A time-stepping dual-reciprocity boundary element method for anisotropic heat diffusion subject to specification of energy

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Abstract

A two-dimensional problem which requires determining the non-steady temperature distribution in a thermally anisotropic body and a temperature control function on a certain part of the boundary (of the body) given that the total amount of heat energy in the body is known at all time is considered. It is solved numerically using a time-stepping dual-reciprocity boundary element method. Numerical results are obtained for specific test problems.

Key words: Heat diffusion, specification of energy, boundary element method, dual-reciprocity method.

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1 Introduction

The problem of finding the non-steady temperature distribution in a finite body which contains a specified amount of heat energy is considered. Mathematically, it requires the solution of a parabolic partial differential equation subject to a non-local condition in the form of a domain integral which gives the total amount of heat energy in the body. On a certain part of the boundary of the body, the temperature is expressed in terms of an unknown temperature control function to be determined. At each and every point on the remaining part of the boundary, either the temperature or the heat flux (not both) is given.

Many authors have used the finite-difference method to solve the problem, usually for a two-dimensional thermally isotropic body occupying a square region with boundary conditions that involve only the temperature, e.g. Gumel *et al* [1], Noye *et al* [2], Cannon *et al* [3] and Wang and Lin [4].

Recently, using the Laplace transformation, Ang [5] and Ang and Gumel [6] applied the boundary element method to solve numerically the problem for two- and three-dimensional thermally isotropic bodies which have arbitrary shapes and rather general boundary conditions.

In the present paper, a time-stepping boundary element method is proposed for solving the problem numerically for a two-dimensional thermally anisotropic body. Instead of using the modified Bessel function as a fundamental solution for the boundary element method (as in Ang [5]), a simpler fundamental solution in the form of a logarithmic function (i.e. one for an elliptic partial differential equation) is used here.

The choice of a simpler fundamental solution immediately gives rise to an undesirable feature: the presence of a domain integral (over the region occupied by the thermally anisotropic body) with an unknown integrand in the integral formulation of the problem. However, the domain integral can be easily approximated as a boundary integral by applying the dual-reciprocity method. It is also possible to re-express the non-local condition in terms of a boundary integral with an integrand containing the heat flux. Thus, in the proposed dual-reciprocity boundary element method, even though a simpler fundamental solution is used, it is not necessary to discretize the region occupied by the thermally anisotropic body into small cells. Only the boundary of the body needs to be approximated using boundary elements. The method is applied to solve specific test problems.

The dual-reciprocity boundary element method was originally introduced by Brebbia and Nardini [7] and Partridge and Brebbia [8] for the numerical solution of dynamic problems in solid mechanics. The method has now been successfully extended to a wide range of heat diffusion problems in engineering. For some examples of those problems, refer to Zhu *et al* [9], Profit *et al* [10], Ang [11], Ang *et al* [12] and other relevant references therein.

2 Statement of the problem

With reference to a Cartesian frame denoted by $0x_1x_2x_3$, consider a thermally anisotropic body with a geometry that does not vary along the x_3-axis. On the $0x_1x_2$ plane, the body occupies the region R bounded by a simple closed curve C.

The temperature in the body is assumed to be independent of x_3. The thermal behaviour of the body is then governed by the parabolic partial differential equation

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\lambda_{ij}\frac{\partial^2 T}{\partial x_i \partial x_j} = \rho c\frac{\partial T}{\partial t}, \tag{1}$$

where λ_{ij} are the thermal conductivity coefficients satisfying the symmetry relation $\lambda_{ij} = \lambda_{ji}$ and the strict inequality $\lambda_{12}^2 - \lambda_{11}\lambda_{22} < 0$, $T(x_1, x_2, t)$ is the temperature at the point (x_1, x_2) at time $t \geq 0$ and ρ and c are respectively the density and the specific heat capacity of the body.

In the present article, λ_{ij}, ρ and c are taken to be constant in R.

The total amount of heat energy in R (per unit length along the x_3-axis) at any time $t \geq 0$ is given by the double integral

$$\rho c \iint_R [T(x_1, x_2, t) - T_0]\, dx_1 dx_2, \tag{2}$$

where T_0 is a given reference temperature which depends on the scale used to measure the temperature. For example, if the temperature is in degree celcius then $T_0 \simeq -273.13^{\mathrm{o}}\mathrm{C}$.

