tag{12}$$

and

$$k_{ij}^{(0)}\frac{\partial^2 W}{\partial x_i \partial x_j} = 0. \tag{13}$$

Similarly, letting

$$u_k(x_1, x_2) = v_k(x_1, x_2) + w_k(x_1, x_2), \tag{14}$$

we replace the elliptic system of partial differential equations in (10) by

$$\frac{\partial}{\partial x_j}(c_{ijkr}^{(0)}\frac{\partial v_k}{\partial x_r} + c_{ijkr}^{(1)}\frac{\partial u_k}{\partial x_r} - \beta_{ij}T) = -f_i, \tag{15}$$

and

$$c_{ijkr}^{(0)}\frac{\partial^2 w_k}{\partial x_j \partial x_r} = 0. \tag{16}$$

The solution approach for the boundary values problems in Section 2 follows that in Ang [3, 4]. As pointed out in Section 1, the variations of the anisotropic elastic moduli in [3] for elastostatic deformations follow the restrictive form in (3) (guided by only a single grading function). In the current paper, the material properties k_{ij}, β_{ij} and c_{ijkr} may be individually given by any smoothly varying functions.

We employ a meshless technique based on radial basis functions to approximate (12) and (15) as linear algebraic equations. The partial differential equations in (13) and (16) are recast in terms of the standard boundary integral equations given in Clements [12]. The boundary integral equations are discretized together with the boundary conditions into linear algbraic equations. The values of the temperature T and the displacements u_k at chosen collocation points in $\mathcal{R} \cup \mathcal{C}$ appear as unknowns to be determined from the resulting system of linear algebraic equations.

4 Radial basis function approximations

For the meshless technique for approximating (12) and (15), we choose P well spaced out collocation points in $\mathcal{R} \cup \mathcal{C}$. The chosen points are denoted by $(\xi_1^{(1)}, \xi_2^{(1)})$, $(\xi_1^{(2)}, \xi_2^{(2)})$, $\cdots$, $(\xi_1^{(P-1)}, \xi_2^{(P-2)})$ and $(\xi_1^{(P)}, \xi_2^{(P)})$, where $\xi_i^{(j)}$ is the x_i coordinate of the j-th collocation point.

We may approximate a smoothly varying function $g(x_1, x_2)$ in $\mathcal{R} \cup \mathcal{C}$ by

$$g(x_1, x_2) \simeq \sum_{r=1}^{P} a^{(r)} \rho^{(r)}(x_1, x_2), \tag{17}$$

where $a^{(r)}$ are constant coefficients and $\rho^{(r)}(x_1, x_2)$ is a radial basis function centered about $(\xi_1^{(r)}, \xi_2^{(r)})$.

If we collocate (17) at $(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1, 2, \cdots , P$ and invert the resulting linear equations for the constants $a^{(r)}$, we obtain

$$a^{(r)} \simeq \sum_{m=1}^{P} \varphi^{(rm)} g^{(m)}, \tag{18}$$

where $g^{(m)} = g(\xi_1^{(m)}, \xi_2^{(m)})$ and $\varphi^{(rm)}$ are constants defined by

$$\sum_{m=1}^{P} \varphi^{(rm)} \rho^{(s)}(\xi_1^{(m)}, \xi_2^{(m)}) = \begin{cases} 1 & \text{if} \quad r = s, \\ 0 & \text{if} \quad r \neq s. \end{cases} \tag{19}$$

Substituting (18) into (17) yields

$$g(x_1, x_2) \simeq \sum_{r=1}^{P} \sum_{m=1}^{P} \varphi^{(rm)} g^{(m)} \rho^{(r)}(x_1, x_2). \tag{20}$$

If the radial basis function $\rho^{(r)}(x_1, x_2)$ is required to be partially differentiable once with respect to x_1 or x_2, we may use the radial basis function proposed in Zhang and Zhu [38], that is,

$$\rho^{(r)}(x_1, x_2) = 1 + (x_1 - \xi_1^{(r)})^2 + (x_2 - \xi_2^{(r)})^2 + ((x_1 - \xi_1^{(r)})^2 + (x_2 - \xi_2^{(r)})^2)^{3/2}. \tag{21}$$

Alternatively, we may consider taking $\rho^{(r)}(x_1, x_2)$ to be the well known multiquadric radial basis function given by (see Ferreira [17] and Sarra [29])

$$\rho^{(r)}(x_1, x_2) = \sqrt{1 + \varepsilon((x_1 - \xi_1^{(r)})^2 + (x_2 - \xi_2^{(r)})^2)}, \tag{22}$$

where ε is a given positive real number. One may refer to Menandro [23] for a more recent account on radial basis function approximation.

4.1 Meshless approximation of (12)

From (20), we make the approximations

$$\begin{aligned} T(x_1, x_2) &\simeq \sum_{r=1}^{P} \sum_{m=1}^{P} \varphi^{(rm)} T^{(m)} \rho^{(r)}(x_1, x_2), \\ V(x_1, x_2) &\simeq \sum_{r=1}^{P} \sum_{m=1}^{P} \varphi^{(rm)} V^{(m)} \rho^{(r)}(x_1, x_2), \end{aligned} \tag{23}$$

where $T^{(m)} = T(\xi_1^{(m)}, \xi_2^{(m)})$ and $V^{(m)} = V(\xi_1^{(m)}, \xi_2^{(m)})$.

It follows that

$$\begin{aligned} & k_{ij}^{(0)} \frac{\partial}{\partial x_j}(V(x_1, x_2)) + k_{ij}^{(1)}(x_1, x_2) \frac{\partial}{\partial x_j}(T(x_1, x_2)) \\ \simeq & \sum_{m=1}^{P} (\theta_i^{(m)}(x_1, x_2) V^{(m)} + \phi_i^{(m)}(x_1, x_2) T^{(m)}), \end{aligned} \quad (24)$$

where

$$\begin{aligned} \theta_i^{(m)}(x_1, x_2) &= k_{ij}^{(0)} \sum_{r=1}^{P} \varphi^{(rm)} \frac{\partial}{\partial x_j}(\rho^{(r)}(x_1, x_2)), \\ \phi_i^{(m)}(x_1, x_2) &= k_{ij}^{(1)}(x_1, x_2) \sum_{r=1}^{P} \varphi^{(rm)} \frac{\partial}{\partial x_j}(\rho^{(r)}(x_1, x_2)). \end{aligned} \quad (25)$$

Furthermore, if we approximate the left hand side of (24) by

$$\begin{aligned} & k_{ij}^{(0)} \frac{\partial}{\partial x_j}(V(x_1, x_2)) + k_{ij}^{(1)}(x_1, x_2) \frac{\partial}{\partial x_j}(T(x_1, x_2) \\ \simeq & \sum_{r=1}^{P} b_i^{(r)} \rho^{(r)}(x_1, x_2), \end{aligned} \quad (26)$$

and collocate (24) at $(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1, 2, \cdots, P$, we obtain

$$\sum_{r=1}^{P} b_i^{(r)} \rho^{(r)}(\xi_1^{(n)}, \xi_2^{(n)}) = \sum_{m=1}^{P} (\theta_i^{(nm)} V^{(m)} + \phi_i^{(nm)} T^{(m)}) \text{ for } n = 1, 2, \cdots, P, \quad (27)$$

where $\theta_i^{(nm)} = \theta_i^{(m)}(\xi_1^{(n)}, \xi_2^{(n)})$ and $\phi_i^{(nm)} = \phi_i^{(m)}(\xi_1^{(n)}, \xi_2^{(n)})$.

The equations in (27) may be inverted to obtain

$$b_i^{(r)} = \sum_{m=1}^{P} (\gamma_i^{(rm)} V^{(m)} + \eta_i^{(rm)} T^{(m)}), \quad (28)$$

where

$$\begin{aligned} \gamma_i^{(rm)} &= \sum_{n=1}^{P} \theta_i^{(nm)} \varphi^{(rn)}, \\ \eta_i^{(rm)} &= \sum_{n=1}^{P} \phi_i^{(nm)} \varphi^{(rn)}. \end{aligned} \quad (29)$$

From (26), the partial differential equation in (12) may be approximately written as

$$\sum_{r=1}^{P} b_i^{(r)} \frac{\partial}{\partial x_i}(\rho^{(r)}(x_1, x_2)) = -Q(x_1, x_2). \tag{30}$$

If we substitute (28) into (30) and collocate the equation at $(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1, 2, \cdots, P$, we obtain

$$\sum_{m=1}^{P} (\mu^{(nm)} V^{(m)} + \omega^{(nm)} T^{(m)}) = -Q(\xi_1^{(n)}, \xi_2^{(n)}) \text{ for } n = 1, 2, \cdots, P, \tag{31}$$

where

$$\begin{aligned} \mu^{(nm)} &= \sum_{r=1}^{P} \gamma_i^{(rm)} \frac{\partial}{\partial x_i}(\rho^{(r)}(x_1, x_2))\bigg|_{(x_1,x_2)=(\xi_1^{(n)},\xi_2^{(n)})}, \\ \omega^{(nm)} &= \sum_{r=1}^{P} \eta_i^{(rm)} \frac{\partial}{\partial x_i}(\rho^{(r)}(x_1, x_2))\bigg|_{(x_1,x_2)=(\xi_1^{(n)},\xi_2^{(n)})}. \end{aligned} \tag{32}$$

The linear algebraic equations in (31) with unknowns $T^{(m)}$ and $V^{(m)}$ ($m = 1, 2, \cdots, P$) may be regarded as a radial basis function approximation of the partial differential equation (12) in $\mathcal{R} \cup \mathcal{C}$.