A thermal problem of some practical interest is to determine the temperature distribution in R and a temperature control function on a certain part of C given that the total heat energy in R as given by the double integral in (2) is specified at all time $t \geq 0$. More specifically, the problem requires the solution of (1) in R subject to the initial-boundary conditions

$$T(x_1, x_2, 0) = f(x_1, x_2) \text{ for } (x_1, x_2) \in R, \tag{3}$$

$$\begin{aligned} T(x_1, x_2, t) &= k(x_1, x_2)q(t) \text{ for } (x_1, x_2) \in C_1 \text{ and } t \geq 0, & (4) \\ T(x_1, x_2, t) &= g(x_1, x_2, t) \text{ for } (x_1, x_2) \in C_2 \text{ and } t \geq 0, & (5) \\ H(x_1, x_2, t) &= v(x_1, x_2, t) \text{ for } (x_1, x_2) \in C_3 \text{ and } t \geq 0, & (6) \end{aligned}$$

and the non-local condition

$$\rho c \iint_R [T(x_1, x_2, t) - T_0] dx_1 dx_2 = E(t) \text{ for } t \geq 0, \tag{7}$$

where f, g, k, v and E are given functions assumed to be suitably prescribed in such a way that the initial condition (3) is compatible with the boundary and non-local conditions (4)-(7) at $t = 0$, $q(t)$ is an unknown (control) function to be determined, C_1, C_2 and C_3 are non-intersecting curves such that $C_1 \cup C_2 \cup C_3 = C$ and $H(x_1, x_2, t)$ is the heat flux defined by

$$H(x_1, x_2, t) = -\sum_{i=1}^{2} \sum_{j=1}^{2} \lambda_{ij} n_i(x_1, x_2) \frac{\partial T}{\partial x_j}, \tag{8}$$

with $[n_1(x_1, x_2), n_2(x_1, x_2)]$ being the unit normal outward vector to R at the point (x_1, x_2) on C.

It is clear that (7) gives the total amount of heat energy present in R at any time $t \geq 0$. Condition (4) implies that the temperature is not completely known on C_1 and the control function $q(t)$ must be chosen in such a way that (7) is satisfied.

3 Integral formulation

The partial differential equation (1) may be used to derive the integral equation

$$\begin{aligned}\gamma(\xi_1, \xi_2)T(\xi_1, \xi_2, t) = \rho c \iint\limits_R \Phi(x_1, x_2, \xi_1, \xi_2)\frac{\partial}{\partial t}[T(x_1, x_2, t)]dx_1 dx_2 \\ + \int\limits_C [T(x_1, x_2, t)\Gamma(x_1, x_2, \xi_1, \xi_2) \\ + \Phi(x_1, x_2, \xi_1, \xi_2)H(x_1, x_2, t)]ds(x_1, x_2), \quad (9)\end{aligned}$$

where $\gamma(\xi_1, \xi_2) = 0$ if $(\xi_1, \xi_2) \notin R \cup C$, $\gamma(\xi_1, \xi_2) = 1$ if $(\xi_1, \xi_2) \in R$, $0 < \gamma(\xi_1, \xi_2) < 1$ if $(\xi_1, \xi_2) \in C$ $[\gamma(\xi_1, \xi_2) = 1/2$ if (ξ_1, ξ_2) lies on a smooth part of $C]$ and

$$\begin{aligned}\Phi(x_1, x_2, \xi_1, \xi_2) &= \frac{1}{2\pi\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}}\,\mathrm{Re}\{\ln(x_1 - \xi_1 + \tau[x_2 - \xi_2])\}, \\ \Gamma(x_1, x_2, \xi_1, \xi_2) &= \frac{1}{2\pi\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}}\,\mathrm{Re}\left\{\frac{L(x_1, x_2)}{(x_1 - \xi_1 + \tau[x_2 - \xi_2])}\right\},\end{aligned}$$

$$\begin{aligned}L(x_1, x_2) &= (\lambda_{11} + \tau\lambda_{12})n_1(x_1, x_2) + (\lambda_{21} + \tau\lambda_{22})n_2(x_1, x_2), \\ \tau &= \frac{-\lambda_{12} + i\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}}{\lambda_{22}} \quad (i = \sqrt{-1}). \quad (10)\end{aligned}$$

To derive (9) from (1), one may use a reciprocal theorem given in Clements [13] for a particular system of elliptic partial differential equations. Note that τ is never real since $\lambda_{12}^2 - \lambda_{11}\lambda_{22} < 0$.

If one differentiates (7) with respect to t and applies (1) together with the Gauss-Ostrogradskii theorem, one obtains:

$$\int_C H(x_1, x_2, t) ds(x_1, x_2) = -E'(t) \text{ for } t \geq 0, \tag{11}$$

where the prime denotes differentiation with respect to the relevant argument of the function.

In view of the assumption that the initial condition (3) is compatible with the non-local (7) at $t = 0$, $E(t)$ is not discontinuous at $t = 0$ and $E(0)$ can be determined by integrating $\rho c[f(x, y) - T_0]$ over R. Thus, (7) may be replaced by (11).

In the following section, (9) is used together with (11) to derive a dual-reciprocity boundary element method for the numerical solution of the problem described in Section 2.

4 Dual-reciprocity boundary element method

For the dual-reciprocity boundary element method, the curves C_1, C_2 and C_3 are discretized into N_1, N_2 and N_3 straight line elements respectively. Let us denote those elements on C_1 by $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N_1-1)}$ and $C^{(N_1)}$, those on C_2 by $C^{(N_1+1)}$, $C^{(N_1+2)}$, $\cdots$, $C^{(N_1+N_2-1)}$ and $C^{(N_1+N_2)}$ and those on C_3 by $C^{(N_1+N_2+1)}$, $C^{(N_1+N_2+2)}$, $\cdots$, $C^{(N_1+N_2+N_3-1)}$ and $C^{(N_1+N_2+N_3)}$. The starting and ending points of the boundary element $C^{(m)}$ are given by $(a_1^{(m)}, a_2^{(m)})$ and $(b_1^{(m)}, b_2^{(m)})$ respectively. The total number of boundary elements used is therefore $N = N_1 + N_2 + N_3$.

For an accurate approximation, T and H are approximated using discontinuous linear boundary elements. Details on the implementation of such boundary elements may be found in París and Cañas [14].

For the discontinuous linear boundary elements, two points $(\eta_1^{(m)}, \eta_2^{(m)})$ and $(\eta_1^{(N+m)}, \eta_2^{(N+m)})$ on $C^{(m)}$ are chosen as follows:

$$\left.\begin{array}{r} \eta_i^{(m)} = a_i^{(m)} + r(b_i^{(m)} - a_i^{(m)}) \\ \eta_i^{(N+m)} = b_i^{(m)} - r(b_i^{(m)} - a_i^{(m)}) \end{array}\right\} \text{ for a given } r \in (0, \frac{1}{2}). \tag{12}$$

If the temperature T has values $T^{(m)}(t)$ and $T^{(N+m)}(t)$ at $(\eta_1^{(m)},\ \eta_2^{(m)})$ and $(\eta_1^{(N+m)}, \eta_2^{(N+m)})$ respectively then one makes the approximation:

$$\begin{aligned} T(x_1, x_2, t) \simeq &[1 - d^{(m)}(x_1, x_2)]T^{(m)}(t) \\ &+ d^{(m)}(x_1, x_2)T^{(N+m)}(t) \text{ for } (x_1, x_2) \in C^{(m)}, \end{aligned} \tag{13}$$

where

$$d^{(m)}(x_1, x_2) = \frac{\sqrt{(x_1 - a_1^{(m)})^2 + (x_2 - a_2^{(m)})^2} - r\ell^{(m)}}{(1 - 2r)\ell^{(m)}}. \tag{14}$$

Similarly, for the heat flux H, if its values are given by $H^{(m)}(t)$ and $H^{(N+m)}(t)$ at $(\eta_1^{(m)},\ \eta_2^{(m)})$ and $(\eta_1^{(N+m)}, \eta_2^{(N+m)})$ respectively, then

$$\begin{aligned} H(x_1, x_2, t) \simeq &[1 - d^{(m)}(x_1, x_2)]H^{(m)}(t) \\ &+ d^{(m)}(x_1, x_2)H^{(N+m)}(t) \text{ for } (x_1, x_2) \in C^{(m)}. \end{aligned} \tag{15}$$