4.2 Meshless approximation of (15)

If we make the approximations

$$\begin{aligned} u_k(x_1, x_2) &\simeq \sum_{s=1}^{P} \sum_{m=1}^{P} \varphi^{(sm)} u_k^{(m)} \rho^{(s)}(x_1, x_2), \\ v_k(x_1, x_2) &\simeq \sum_{s=1}^{P} \sum_{m=1}^{P} \varphi^{(sm)} v_k^{(m)} \rho^{(s)}(x_1, x_2), \end{aligned} \tag{33}$$

and

$$\begin{aligned} & c_{ijkr}^{(0)} \frac{\partial}{\partial x_r}(v_k(x_1, x_2)) + c_{ijkr}^{(1)}(x_1, x_2) \frac{\partial}{\partial x_r}(u_k(x_1, x_2)) - \beta_{ij} T(x_1, x_2) \\ \simeq & \sum_{s=1}^{P} d_{ij}^{(s)} \rho^{(s)}(x_1, x_2), \end{aligned} \tag{34}$$

where $u_k^{(m)} = u_k(\xi_1^{(m)}, \xi_2^{(m)})$, $v_k^{(m)} = v_k(\xi_1^{(m)}, \xi_2^{(m)})$ and $d_{ij}^{(s)}$ are constant coefficients, we may substitute (33) into the left hand side of (34) and collocate the resulting equations at $(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1, 2, \cdots, P$ to obtain

$$\sum_{s=1}^{P} d_{ij}^{(s)} \rho^{(s)}(\xi_1^{(n)}, \xi_2^{(n)}) = \sum_{m=1}^{P} (A_{ijk}^{(nm)} v_k^{(m)} + B_{ijk}^{(nm)} u_k^{(m)}) - \beta_{ij}(\xi_1^{(n)}, \xi_2^{(n)}) T^{(n)}, \quad (35)$$

with $A_{ijk}^{(nm)}$ and $B_{ijk}^{(nm)}$ given by

$$\begin{aligned} A_{ijk}^{(nm)} &= c_{ijkr}^{(0)} \sum_{s=1}^{P} \varphi^{(sm)} \left. \frac{\partial}{\partial x_r}(\rho^{(s)}(x_1, x_2)) \right|_{(x_1,x_2)=(\xi_1^{(n)},\xi_2^{(n)})}, \\ B_{ijk}^{(nm)} &= c_{ijkr}^{(1)}(\xi_1^{(n)}, \xi_2^{(n)}) \sum_{s=1}^{P} \varphi^{(sm)} \left. \frac{\partial}{\partial x_r}(\rho^{(s)}(x_1, x_2)) \right|_{(x_1,x_2)=(\xi_1^{(n)},\xi_2^{(n)})}. \end{aligned} \quad (36)$$

We may invert (35) to obtain

$$d_{ij}^{(s)} = \sum_{m=1}^{P} (D_{ijk}^{(sm)} v_k^{(m)} + E_{ijk}^{(sm)} u_k^{(m)}) - \sum_{m=1}^{P} \varphi^{(sm)} \beta_{ij}(\xi_1^{(m)}, \xi_2^{(m)}) T^{(m)}, \quad (37)$$

where

$$\begin{aligned} D_{ijk}^{(sm)} &= \sum_{n=1}^{P} \varphi^{(sn)} A_{ijk}^{(nm)}, \\ E_{ijk}^{(sm)} &= \sum_{n=1}^{P} \varphi^{(sn)} B_{ijk}^{(nm)}. \end{aligned} \quad (38)$$

From (34) and (37), we find that (15) may be collocated at $(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1, 2, \cdots, P$ to give

$$\sum_{m=1}^{P} (H_{ik}^{(nm)} v_k^{(m)} + I_{ik}^{(nm)} u_k^{(m)}) = \sum_{m=1}^{P} Z_i^{(nm)} T^{(m)} - f_i^{(n)} \text{ for } n = 1, 2, \cdots, P, \quad (39)$$

where $f_i^{(n)} = f_i(\xi_1^{(n)}, \xi_2^{(n)})$ and

$$\begin{aligned} H_{ik}^{(nm)} &= \sum_{s=1}^{P} D_{ijk}^{(sm)} \frac{\partial}{\partial x_j}(\rho^{(s)}(x_1, x_2))\Big|_{(x_1,x_2)=(\xi_1^{(n)},\xi_2^{(n)})}, \\ I_{ik}^{(nm)} &= \sum_{s=1}^{P} E_{ijk}^{(sm)} \frac{\partial}{\partial x_j}(\rho^{(s)}(x_1, x_2))\Big|_{(x_1,x_2)=(\xi_1^{(n)},\xi_2^{(n)})}, \\ Z_i^{(nm)} &= \beta_{ij}(\xi_1^{(m)}, \xi_2^{(m)}) \sum_{s=1}^{P} \varphi^{(sm)} \frac{\partial}{\partial x_j}(\rho^{(s)}(x_1, x_2))\Big|_{(x_1,x_2)=(\xi_1^{(n)},\xi_2^{(n)})}. \end{aligned} \quad (40)$$

The linear algebraic equations in (39) with unknowns $u_k^{(m)}$, $v_k^{(m)}$ and $T^{(m)}$ ($m = 1, 2, \cdots, P$) may be regarded as a radial basis function approximation of the partial differential equation (15) in $\mathcal{R} \cup \mathcal{C}$.

5 Boundary integral approximations

Boundary integral equations which can be discretized to obtain linear algebraic equations for approximating the solutions of the homogeneous elliptic partial differential equations in (13) and (16) are given in Clements [12].

5.1 Approximation of the boundary integral solution of (13)

The boundary integral solution of (13) is given by

$$\begin{aligned} &\lambda(\xi_1, \xi_2) W(\xi_1, \xi_2) \\ &= \int_{\mathcal{C}} (\Gamma(x_1, x_2, \xi_1, \xi_2) W(x_1, x_2) \\ &- \Phi(x_1, x_2, \xi_1, \xi_2) k_{ij}^{(0)} n_i(x_1, x_2) \frac{\partial}{\partial x_j}(W(x_1, x_2))) ds(x_1, x_2), \end{aligned} \quad (41)$$

where $\lambda(\xi_1, \xi_2)$ is such that $\lambda(\xi_1, \xi_2) = 1$ if (ξ_1, ξ_2) lies in the interior of the solution domain $\mathcal{R}$ bounded by the curve $\mathcal{C}$ and $\lambda(\xi_1, \xi_2) = 1/2$ if (ξ_1, ξ_2) lies on a smooth part of the curve $\mathcal{C}$, and

$$\Phi(x_1,x_2,\xi_1,\xi_2) = \frac{1}{2\pi\sqrt{k_{11}^{(0)}k_{22}^{(0)}-(k_{12}^{(0)})^2}}\,\mathrm{Re}\{\ln(x_1-\xi_1+\tau[x_2-\xi_2])\},$$
$$\Gamma(x_1,x_2,\xi_1,\xi_2) = \frac{1}{2\pi\sqrt{k_{11}^{(0)}k_{22}^{(0)}-(k_{12}^{(0)})^2}}\,\mathrm{Re}\left\{\frac{L(x_1,x_2)}{(x_1-\xi_1+\tau[x_2-\xi_2])}\right\},$$
$$L(x_1,x_2) = (k_{11}^{(0)}+\tau k_{12}^{(0)})n_1(x_1,x_2)+(k_{21}^{(0)}+\tau k_{22}^{(0)})n_2(x_1,x_2),$$
$$\tau = \frac{-k_{12}^{(0)}+i\sqrt{k_{11}^{(0)}k_{22}^{(0)}-(k_{12}^{(0)})^2}}{k_{22}^{(0)}} \quad (i=\sqrt{-1}), \qquad (42)$$

where Re denotes the real part of a complex number.