From (13) and (15), (9) approximately becomes

$$\begin{aligned} &\gamma(\xi_1, \xi_2)T(\xi_1, \xi_2, t) \\ &= \rho c \iint\limits_R \Phi(x_1, x_2, \xi_1, \xi_2)\frac{\partial}{\partial t}[T(x_1, x_2, t)]dx_1dx_2 \\ &+ \sum_{m=1}^{N} \left\{T^{(m)}(t)\Omega_1^{(m)}(\xi_1, \xi_2) + T^{(N+m)}(t)\Omega_2^{(m)}(\xi_1, \xi_2)\right\} \\ &+ \sum_{m=1}^{N} \left\{H^{(m)}(t)\Omega_3^{(m)}(\xi_1, \xi_2) + H^{(N+m)}(t)\Omega_4^{(m)}(\xi_1, \xi_2)\right\}, \end{aligned} \tag{16}$$

where

$$\Omega_1^{(m)}(\xi_1, \xi_2) = \int\limits_{C^{(m)}} [1 - d^{(m)}(x_1, x_2)]\Gamma(x_1, x_2, \xi_1, \xi_2)ds(x_1, x_2),$$

$$\Omega_2^{(m)}(\xi_1, \xi_2) = \int\limits_{C^{(m)}} d^{(m)}(x_1, x_2)\Gamma(x_1, x_2, \xi_1, \xi_2)ds(x_1, x_2),$$

$$\Omega_3^{(m)}(\xi_1, \xi_2) = \int\limits_{C^{(m)}} [1 - d^{(m)}(x_1, x_2)]\Phi(x_1, x_2, \xi_1, \xi_2) ds(x_1, x_2),$$

$$\Omega_4^{(m)}(\xi_1, \xi_2) = \int\limits_{C^{(m)}} d^{(m)}(x_1, x_2)\Phi(x_1, x_2, \xi_1, \xi_2) ds(x_1, x_2). \tag{17}$$

The line integrals in (17) can be evaluated exactly. Analytical formulae for computing these integrals are given in the Appendix.

To deal with the domain integral in (16), P well spaced out points in the interior of R are selected. These points are denoted by $(\eta_1^{(2N+1)}, \eta_2^{(2N+1)})$, $(\eta_1^{(2N+2)}, \eta_2^{(2N+2)})$, $\cdots$, $(\eta_1^{(2N+P-1)}, \eta_2^{(2N+P-1)})$ and $(\eta_1^{(2N+P)}, \eta_2^{(2N+P)})$. One then approximates the expression $\rho c \partial T/\partial t$ in (16) using radial basis functions as follows:

$$\rho c \frac{\partial}{\partial t}[T(x_1, x_2, t)] \simeq \sum_{j=1}^{2N+P} \mu^{(j)}(t)\sigma^{(j)}(x_1, x_2), \tag{18}$$

where $\mu^{(j)}(t)$ are unknown parameters to be determined and the radial basis functions $\sigma^{(j)}(x_1, x_2)$ are given by

$$\begin{aligned}\sigma^{(j)}(x_1, x_2) = 1 &+ \left([x_1 - \eta_1^{(j)} + \text{Re}\{\tau\}\{x_2 - \eta_2^{(j)}\}]^2 + [\text{Im}\{\tau\}\{x_2 - \eta_2^{(j)}\}]^2\right) \\ &+ \left([x_1 - \eta_1^{(j)} + \text{Re}\{\tau\}\{x_2 - \eta_2^{(j)}\}]^2 + [\text{Im}\{\tau\}\{x_2 - \eta_2^{(j)}\}]^2\right)^{3/2} \\ &\text{for } j = 1, 2, \cdots, 2N + P.\end{aligned} \tag{19}$$

Note that $(\eta_1^{(1)}, \eta_2^{(1)})$, $(\eta_1^{(2)}, \eta_2^{(2)})$, $\cdots$, $(\eta_1^{(2N-1)}, \eta_2^{(2N-1)})$ and $(\eta_1^{(2N)}, \eta_2^{(2N)})$ are points on the boundary elements as defined in (12).

The radial basis functions in (19) were used by Ang *et al* [15] in formulating a dual-reciprocity boundary element method for the numerical solution of problems involving nonhomogeneous anisotropic media. For isotropic heat conduction in homogeneous bodies, the thermal conductivity coefficients are constants such that $\lambda_{11} = \lambda_{22}$ and $\lambda_{12} = \lambda_{21} = 0$ (hence $\tau = i$) and (19) can be reduced to give the radial basis functions originally proposed by Zhang and Zhu [16].

One can let (x_1, x_2) in (18) to be given by $(\eta_1^{(k)}, \eta_2^{(k)})$ for $k = 1,\ 2,\ \cdots,$ $2N + P$ to set up a system of $2N + P$ linear algebraic equations that can be inverted to give

$$\mu^{(j)}(t) = \rho c \sum_{k=1}^{2N+P} \chi^{(kj)} \frac{d}{dt}[T^{(k)}(t)] \text{ for } j = 1, 2, \cdots, 2N + P, \tag{20}$$

where $T^{(k)}(t) = T(\eta_1^{(k)}, \eta_2^{(k)}, t)$ for $k = 1, 2,\ \cdots, 2N + P$, and $\chi^{(kj)}$ are constants defined by

$$\sum_{k=1}^{2N+P} \sigma^{(j)}(\eta_1^{(k)}, \eta_2^{(k)}) \chi^{(km)} = \delta^{(jm)} \text{ for } j, m = 1, 2, \cdots, 2N + P, \tag{21}$$

$$\delta^{(jm)} = \begin{cases} 1 & \text{if} \quad j = m, \\ 0 & \text{if} \quad j \neq m. \end{cases} \tag{22}$$

With (18), the double integral in (16) can now be approximated using

$$\begin{aligned} &\rho c \iint\limits_R \Phi(x_1, x_2, \xi_1, \xi_2) \frac{\partial}{\partial t}[T(x_1, x_2, t)] dx_1 dx_2 \\ &\simeq \rho c \sum_{k=1}^{2N+P} \frac{d}{dt}[T^{(k)}(t)] \sum_{j=1}^{2N+P} \chi^{(kj)} \Psi^{(j)}(\xi_1, \xi_2) \end{aligned} \tag{23}$$

where

$$\begin{aligned} \Psi^{(j)}(\xi_1, \xi_2) = {} & \gamma(\xi_1, \xi_2)\theta^{(j)}(\xi_1, \xi_2) - \int\limits_C \Phi(x_1, x_2, \xi_1, \xi_2) \beta^{(j)}(x_1, x_2) ds(x_1, x_2) \\ & - \int\limits_C \theta^{(j)}(x_1, x_2) \Gamma(x_1, x_2, \xi_1, \xi_2) ds(x_1, x_2) \\ & \text{for } j = 1, 2, \cdots, 2N + P. \end{aligned} \tag{24}$$

$$\left(\frac{\lambda_{11}\lambda_{22}-\lambda_{12}^2}{\lambda_{22}}\right)\theta^{(j)}(x_1,x_2)$$
$$=\frac{1}{4}\left([x_1-\eta_1^{(j)}+\mathrm{Re}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2+[\mathrm{Im}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2\right)$$
$$+\frac{1}{16}\left([x_1-\eta_1^{(j)}+\mathrm{Re}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2+[\mathrm{Im}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2\right)^2$$
$$+\frac{1}{25}\left([x_1-\eta_1^{(j)}+\mathrm{Re}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2+[\mathrm{Im}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2\right)^{5/2}. \quad (25)$$

$$\beta^{(j)}(x_1,x_2)=-\sum_{i=1}^{2}\sum_{k=1}^{2}\lambda_{ik}n_i(x_1,x_2)\frac{\partial\theta^{(j)}}{\partial x_k}. \quad (26)$$

The function $\Psi^{(j)}(\xi_1,\xi_2)$ in (24) can be computed approximately using

$$\begin{aligned}\Psi^{(j)}(\xi_1,\xi_2) &\simeq \gamma(\xi_1,\xi_2)\theta^{(j)}(\xi_1,\xi_2)\\ &-\sum_{m=1}^{N}\{\beta^{(j)}(\eta_1^{(m)},\eta_2^{(m)})\Omega_3^{(m)}(\xi_1,\xi_2)\\ &-\beta^{(j)}(\eta_1^{(N+m)},\eta_2^{(N+m)})\Omega_4^{(m)}(\xi_1,\xi_2)\\ &-\theta^{(j)}(\eta_1^{(m)},\eta_2^{(m)})\Omega_1^{(m)}(\xi_1,\xi_2)\\ &-\theta^{(j)}(\eta_1^{(N+m)},\eta_2^{(N+m)})\Omega_2^{(m)}(\xi_1,\xi_2)\}. \end{aligned} \quad (27)$$

If one makes the approximation (for $k=1,\,2,\,\cdots,\,2N+P$)

$$\begin{aligned}T^{(k)}(t) &\simeq \frac{1}{2}[T^{(k)}(t+\frac{1}{2}\Delta t)+T^{(k)}(t-\frac{1}{2}\Delta t)],\\ \frac{d}{dt}[T^{(k)}(t)] &\simeq \frac{T^{(k)}(t+\frac{1}{2}\Delta t)-T^{(k)}(t-\frac{1}{2}\Delta t)}{\Delta t},\end{aligned} \quad (28)$$

then letting (ξ_1,ξ_2) in (16) and (23) be given by $(\eta_1^{(n)},\eta_2^{(n)})$ for $n=1,\,2,\,\cdots,$