From (11), we rewrite (41) as

$$\begin{aligned}&\lambda(\xi_1,\xi_2)(T(\xi_1,\xi_2)-V(\xi_1,\xi_2))\\ &=\int_{\mathcal{C}}(\Gamma(x_1,x_2,\xi_1,\xi_2)(T(x_1,x_2)-V(x_1,x_2))\\ &-\Phi(x_1,x_2,\xi_1,\xi_2)(S(x_1,x_2)-U(x_1,x_2)))ds(x_1,x_2),\end{aligned} \qquad (43)$$

where

$$\begin{aligned}S(x_1,x_2) &= k_{ij}^{(0)}n_i(x_1,x_2)\frac{\partial}{\partial x_j}(T(x_1,x_2)),\\ U(x_1,x_2) &= k_{ij}^{(0)}n_i(x_1,x_2)\frac{\partial}{\partial x_j}(V(x_1,x_2)).\end{aligned} \qquad (44)$$

To approximate (43) as linear algebraic equations, we discretize the boundary $\mathcal{C}$ into M straight line elements denoted by $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(M-1)}$ and $C^{(M)}$. The midpoint of $C^{(m)}$ is denoted by $(\xi_1^{(m)},\xi_2^{(m)})$. We take the first M collocation points for the radial basis function approximations in Section 4 to be the midpoints of the M straight line elements. The remaining collocation points are in the interior of the solution domain of the boundary value problem under consideration. If the number of interior collocation points is N, then the integer P in Section 4 is given by $M+N$.

We make the approximations

$$C \simeq C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(M-1)} \cup C^{(M)}, \tag{45}$$

and

$$\left.\begin{array}{r} T(x_1,x_2) \simeq T^{(m)} \\ V(x_1,x_2) \simeq V^{(m)} \\ S(x_1,x_2) \simeq S^{(m)} \\ U(x_1,x_2) \simeq U^{(m)} \end{array}\right\} \text{ for } (x_1,x_2) \in C^{(m)}, \tag{46}$$

where $S^{(m)} = S(\xi_1^{(m)}, \xi_2^{(m)})$ and $U^{(m)} = U(\xi_1^{(m)}, \xi_2^{(m)})$ for $m = 1, 2, \cdots, M$. Note that $T^{(n)}$ and $V^{(n)}$ are respectively the values of $T(x_1, x_2)$ and $V(x_1, x_2)$ at the n-th collocation point as defined in Section 4.

Substituting (45) and (46) into (43) and collocating the resulting equation at $(x_1, x_2) = (\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1, 2, \cdots, M+N$, we obtain the linear algebraic equations

$$\begin{aligned} &\lambda(\xi_1^{(n)}, \xi_2^{(n)})(T^{(n)} - V^{(n)}) \\ &= \sum_{m=1}^{M} (T^{(m)} - V^{(m)}) \int\limits_{C^{(m)}} \Gamma(x_1, x_2, \xi_1^{(n)}, \xi_2^{(n)}) ds(x_1, x_2) \\ &- \sum_{m=1}^{M} (S^{(m)} - U^{(m)}) \int\limits_{C^{(m)}} \Phi(x_1, x_2, \xi_1^{(n)}, \xi_2^{(n)}) ds(x_1, x_2) \\ &\text{for } n = 1, 2, \cdots, M+N. \end{aligned} \tag{47}$$

Note that $\lambda(\xi_1^{(n)}, \xi_2^{(n)}) = 1/2$ for $n = 1, 2, \cdots, M$ and $\lambda(\xi_1^{(n)}, \xi_2^{(n)}) = 1$ for $n = M+1, M+2, \cdots, M+N$. The integrals over $C^{(m)}$ can be readily evaluated by using a numerical integration formula (see Clements [12]). Alternatively, analytical formulae may also be derived for the integrals as in Ang [1].

As explained in Section 6 below, the linear algebraic equations in (47) containing the unknowns $T^{(n)}$, $V^{(n)}$, $S^{(m)}$ and $U^{(m)}$ ($n = 1, 2, \cdots, M+N$ and $m = 1, 2, \cdots, M$) can be used together with the boundary conditions in (4) and the linear algebraic equations in (31) to solve approximately the first boundary value problem in Section 2.

5.2 Approximation of the boundary integral solution of (16)

The boundary integral equations for (16) are given by

$$\lambda(\xi_1,\xi_2)w_k(\xi_1,\xi_2) = \int\limits_{C} (w_r(x_1,x_2)\Gamma_{rk}(x_1,x_2;\xi_1,\xi_2) - q_r(x_1,x_2)\Phi_{rk}(x_1,x_2;\xi_1,\xi_2))ds(x_1,x_2), \quad (48)$$

where the functions $q_r(x_1,x_2)$ are defined by

$$q_r(x_1,x_2) = c^{(0)}_{rjkp} n_j(x_1,x_2)\frac{\partial w_k}{\partial x_p}, \quad (49)$$

and $\Phi_{rk}(x_1,x_2;\xi_1,\xi_2)$ and $\Gamma_{rk}(x_1,x_2;\xi_1,\xi_2)$ are given by

$$\Phi_{rk}(x_1,x_2;\xi_1,\xi_2) = \frac{1}{2\pi}\operatorname{Re}\{\sum_{\alpha=1}^{2} A_{r\alpha}N_{\alpha p}\ln([x_1-\xi_1]+\tau_\alpha[x_2-\xi_2])\}D_{pk},$$
$$\Gamma_{rk}(x_1,x_2;\xi_1,\xi_2) = \frac{1}{2\pi}\operatorname{Re}\{\sum_{\alpha=1}^{2} \frac{n_j(x_1,x_2)L_{rj\alpha}N_{\alpha p}}{[x_1-\xi_1]+\tau_\alpha[x_2-\xi_2]}\}D_{pk}, \quad (50)$$

with $A_{r\alpha}$, $N_{\alpha p}$, τ_α, D_{pk} and $L_{rj\alpha}$ being complex constants related to $c^{(0)}_{ijkp}$.

As detailed in Ang [2] and Clements [12], τ_1 and τ_2 are distinct complex numbers having positive imaginary parts and satisfying the quartic equation in τ given by

$$\det(c^{(0)}_{i1r1} + (c^{(0)}_{i1r2} + c^{(0)}_{i2r1})\tau + c^{(0)}_{i2r2}\tau^2) = 0, \quad (51)$$

$A_{r\alpha}$ are non-trivial solutions of the homogeneous systems

$$(c^{(0)}_{i1r1} + (c^{(0)}_{i1r2} + c^{(0)}_{i2r1})\tau_a + c^{(0)}_{i2r2}\tau_\alpha^2)A_{r\alpha} = 0, \quad (52)$$

$L_{rj\alpha}$ is defined by

$$L_{rj\alpha} = (c^{(0)}_{rjk1} + \tau_\alpha c^{(0)}_{rjk2}) A_{k\alpha}, \tag{53}$$

and $N_{\alpha p}$ and D_{pk} by

$$\sum_{\alpha=1}^{2} A_{i\alpha} N_{ap} = \delta_{ip} \text{ and } \sum_{\alpha=1}^{2} \mathrm{Im}\{L_{i2\alpha} N_{ap}\} D_{rp} = \delta_{ip}, \tag{54}$$

where δ_{ip} is the Kronecker-delta.

From (14), the boundary integral equations in (48) can be rewritten as

$$\begin{aligned}
&\lambda(\xi_1,\xi_2)(u_k(\xi_1,\xi_2) - v_k(\xi_1,\xi_2)) \\
&= \int\limits_{\mathcal{C}} ((u_r(x_1,x_2) - v_r(x_1,x_2))\Gamma_{rk}(x_1,x_2;\xi_1,\xi_2) \\
&- (s_r(x_1,x_2) - z_r(x_1,x_2))\Phi_{rk}(x_1,x_2;\xi_1,\xi_2))ds(x_1,x_2),
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
s_r(x_1,x_2) &= c^{(0)}_{rjkp} n_j(x_1,x_2)\frac{\partial u_k}{\partial x_p}, \\
z_r(x_1,x_2) &= c^{(0)}_{rjkp} n_j(x_1,x_2)\frac{\partial v_k}{\partial x_p}.
\end{aligned} \tag{56}$$

Proceeding as in the previous subsection, we discretize (48) and collocate the resulting equations at $(x_1,x_2) = (\xi_1^{(n)}, \xi_2^{(n)})$ for $n = 1, 2, \cdots, M+N$ to obtain

$$\begin{aligned}
&\lambda(\xi_1^{(n)},\xi_2^{(n)})(u_k^{(n)} - v_k^{(n)}) \\
&= \sum_{m=1}^{M} (u_r^{(m)} - v_r^{(m)}) \int\limits_{C^{(m)}} \Gamma_{rk}(x_1,x_2,\xi_1^{(n)},\xi_2^{(n)})ds(x_1,x_2) \\
&- \sum_{m=1}^{M} (s_r^{(m)} - z_r^{(m)}) \int\limits_{C^{(m)}} \Phi_{rk}(x_1,x_2,\xi_1^{(n)},\xi_2^{(n)})ds(x_1,x_2) \\
&\text{for } n = 1, 2, \cdots, M+N,
\end{aligned} \tag{57}$$

where $s_r(\xi_1^{(m)}, \xi_2^{(m)}) \simeq s_r^{(m)}$ and $z_r(\xi_1^{(m)}, \xi_2^{(m)}) \simeq z_r^{(m)}$ $(m = 1, 2, \cdots, M)$.

As outlined in Section 6, the linear algebraic equations in (57) containing the unknowns $u_r^{(n)}$, $v_r^{(n)}$, $s_r^{(m)}$ and $z_r^{(m)}$ ($n = 1, 2, \cdots, M + N$ and $m = 1, 2, \cdots, M$) can be used together with the boundary conditions in (7) and the linear algebraic equations in (39) to solve approximately the second boundary value problem in Section 2.

6 Numerical procedures

Numerical procedures based on boundary integral equations and radial basis function approximations given above are outlined here for solving numerically the boundary value problems in Section 2.

6.1 Thermal problem

From the boundary conditions in (4), we obtain

$$T^{(m)} = \Omega(\xi_1^{(m)}, \xi_2^{(m)}) \text{ if } T \text{ is specified on } C^{(m)}, \tag{58}$$

or

$$S^{(m)} + \sum_{q=1}^{M+N} Y^{(mq)} T^{(q)} = \Psi(\xi_1^{(m)}, \xi_2^{(m)}) \text{ if } F \text{ is specified on } C^{(m)}, \tag{59}$$

with the constant coefficients $Y^{(mq)}$ defined by

$$Y^{(mq)} = k_{ij}^{(1)}(\xi_1^{(m)}, \xi_2^{(m)}) n_i^{(m)} \sum_{r=1}^{M+N} \varphi^{(rq)} \frac{\partial}{\partial x_j}(\rho^{(r)}(x_1, x_2))\bigg|_{(x_1,x_2)=(\xi_1^{(m)},\xi_2^{(m)})}, \tag{60}$$

where $n_i^{(m)}$ is the x_i component of the unit vector that is normal to $C^{(m)}$ and that points out of the solution domain. Note that (59) is derived using (23) and (44).

Another M linear algebraic equations may be derived by using (23) and (44). They are given by

$$U^{(m)} - \sum_{q=1}^{M+N} R^{(mq)} V^{(q)} = 0 \text{ for } m = 1, 2, \cdots, M, \tag{61}$$

where

$$R^{(mq)} = k_{ij}^{(0)} n_i^{(m)} \sum_{r=1}^{M+N} \varphi^{(rq)} \frac{\partial}{\partial x_j} (\rho^{(r)}(x_1, x_2)) \bigg|_{(x_1,x_2)=(\xi_1^{(m)},\xi_2^{(m)})}. \tag{62}$$

For the approximate solution of the first boundary value problem in Section 2, we solve $4M + 2N$ linear algebraic equations given by (31) (with $P = M + N$), (47), (58), (59) and (61) for $4M + 2N$ unknowns $T^{(n)}$, $V^{(n)}$, $S^{(m)}$ and $U^{(m)}$ ($n = 1, 2, \cdots, M + N$ and $m = 1, 2, \cdots, M$).

6.2 Thermoelastostatic problem

From (7), we obtain

$$u_k^{(m)} = F_k(\xi_1^{(m)}, \xi_2^{(m)}) \text{ if } u_k \text{ is specified on } C^{(m)}, \tag{63}$$

and

$$s_k^{(m)} + \sum_{q=1}^{M+N} K_{kp}^{(mq)} u_p^{(q)} = \beta_{ki}(\xi_1^{(m)}, \xi_2^{(m)}) n_i^{(m)} T^{(m)} + G_k(\xi_1^{(m)}, \xi_2^{(m)})$$
$$\text{if } t_k \text{ is specified on } C^{(m)}, \tag{64}$$

with the constant coefficients $Y^{(mq)}$ defined by

$$K_{kp}^{(mq)} = c_{kipj}^{(1)}(\xi_1^{(m)}, \xi_2^{(m)}) n_i^{(m)} \sum_{r=1}^{M+N} \varphi^{(rq)} \frac{\partial}{\partial x_j} (\rho^{(r)}(x_1, x_2)) \bigg|_{(x_1,x_2)=(\xi_1^{(m)},\xi_2^{(m)})}, \tag{65}$$

Note that (64) is derived using (23) and (44).

Also, from (23) and (44), we obtain

$$z_k^{(m)} - \sum_{q=1}^{M+N} J_{kp}^{(mq)} v_p^{(q)} = 0 \text{ for } m = 1, 2, \cdots, M, \tag{66}$$

where

$$J_{kp}^{(mq)} = c_{kipj}^{(0)} n_i^{(m)} \sum_{r=1}^{M+N} \varphi^{(rq)} \frac{\partial}{\partial x_j}(\rho^{(r)}(x_1, x_2))\bigg|_{(x_1,x_2)=(\xi_1^{(m)},\xi_2^{(m)})}. \tag{67}$$

Assume that we have found the value of $T^{(n)}$ for $n = 1, 2, \cdots, M+N$. For the second boundary value problem in Section 2, we solve $8M + 4N$ linear algebraic equations given by (39) (with $P = M + N$), (57), (65) and (66) for $8M + 4N$ unknowns $u_k^{(n)}$, $v_k^{(n)}$, $s_k^{(m)}$ and $z_k^{(m)}$ ($n = 1, 2, \cdots, M + N$ and $m = 1, 2, \cdots, M$).

7 Specific problems

The numerical procedures outlined in Section 6 are applied here to solve some specific problems. The radial basis function in (21) is chosen to carry out the numerical calculations in all the specific problems below.

The choice of radial basis functions to use is an interesting question. Other radial basis functions such as the one in (22) and many others in Chu, Wang, Zhong and He [10], Fasshauer [16], Sarler and Vertnik [28] and references therein may also be used. However, radial basis functions like (22) contain a free parameter ε. We find that the choice of the free parameter affects the accuracy of the numerical values for some of the problems below in a significant way. The dependence of the accuracy on the free parameter is also reported in Dehghan, Abbaszadeh and Mohebbi [15]. The optimal value of the free parameter to use for obtaining accurate numerical solutions may depend on factors such as the specific problems solved, distribution of collocation points and computational precision (Ooi, Ooi and Ang [25]). We choose to use (21) as it is easy to use and it consistently delivers numerical solutions of reasonably good accuracy in all the problems below.

Problem 1. Take all the non-zero coefficients of the partial differential equations in (1) and (2) to be given by

$$
\begin{aligned}
k_{11} &= 1+x_1,\ k_{12}=k_{21}=\frac{1}{2}x_2,\ k_{22}=2-x_2,\ Q=\frac{1}{2}, \\
c_{1111} &= 1+x_1+x_2,\ c_{2222}=\frac{3}{2}+x_1+x_2,\ c_{1122}=c_{2211}=\frac{1}{2}+x_1, \\
c_{1212} &= c_{2121}=c_{1221}=c_{2112}=\frac{1}{4}+\frac{1}{2}x_2, \\
\beta_{11} &= 1+x_1,\ \beta_{12}=\beta_{21}=x_1+x_2,\ \beta_{22}=1+x_2, \\
f_1(x_1,x_2) &= \frac{3}{2}x_2+\frac{1}{3},\ f_2(x_1,x_2)=\frac{1}{2}x_1+\frac{7}{3}x_2+\frac{9}{4},
\end{aligned}
$$

the solution domain to be the square region $0 < x_1 < 1$, $0 < x_2 < 1$, and the boundary conditions as

$$
\left.
\begin{aligned}
T(x_1,0) &= 1+\frac{1}{3}x_1 \\
T(x_1,1) &= 2+\frac{1}{3}x_1
\end{aligned}
\right\} \text{ for } 0 < x_1 < 1,
$$

$$
\left.
\begin{aligned}
F(0,x_2) &= -\frac{1}{3}-\frac{1}{2}x_2 \\
F(1,x_2) &= \frac{2}{3}+\frac{1}{2}x_2
\end{aligned}
\right\} \text{ for } 0 < x_2 < 1,
$$

$$
\left.
\begin{aligned}
u_k(x_1,0) &= \frac{1}{2}\delta_{k1}x_1^2+\delta_{k2}x_1 \\
t_k(x_1,1) &= -\delta_{k1}(\frac{1}{3}x_1^2+\frac{7}{3}x_1+\frac{1}{2}) \\
&\quad +\delta_{k2}(x_1^2+\frac{1}{3}x_1-\frac{11}{4})
\end{aligned}
\right\} \text{ for } 0 < x_1 < 1,
$$

$$\left.\begin{array}{rcl} u_k(0,x_2) & = & \delta_{k1}x_2 + \frac{1}{4}\delta_{k2}x_2^2 \\ t_k(1,x_2) & = & -\delta_{k1}(\frac{1}{4}x_2 + \frac{2}{3}) \\ & & -\ \delta_{k2}(x_2^2 + \frac{4}{3}x_2 + \frac{5}{6}) \end{array}\right\} \text{ for } 0 < x_2 < 1.$$