$2N + P$, one finds that

$$
\begin{aligned}
&\frac{1}{2}\gamma(\eta_1^{(n)}, \eta_2^{(n)})[T^{(n)}(t + \frac{1}{2}\Delta t) + T^{(n)}(t - \frac{1}{2}\Delta t)] \\
&= \frac{\rho c}{\Delta t} \sum_{k=1}^{2N+P} [T^{(k)}(t + \frac{1}{2}\Delta t) - T^{(k)}(t - \frac{1}{2}\Delta t)] \sum_{j=1}^{2N+P} \chi^{(kj)} \Psi^{(j)}(\eta_1^{(n)}, \eta_2^{(n)}) \\
&+ \frac{1}{2}\sum_{m=1}^{N} \left\{ [T^{(m)}(t + \frac{1}{2}\Delta t) + T^{(m)}(t - \frac{1}{2}\Delta t)]\Omega_1^{(m)}(\eta_1^{(n)}, \eta_2^{(n)}) \right. \\
&\left. + [T^{(N+m)}(t + \frac{1}{2}\Delta t) + T^{(N+m)}(t - \frac{1}{2}\Delta t)]\Omega_2^{(m)}(\eta_1^{(n)}, \eta_2^{(n)}) \right\} \\
&+ \sum_{m=1}^{N} \left\{ H^{(m)}(t)\Omega_3^{(m)}(\eta_1^{(n)}, \eta_2^{(n)}) + H^{(N+m)}(t)\Omega_4^{(m)}(\eta_1^{(n)}, \eta_2^{(n)}) \right\} \\
&\qquad\qquad \text{for } n = 1, 2, \cdots, 2N + P. \qquad (29)
\end{aligned}
$$

Conditions (4), (5) and (6) provide partial information for the temperature and heat flux on the boundary elements. According to (6), the heat flux is known on $C^{(N_1+N_2+1)}$, $C^{(N_1+N_2+2)}$, $\cdots$, $C^{(N_1+N_2+N_3-1)}$ and $C^{(N_1+N_2+N_3)}$. Thus, for $m = N_1 + N_2 + 1$, $N_1 + N_2 + 2$, $\cdots$, $N_1 + N_2 + N_3$, $H^{(m)}(t)$ and $H^{(N+m)}(t)$ are known but not $T^{(m)}(t)$ and $T^{(N+m)}(t)$. From (5), $T^{(m)}(t)$ and $T^{(N+m)}(t)$ but not $H^{(m)}(t)$ and $H^{(N+m)}(t)$ are known for $m = N_1 + 1$, $N_1 + 2$, $\cdots$, $N_1 + N_2$. The functions $H^{(m)}(t)$ and $H^{(N+m)}(t)$ are also not known for $m = 1, 2, \cdots, N_1$. From (4), one can write $T^{(m)}(t) = k(\eta_1^{(m)}, \eta_2^{(m)})q(t)$ and $T^{(N+m)}(t) = k(\eta_1^{(N+m)}, \eta_2^{(N+m)})q(t)$, where $k(x_1, x_2)$ is a given function and $q(t)$ is the unknown control function to be determined, for $m = 1, 2, \cdots$, N_1. Lastly, the temperature is not known at all points in the interior of the region R. Thus, $T^{(k)}(t)$ is not known for $k = 2N + 1$, $2N + 2$, $\cdots$, $2N + P$. [Note that $N = N_1 + N_2 + N_3$.]

It is clear then that if $T^{(k)}(t - \frac{1}{2}\Delta t)$ and $q(t - \frac{1}{2}\Delta t)$ are assumed to be known then the system (29) constitutes a system of $2N + P$ linear algebraic equations with $2N + P + 1$ unknowns given by $H^{(m)}(t)$ and $H^{(N+m)}(t)$ for $m = 1, 2, \cdots, N_1 + N_2$, $T^{(k)}(t + \frac{1}{2}\Delta t)$ for $k = N_1 + N_2 + 1$, $N_1 + N_2 + 2$, $\cdots$, $2N$, $2N + 1$, $\cdots$, $2N + P$ and $q(t + \frac{1}{2}\Delta t)$. To complete the formulation, another equation is needed. This comes from (11) which can be discretized to give

$$\sum_{m=1}^{N} \ell^{(m)} \left\{ H^{(m)}(t) + H^{(N+m)}(t) \right\} = -2E'(t). \tag{30}$$

Thus, using the initial condition (3), one can solve the system given by (29) and (30) for the unknowns at consecutive time levels $t = \frac{1}{2}(2n-1)\Delta t$ for $n = 1, 2, \cdots$.