For the numerical solution of the boundary value problem above, each side of the square solution domain is discretized into M_0 equal length elements and the interior collocation points are chosen to be well spaced out points given by $(i/(N_0+1), j/(N_0+1))$ for $i,j = 1,2,\cdots,N_0$. Thus, $M = 4M_0$ and $N = N_0^2$. We take $k_{ij}^{(0)}$ and $c_{ijkp}^{(0)}$ to be the average values of k_{ij} and c_{ijkp} respectively at all the interior collocation points, that is,

$$\begin{array}{rcl} k_{ij}^{(0)} & = & \frac{1}{N_0^2}\sum_{i=1}^{N_0}\sum_{j=1}^{N_0} k_{ij}(\frac{i}{N_0+1}, \frac{j}{N_0+1}), \\ c_{ijkp}^{(0)} & = & \frac{1}{N_0^2}\sum_{i=1}^{N_0}\sum_{j=1}^{N_0} c_{ijkp}(\frac{i}{N_0+1}, \frac{j}{N_0+1}). \end{array}$$

In Table 1, we compare the numerical values of the temperature T obtained using $(M_0, N_0) = (10, 9)$ and $(M_0, N_0) = (20, 19)$ with the analytic solution

$$T(x_1, x_2) = 1 + \frac{1}{3}x_1 + x_2,$$

at selected interior collocation points. From the average absolute error (AAE) of each set of the numerical values, it is obvious that there is a significant improvement in accuracy of the numerical results when the calculation is refined by increasing the boundary elements and interior collocation points.

Table 1. Numerical and analytic values of the temperature.

(x_1, x_2)	$M_0 = 10$ $N_0 = 9$	$M_0 = 20$ $N_0 = 19$	Analytic
$(0.20, 0.20)$	1.2657	1.2664	1.2667
$(0.20, 0.40)$	1.4670	1.4667	1.4667
$(0.20, 0.60)$	1.6688	1.6671	1.6667
$(0.20, 0.80)$	1.8698	1.8673	1.8667
$(0.40, 0.20)$	1.3335	1.3333	1.3333
$(0.40, 0.40)$	1.5342	1.5335	1.5333
$(0.40, 0.60)$	1.7349	1.7337	1.7333
$(0.40, 0.80)$	1.9349	1.9337	1.9333
$(0.60, 0.20)$	1.4002	1.3999	1.4000
$(0.60, 0.40)$	1.6010	1.6002	1.6000
$(0.60, 0.60)$	1.8021	1.8004	1.8000
$(0.60, 0.80)$	2.0021	2.0005	2.0000
$(0.80, 0.20)$	1.4659	1.4664	1.4667
$(0.80, 0.40)$	1.6672	1.6667	1.6667
$(0.80, 0.60)$	1.8694	1.8672	1.8667
$(0.80, 0.80)$	2.0718	2.0678	2.0667
AAE	1.6×10^{-3}	3.4×10^{-4}	–

In Table 2, the numerical values of the displacement u_k obtained using $(M_0, N_0) = (10, 9)$ and $(M_0, N_0) = (20, 19)$ are compared with the analytic solution

$$u_k(x_1, x_2) = (\frac{1}{2}x_1^2 + x_2)\delta_{k1} + (x_1 + \frac{1}{4}x_2^2)\delta_{k2},$$

at selected interior collocation points. The absolute error of the numerical displacement at each point is calculated using the formula

$$\text{“absolute error”} = \sqrt{(u_1^{(\text{numerical})} - u_1^{(\text{analytic})})^2 + (u_2^{(\text{numerical})} - u_2^{(\text{analytic})})^2}.$$

Table 2. Numerical and analytic values of the displacement.

(x_1, x_2)	$M_0 = 10$ $N_0 = 9$	$M_0 = 20$ $N_0 = 19$	Analytic	
$(0.20, 0.20)$	0.2207	0.2202	0.2200	u_1
	0.2128	0.2111	0.2100	u_2
$(0.20, 0.40)$	0.4217	0.4205	0.4200	u_1
	0.2460	0.2418	0.2400	u_2
$(0.20, 0.60)$	0.6237	0.6210	0.6200	u_1
	0.2980	0.2922	0.2900	u_2
$(0.20, 0.80)$	0.8264	0.8217	0.8200	u_1
	0.3698	0.3626	0.3600	u_2
$(0.40, 0.20)$	0.2791	0.2797	0.2800	u_1
	0.4098	0.4104	0.4100	u_2
$(0.40, 0.40)$	0.4802	0.4800	0.4800	u_1
	0.4426	0.4411	0.4400	u_2
$(0.40, 0.60)$	0.6836	0.6809	0.6800	u_1
	0.4950	0.4916	0.4900	u_2
$(0.40, 0.80)$	0.8899	0.8825	0.8800	u_1
	0.5664	0.5619	0.5600	u_2
$(0.60, 0.20)$	0.3780	0.3795	0.3800	u_1
	0.6077	0.6099	0.6100	u_2
$(0.60, 0.40)$	0.5789	0.5797	0.5800	u_1
	0.6383	0.6402	0.6400	u_2
$(0.60, 0.60)$	0.7831	0.7808	0.7800	u_1
	0.6891	0.6904	0.6900	u_2
$(0.60, 0.80)$	0.9929	0.9831	0.9800	u_1
	0.7603	0.7607	0.7600	u_2
$(0.80, 0.20)$	0.5181	0.5194	0.5200	u_1
	0.8061	0.8097	0.8100	u_2
$(0.80, 0.40)$	0.7194	0.7197	0.7200	u_1
	0.8329	0.8392	0.8400	u_2
$(0.80, 0.60)$	0.9232	0.9208	0.9200	u_1
	0.8803	0.8885	0.8900	u_2
$(0.80, 0.80)$	1.1337	1.1233	1.1200	u_1
	0.9491	0.9583	0.9600	u_2
AAE	6.9×10^{-3}	1.7×10^{-3}	–	

For each set of the numerical values of the displacement at the selected interior points in Table 2, we compute the average absolute error (AAE). The AAE of the numerical values for $(M_0, N_0) = (20, 19)$ is more than four times smaller than that of the numerical values for $(M_0, N_0) = (10, 9)$.

As constant elements are used to discretize the boundary integral equations, the error in the numerical calculations is expected to be $O(h)$, where h is a typical length of a boundary element (Ang [1]). To demonstrate this numerically, we take $N_0 = M_0 - 1$, calculate the absolute error of the numerical displacement at the point $(1/2, 1/2)$ for selected values of $h = 1/M_0$ ($M_0 = 5$, 10, 20 and 30) and find the line of best fit through the point plots of $-\log_{10}$(absolute error) against $-\log_{10}(h)$ in Figure 1.

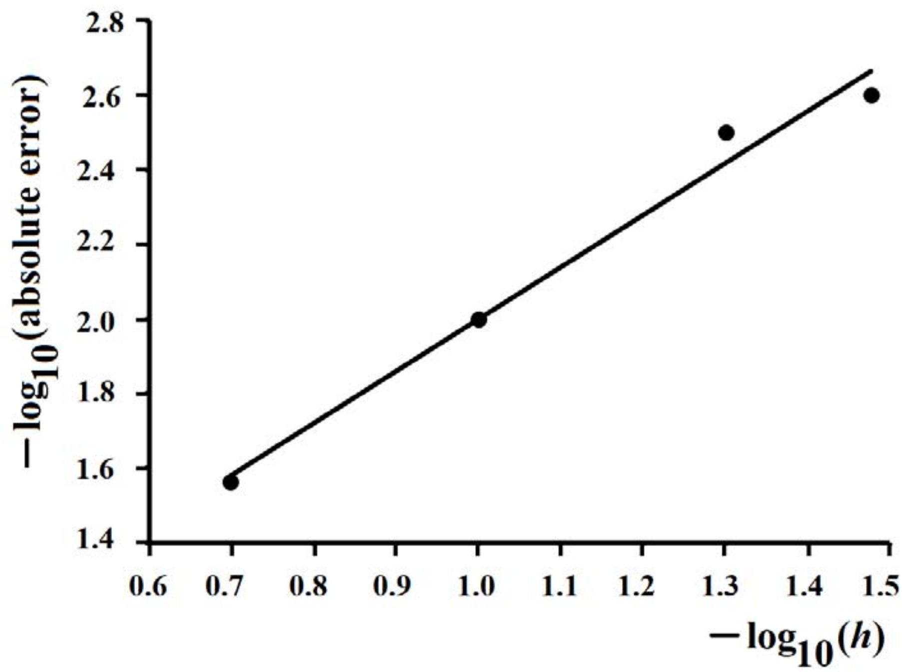

Figure 1. Line of best fit through the point plots of $-\log_{10}$(absolute error) against $-\log_{10}(h)$.

The slope of the line of best fit in Figure 1 is approximately 1.40. This seems to suggest that the absolute error of the numerical displacement at $(1/2, 1/2)$ is $O(h^{1.4})$ which is slightly better than the expected $O(h)$.

Problem 2. All the non-zero coefficients of the partial differential equations in (1) and (2) are given by

$$\begin{aligned}
k_{11} &= e^{-x_1},\ k_{22} = e^{-x_2},\ Q = -2e^{-2x_1-\frac{1}{2}x_2} - \frac{3}{4}e^{-x_1-\frac{3}{2}x_2} + e^{-x_2}, \\
c_{1111} &= \frac{9}{5}e^{x_1},\ c_{2222} = \frac{8}{5}e^{x_1},\ c_{1122} = c_{2211} = \frac{3}{5}e^{x_1}, \\
c_{1212} &= c_{2121} = c_{1221} = c_{2112} = \frac{2}{5}e^{x_1}, \\
\beta_{11} &= e^{x_1+\frac{1}{2}x_2},\ \beta_{22} = e^{\frac{1}{2}x_2},\ \beta_{12} = \beta_{21} = \frac{1}{2}e^{\frac{1}{2}x_1+\frac{1}{2}x_2}, \\
f_1(x_1, x_2) &= x_2 e^{x_1+\frac{1}{2}x_2} + \frac{1}{2}e^{\frac{1}{2}x_1+\frac{1}{2}x_2}(1 + \frac{1}{2}x_2) - \frac{2}{5}(1 + x_2), \\
f_2(x_1, x_2) &= (1 + \frac{1}{2}x_2)e^{\frac{1}{2}x_2} - \frac{1}{4}e^{-\frac{1}{2}x_1} + \frac{1}{4}x_2 e^{\frac{1}{2}x_1+\frac{1}{2}x_2} + \frac{3}{5}x_2 + \frac{8}{5}.
\end{aligned}$$

The boundary value problem here is to solve (1) and (2) in the solution domain $0 < x_1 < 1$, $0 < x_2 < 1$, subject to the boundary conditions

$$\left.\begin{aligned}
T(x_1, 0) &= \frac{1}{2}e^{-x_1} - 1 \\
T(x_1, 1) &= e^{-x_1-\frac{1}{2}} + 1
\end{aligned}\right\} \text{ for } 0 < x_1 < 1,$$

$$\left.\begin{aligned}
T(0, x_2) &= e^{-\frac{1}{2}x_2} + x_2 \\
T(1, x_2) &= e^{-1-\frac{1}{2}x_2} + x_2
\end{aligned}\right\} \text{ for } 0 < x_2 < 1,$$

$$\left.\begin{aligned}
u_k(x_1, 0) &= \delta_{k1}e^{-x_1} + \delta_{k2}e^{-x_1} \\
u_k(x_1, 1) &= \frac{3}{2}\delta_{k1}e^{-x_1} + \frac{1}{2}\delta_{k2}e^{-x_1}
\end{aligned}\right\} \text{ for } 0 < x_1 < 1,$$

$$\left.\begin{aligned}
t_k(0,x_2) &= \delta_{k1}(1 + x_2 e^{\frac{1}{2}x_2} + \frac{9}{5}(1+\frac{1}{2}x_2^2) + \frac{3}{5}x_2) \\
&\quad + \delta_{k2}(-\frac{2}{5}x_2 + \frac{2}{5}(1-\frac{1}{2}x_2^2) \\
&\quad + \frac{1}{2}(1 + x_2 e^{\frac{1}{2}x_2})) \\
t_k(1,x_2) &= \delta_{k1}(-1 - x_2 e^{1+\frac{1}{2}x_2} - \frac{9}{5}(1+\frac{1}{2}x_2^2) - \frac{3}{5}x_2) \\
&\quad + \delta_{k2}(\frac{2}{5}x_2 - \frac{2}{5}(1-\frac{1}{2}x_2^2) \\
&\quad - \frac{1}{2}e^{\frac{1}{2}+\frac{1}{2}x_2}(e^{-1-\frac{1}{2}x_2} + x_2))
\end{aligned}\right\} \text{ for } 0 < x_2 < 1.$$

For the numerical solution of the boundary value problem here, we discretize the boundary into $4M_0$ equal length elements, choose N_0^2 interior collocation points, and compute $k_{ij}^{(0)}$ and $c_{ijkp}^{(0)}$, as described in Problem 1 above. Once the numerical values of the temperature and displacements at all the collocation points (that is, $T^{(m)}$ and $u_k^{(m)}$ for $m = 1, 2, \cdots, M+N$) are obtained, we use the approximations in (23) and (33) to compute approximately the stresses σ_{ij} (as defined in (6)) at chosen points in the interior of the solution domain.

Table 3. Numerical and analytic values of the stresses.

(x_1, x_2)	Numerical	Analytic	
	-3.3644	-3.3699	σ_{11}
$(0.25, 0.25)$	-2.0772	-2.0808	σ_{22}
	-0.8862	-0.8893	σ_{12}
	-4.1549	-4.1494	σ_{11}
$(0.25, 0.50)$	-2.9137	-2.8958	σ_{22}
	-0.9521	-0.9550	σ_{12}
	-5.1594	-5.1574	σ_{11}
$(0.25, 0.75)$	-3.8374	-3.8388	σ_{22}
	-1.0440	-1.0470	σ_{12}
	-3.4554	-3.4733	σ_{11}
$(0.50, 0.25)$	-1.9014	-1.9086	σ_{22}
	-0.8572	-0.8588	σ_{12}
	-4.3779	-4.3835	σ_{11}
$(0.50, 0.50)$	-2.7375	-2.7235	σ_{22}
	-0.9484	-0.9516	σ_{12}
	-5.5407	-5.5554	σ_{11}
$(0.50, 0.75)$	-3.6595	-3.6665	σ_{22}
	-1.0748	-1.0775	σ_{12}
	-3.6051	-3.6060	σ_{11}
$(0.75, 0.25)$	-1.7696	-1.7744	σ_{22}
	-0.8346	-0.8372	σ_{12}
	-4.6992	-4.6841	σ_{11}
$(0.75, 0.50)$	-2.6114	-2.5894	σ_{22}
	-0.9564	-0.9607	σ_{12}
	-6.0667	-6.0664	σ_{11}
$(0.75, 0.75)$	-3.5271	-3.5324	σ_{22}
	-1.1206	-1.1250	σ_{12}

At selected points, Table 3 compares the numerical values of the stresses obtained using $M_0 = 40$ and $N_0 = 20$ with the values calculated from the analytic solution given by

$$\begin{aligned} T(x_1, x_2) &= e^{-x_1 - \frac{1}{2}x_2} + x_2, \\ u_k(x_1, x_2) &= \delta_{k1} e^{-x_1}(1 + \frac{1}{2}x_2^2) + \delta_{k2} e^{-x_1}(1 - \frac{1}{2}x_2^2). \end{aligned}$$

The numerical results in the table shows that the stresses can be computed numerically as described above.

Problem 3. Consider an isotropic material occupying the quarter annular region $1 < x_1^2 + x_2^2 < 4$, $x_1 > 0$, $x_2 > 0$, as sketched in Figure 2. The heat conduction coefficients and the stress-temperature coefficients of the isotropic material are taken to be given by

$$k_{ij} = \frac{\delta_{ij}}{\sqrt{x_1^2 + x_2^2}} \text{ and } \beta_{ij} = \frac{\delta_{ij}}{\sqrt{x_1^2 + x_2^2}},$$

and all the non-zero elastic moduli by

$$\begin{aligned} c_{1111} &= c_{2222} = \frac{6}{5(x_1^2 + x_2^2)^{1/2}},\ c_{1122} = c_{2211} = \frac{2}{5(x_1^2 + x_2^2)^{1/2}}, \\ c_{1212} &= c_{2121} = c_{1221} = c_{2112} = \frac{2}{5(x_1^2 + x_2^2)^{1/2}}. \end{aligned}$$

Note that the above elastic moduli correspond to those for an isotropic material with Young's modulus $E = 1/(x_1^2 + x_2^2)^{1/2}$ and Poisson ratio $\nu = 1/4$.