Once $T^{(k)}(t)$ are determined numerically for $k = 1, 2, \cdots, 2N+P$, the temperature at any point in the solution domain can be computed approximately using

$$T(x_1, x_2, t) \simeq \sum_{k=1}^{2N+P} T^{(k)}(t) \sum_{j=1}^{2N+P} \chi^{(kj)} \sigma^{(j)}(x_1, x_2). \tag{31}$$

5 Specific problems

Problem 1. For a specific problem, let us take the case where the body lies in the region R given by

$$R = \{(x_1, x_2) : x_1^2 + x_2^2 < 1,\ x_1 > 0,\ x_2 > 0\}. \tag{32}$$

The thermal behaviour of the body is governed by the partial differential equation

$$\frac{5}{9}\frac{\partial^2 T}{\partial x_1^2} + \frac{2}{9}\frac{\partial^2 T}{\partial x_1 \partial x_2} + \frac{2}{9}\frac{\partial^2 T}{\partial x_2^2} = \frac{\partial T}{\partial t}, \tag{33}$$

in which $\lambda_{11} = 5/9$, $\lambda_{12} = \lambda_{21} = 1/9$, $\lambda_{22} = 2/9$ and $\rho c = 1$.

The initial temperature is given by

$$T(x_1, x_2, 0) = \cos(x_1 + x_2) + \sin(\frac{3}{2}x_2)\exp(x_1 - \frac{1}{2}x_2). \tag{34}$$

and the conditions on the boundary of R are given by

$$T(0,x_2,t)=\cos(x_2)\exp(-t)+\sin(\frac{3}{2}x_2)\exp(-\frac{1}{2}x_2) \quad \text{for } 0<x_2<1 \text{ and } t>0, \tag{35}$$

$$T(x_1,0,t)=\cos(x_1)\,q(t) \text{ for } 0<x_1<1 \text{ and } t>0, \tag{36}$$

$$\begin{aligned} H(x_1,x_2,t) = & (\frac{5}{9}x_1+\frac{1}{9}x_2)[-\sin(x_1+x_2)\exp(-t) \\ & + \sin(\frac{3}{2}x_2)\exp(x_1-\frac{1}{2}x_2)] \\ & + (\frac{1}{9}x_1+\frac{2}{9}x_2)[-\sin(x_1+x_2)\exp(-t) \\ & + \frac{3}{2}\cos(\frac{3}{2}x_2)\exp(x_1-\frac{1}{2}x_2) \\ & - \frac{1}{2}\sin(\frac{3}{2}x_2)\exp(x_1-\frac{1}{2}x_2)] \\ & \text{on } x_1^2+x_2^2=1,\ x_1>0,\ x_2>0. \end{aligned} \tag{37}$$

The non-local condition is given by:

$$\iint_R T(x_1,x_2,t)dx_1dx_2=\alpha\exp(-t)+\beta \quad \text{for } t\geq 0, \tag{38}$$

where

$$\begin{aligned} \alpha &= \int_0^{\pi/2}\int_0^1 \cos(r(\cos\theta+\sin\theta))rdrd\theta \simeq 0.493\,014\,650\,9, \\ \beta &= \int_0^{\pi/2}\int_0^1 \sin(\frac{3}{2}r\sin\theta)\exp(r(\cos\theta-\frac{1}{2}\sin\theta))rdrd\theta \simeq 0.499\,472\,639. \end{aligned} \tag{39}$$

It is easy to check that the exact solution for the problem defined by (33)-(38) is given by

$$T(x_1,x_2,t)=\cos(x_1+x_2)\exp(-t)+\sin(\frac{3}{2}x_2)\exp(x_1-\frac{1}{2}x_2) \quad \text{with } q(t)=\exp(-t). \tag{40}$$

A comparison of (35)-(37) with (4)-(7) shows that

$$\begin{aligned} C_1 &= \{(x_1, x_2) : \ 0 < x_1 < 1, \ x_2 = 0\}, \\ C_2 &= \{(x_1, x_2) : \ 0 < x_2 < 1, \ x_1 = 0\}, \\ C_3 &= \{(x_1, x_2) : \ x_1^2 + x_2^2 = 1, \, x_1 > 0, \ x_2 > 0\}, \end{aligned}$$

for the test problem under consideration.