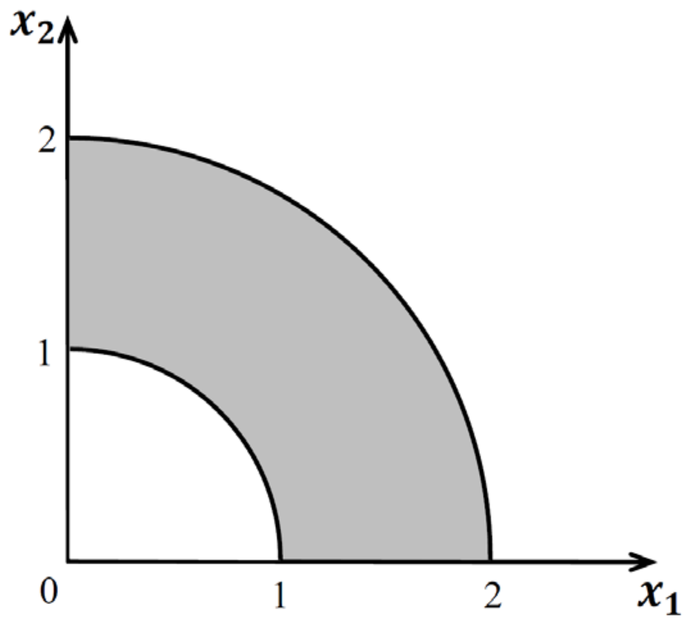

Figure 2. A sketch of the solution domain for Problem 3.

The problem of interest here is to solve (1) and (2) in the quarter annular region, with the isotropic material properties given above and with the heat source $Q = -6$ and body force f_i given by

$$\begin{aligned} f_1 &= 2x_1 + \frac{1}{10(x_1^2 + x_2^2)^2}((4\pi x_1^2 - 4\pi x_2^2 \\ &\quad -2\pi^2 x_1 x_2 (x_1^2 + x_2^2)^{1/2}) \cos(\frac{\pi}{2}(x_1^2 + x_2^2)^{1/2}) \\ &\quad +(\pi^2(3x_1^2 + x_2^2)(x_1^2 + x_2^2)^{1/2} + 8\pi x_1 x_2) \sin(\frac{\pi}{2}(x_1^2 + x_2^2)^{1/2})), \end{aligned}$$

$$
\begin{aligned}
f_2 &= 2x_2 - \frac{1}{10(x_1^2+x_2^2)^2}((4\pi x_1^2 - 4\pi x_2^2 \\
&\quad -2\pi^2 x_1 x_2 (x_1^2+x_2^2)^{1/2}) \sin(\frac{\pi}{2}(x_1^2+x_2^2)^{1/2}) \\
&\quad +(\pi^2(3x_2^2+x_1^2)(x_1^2+x_2^2)^{1/2} - 8\pi x_1 x_2) \cos(\frac{\pi}{2}(x_1^2+x_2^2)^{1/2})),
\end{aligned}
$$

subject to the boundary conditions

$$
\left.\begin{aligned}
F(s,0) &= 0 \\
F(0,s) &= 0
\end{aligned}\right\} \text{ for } 1 < s < 2,
$$

$$
\left.\begin{aligned}
T(x_1,x_2) = 1 \text{ on } x_1^2+x_2^2 = 1 \\
T(x_1,x_2) = 8 \text{ on } x_1^2+x_2^2 = 4
\end{aligned}\right\} \text{ for } x_1 > 0,\ x_2 > 0,
$$

$$
\left.\begin{aligned}
t_k(s,0) &= -\frac{\pi\delta_{k1}\sin(\frac{\pi s}{2})}{5s} + \delta_{k2}(s^2 - \frac{\pi\cos(\frac{\pi s}{2})}{5s}) \\
t_k(0,s) &= \delta_{k1}(s^2 - \frac{\pi\sin(\frac{\pi s}{2})}{5s}) - \frac{\pi\delta_{k2}\cos(\frac{\pi s}{2})}{5s}
\end{aligned}\right\} \text{ for } 1 < s < 2,
$$

$$
\left.\begin{aligned}
u_k(x_1,x_2) = \delta_{k1} \text{ on } x_1^2+x_2^2 = 1 \\
u_k(x_1,x_2) = \delta_{k2} \text{ on } x_1^2+x_2^2 = 4
\end{aligned}\right\} \text{ for } x_1 > 0,\ x_2 > 0.
$$

The elastic moduli for isotropic materials give rise to a degenerate case where the constants $N_{\alpha k}$ and D_{pk} in (50) are not well defined. Nevertheless, the functions Φ_{rk} and Γ_{rk} in (50) (hence the numerical procedure presented here) can be recovered for isotropic materials in a limiting sense by replacing $c_{1122} = c_{2211} = 2/(5(x_1^2+x_2^2)^{1/2})$ with $c_{1122} = c_{2211} = 2(1-\varepsilon)/(5(x_1^2+x_2^2)^{1/2})$ and letting ε tend to zero. For practical purposes, ε may be chosen to be a small number (say, $\varepsilon = 10^{-10}$) to carry out the numerical calculations. Further details on this may be found in Ang [2].

Table 4. Numerical and analytic values of T and u_k on $1 < x_1 < 2$, $x_2 = 0$.

(x_1, x_2)	Numerical	Analytic	
	1.1596	1.1576	T
$(1.05, 0)$	0.9941	0.9969	u_1
	0.0750	0.0785	u_2
	1.5229	1.5208	T
$(1.15, 0)$	0.9692	0.9724	u_1
	0.2298	0.2334	u_2
	1.9557	1.9531	T
$(1.25, 0)$	0.9210	0.9239	u_1
	0.3797	0.3827	u_2
	2.4632	2.4604	T
$(1.35, 0)$	0.8498	0.8526	u_1
	0.5189	0.5225	u_2
	3.0521	3.0486	T
$(1.45, 0)$	0.7579	0.7604	u_1
	0.6465	0.6494	u_2
	3.7274	3.7239	T
$(1.55, 0)$	0.6469	0.6494	u_1
	0.7569	0.7604	u_2
	4.4961	4.4921	T
$(1.65, 0)$	0.5203	0.5225	u_1
	0.8497	0.8526	u_2
	5.3633	5.3594	T
$(1.75, 0)$	0.3805	0.3827	u_1
	0.9202	0.9239	u_2
	6.3356	6.3316	T
$(1.85, 0)$	0.2321	0.2334	u_1
	0.9692	0.9724	u_2
	7.4188	7.4149	T
$(1.95, 0)$	0.0788	0.0785	u_1
	0.9932	0.9969	u_2

The temperature T and the displacement u_k are not known a priori on the side $1 < x_1 < 2$, $x_2 = 0$ of the boundary. Their values at the boundary collocation points on the side appear directly as unknowns in the linear algebraic equations in the numerical formulation here. In Table 4, the values of T and u_k at selected points on the side, which are obtained numerically using 195 boundary elements (of average length of around 0.03 units) and 400 well spaced out interior collocation points, are compared with the values from the analytic solution given by

$$\begin{aligned} T(x_1, x_2) &= (x_1^2 + x_2^2)^{3/2}, \\ u_k(x_1, x_2) &= \delta_{k1} \sin(\frac{\pi}{2}(x_1^2 + x_2^2)^{1/2}) - \delta_{k2} \cos(\frac{\pi}{2}(x_1^2 + x_2^2)^{1/2}). \end{aligned}$$

As in the first two problems, the two sets of values agree closely with each other.

Problem 4. Consider an isotropic block occupying the region $0 < x_1 < a$, $0 < x_2 < a$. The heat conduction coefficients and the stress-temperature coefficients of the block are linearly graded along the x_2 direction as given by

$$k_{ij} = \delta_{ij} k_0 (1 + c x_2) \text{ and } \beta_{ij} = \delta_{ij} \beta_0 (1 + c x_2),$$

where c, k_0 and β_0 are given positive constants. The Poisson's ratio is assumed to be constant and the Young's modulus E is also linearly graded along the x_2 direction as given by

$$E = E_0 (1 + c x_2),$$

where E_0 is a given positive constant.

The horizontal side of the square block at $x_2 = 0$ (the bottom side) is thermally insulated and perfectly attached to a rigid wall so that

$$\left.\begin{array}{rcl} F(x_1, 0) & = & 0 \\ u_k(x_1, 0) & = & 0 \end{array}\right\} \text{ for } 0 < x_1 < a.$$

The top side of the block at $x_2 = a$ has a constant temperature T_0 and is acted upon by a uniform tensile load P_0, that is,

$$\left.\begin{array}{rcl} T(x_1, \ell) & = & T_0 \\ t_k(x_1, 0) & = & \delta_{k2} P_0 \end{array}\right\} \text{ for } 0 < x_1 < a.$$

The two vertical sides of the block at $x_1 = 0$ and $x_1 = a$ have a constant temperature T_1 and are traction free, that is,

$$\left.\begin{array}{rcl} T(0, x_2) & = & T_1 \\ T(a, x_2) & = & T_1 \\ t_k(0, x_2) & = & 0 \\ t_k(a, x_2) & = & 0 \end{array}\right\} \text{ for } 0 < x_2 < a.$$

We discretize the boundary into $4M_0$ equal length elements, choose N_0^2 interior collocation points, and compute $k_{ij}^{(0)}$ and $c_{ijkp}^{(0)}$, as described in Problems 1 and 2 above. For $T_1/T_0 = 2/3$, $ca = 1$ and $T_0\beta_0/P_0 = 1$, we compute the non-dimensionalized temperature T/T_0 and displacement $E_0 u_k(P_0 a)$ by using $M_0 = 40$ and $N_0 = 30$ (160 boundary elements and 900 interior collocation points) and we compare the numerical values with the finite element method. For the finite element solution, the linearly graded block is modeled as consisting of ten horizontal layers, with each layer occupied by a homogeneous material. Specifically, the m-th layer is given by $0 < x_1 < a$, $(m-1)a/10 < x_2 < ma/10$, and the properties of the homogeneous material in the m-th layer are given by $\nu = 1/4$ and k_{ij}, β_{ij} and E given respectively by $\delta_{ij}k_0(1+cx_2)$, $\delta_{ij}\beta_0(1+cx_2)$ and $E_0(1+cx_2)$ evaluated at the midpoint of the layer. The finite element model is solved using the Abaqus commercial software with 1600 elements.

In Figure 3, the non-dimensionalized temperature T/T_0 calculated using the present method based on the boundary element method and radial basis function approximation (BEM-RBF) is compared with the values from the finite element (FEM) model at selected points on the line $x_1/a = 1/2$. The numerical values of $T(a/2, x_2)/T_0$ computed by the BEM-RBF are graphically indistinguishable from those of the FEM model.

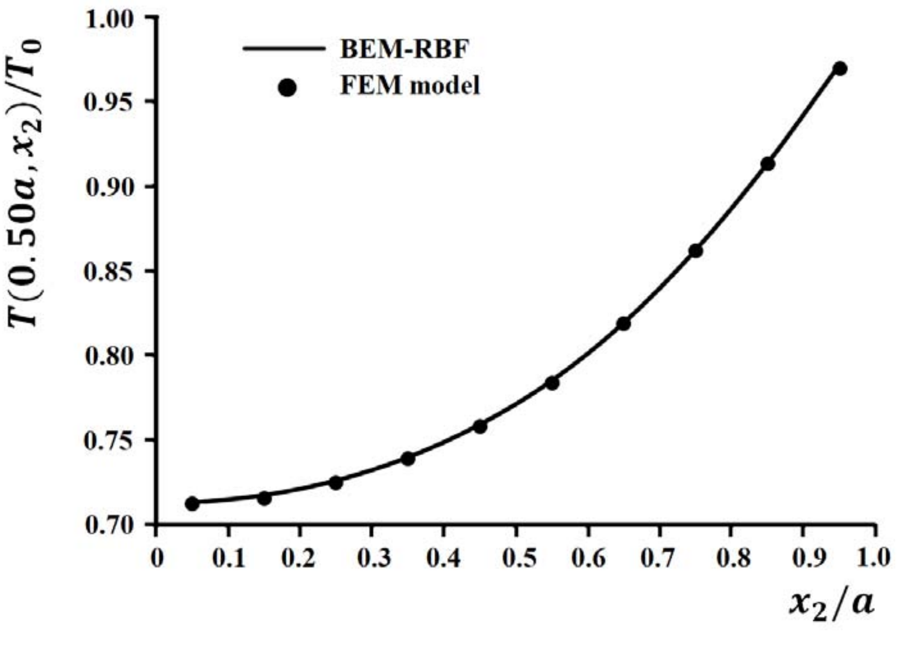

Figure 3. A graphical comparison of the numerical values of T/T_0 at selected points on $x_1/a = 1/2$.

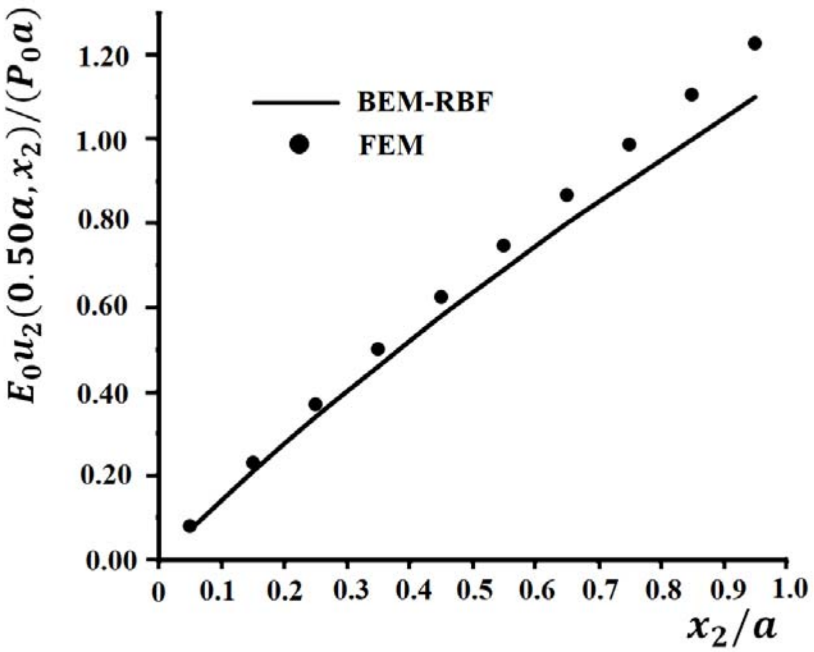

Figure 4. A graphical comparison of the numerical values of $E_0u_2(a/2, x_2)/(P_0a)$ at selected points on $x_1/a = 1/2$.

In Figure 4, we compare the non-dimensionalized displacement $E_0u_2/(P_0a)$ calculated by the BEM-RBF and FEM. (By symmetry, one can see that $u_1 = 0$ on $x_1/a = 1/2$. Numerical calculations from both BEM-RBF and FEM give extremely small $E_0u_1/(P_0a)$, such as 10^{-7}, on $x_1/a = 1/2$.) The two sets of numerical values for $E_0u_2(a/2, x_2)/(P_0a)$ in Figure 4 are quite close to each other. Nevertheless, unlike the comparison for T/T_1 in Figure 3, they show a noticeable difference which is more pronounced near $x_2/a = 1$.

The numerical calculation of the displacement field requires the estimation of the temperature gradient from the numerical values of the temperature. Consequently, the error in the numerical displacement (in both BEM-RBF and FEM or any numerical method in general) may become larger than that in the numerical temperature. This explains why there is a greater difference in the two sets of numerical values for the displacement (in Figure 4) than those for the temperature (in Figure 3). The difference can be reduced by refining both BEM-RBF and FEM numerical calculations and increasing the number of layers in the FEM model. We have also checked that the numerical values from the BEM-RBF calculation are reasonably close to those from the FEM model at other interior points.

8 Summary

A method based on boundary integral equations and radial basis function approximations is proposed for the numerical solution of boundary value problems governed by a system of two-dimensional thermoelastostatic equations with variable coefficients. The equations describe the thermoelastic behaviors of nonhomogeneous anisotropic materials that have properties that vary from point to point in space. The heat conduction coefficients, the elastic moduli and the stress-temperature coefficients of the materials are given by any functions that vary smoothly in space, that is, no restriction is placed on the variation of the thermoelastic properties as long as all requirements by the laws of physics are satisfied.

The proposed numerical procedure is easy to implement in the computer. Several specific problems that have closed form analytic solutions are solved using the numerical method. The numerical solutions obtained agree closely with the analytic solutions and show convergence when the numerical calculation is refined. The numerical procedure is also applied to solve a specific problem that does not have a known closed form analytic solution and the results obtained are compared with the numerical solution from a finite element model.

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