To execute the numerical procedure described in Section 4, $C_1 \cup C_2$ are discretized into $2M$ boundary elements, each of equal length $\ell_1 = 1/M$ units, and the curve C_3 into M boundary elements, each of length $\ell_2 = \sqrt{2 - 2\cos(\pi/[2M])}$ units. Thus, a total of $3M$ boundary elements are employed on the boundary of the solution domain.

The collocation points on the boundary are chosen using (12) with $r = 1/4$. For $m = 1, 2, \cdots, J$ and $n = 1, 2, \cdots, J$, the point $(y^{(m)}, y^{(n)})$, where $y^{(m)} = (2m-1)/(2J)$, is chosen to be an interior collocation point for the dual-reciprocity boundary element method, if $[y^{(m)}]^2 + [y^{(n)}]^2 < (1-\ell_2)^2$.

To obtain some numerical results, the integers J and M are chosen in such a way that $2J = M$ and the size of the time-step is taken to be $\Delta t = 2\ell_1$. Two different sets of numerical results are obtained. Details such as the size of the time-step and the number of boundary elements used for obtaining the two sets of numerical results are as follows:

$$\begin{aligned} &\text{set A}: \quad M = 10, \ J = \ \ 5, \ \ell_1 = 0.10, \ \ell_2 = 0.1569, \ \Delta t = 0.20, \\ &\text{set B}: \quad M = 20, \ J = 10, \ \ell_1 = 0.05, \ \ell_2 = 0.0785, \ \Delta t = 0.10. \end{aligned}$$

There are 13 and 67 interior collocation points in sets A and B respectively.

Table 1 gives the two sets of numerical values of the temperature control function q at selected time t. Similarly, at time $t = 1.0$, the numerical values of the temperature T at selected points in the interior of the solution domain, as obtained by using (31), are shown in Table 2. In general, the numerical values are in reasonably good agreement with the exact solution in (40). It is also obvious that set B delivers significantly more accurate results than set A in both Tables 1 and 2.

Table 1. Numerical and exact values of the temperature control function q at selected time t (Problem 1).

Time t	Set A	Set B	Exact
0.2	0.8260	0.8181	0.8187
0.4	0.6684	0.6697	0.6703
0.6	0.5552	0.5482	0.5488
0.8	0.4487	0.4488	0.4493
1.0	0.3743	0.3675	0.3679
1.2	0.3021	0.3009	0.3012
1.4	0.2535	0.2465	0.2466
1.6	0.2047	0.2019	0.2019
1.8	0.1731	0.1654	0.1653
2.0	0.1400	0.1355	0.1353

Table 2. Numerical and exact values of the temperature T at selected interior points and at $t = 1.0$ (Problem 1).

(x_1, x_2)	Set A	Set B	Exact
(0.125, 0.125)	0.5526	0.5546	0.5549
(0.125, 0.375)	0.8219	0.8235	0.8238
(0.125, 0.625)	0.9359	0.9372	0.9374
(0.125, 0.875)	0.9036	0.9055	0.9061
(0.375, 0.125)	0.5785	0.5778	0.5776
(0.375, 0.375)	0.9132	0.9125	0.9125
(0.375, 0.625)	1.0578	1.0567	1.0568
(0.375, 0.875)	1.0224	1.0239	1.0243
(0.625, 0.125)	0.5957	0.5966	0.5963
(0.625, 0.375)	1.0297	1.0249	1.0248
(0.625, 0.625)	1.2225	1.2180	1.2178
(0.875, 0.125)	0.6290	0.6201	0.6188
(0.875, 0.375)	1.1854	1.1780	1.1766

Problem 2. For another specific problem, the heat diffusion is taken to occur within a square region given by $0 < x_1 < 1$, $0 < x_2 < 1$, with the governing equation given by

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} = \frac{\partial T}{\partial t}, \tag{41}$$

that is, we take $\lambda_{11} = \lambda_{22} = 1$, $\lambda_{12} = 0$ and $\rho c = 1$ in (1).

As for the initial-boundary and non-local conditions, we are required to solve (41) within the square region $0 < x_1 < 1$, $0 < x_2 < 1$, subject to

$$\begin{aligned}
&T(x_1, x_2, 0) = \cos(\frac{\pi}{3}[x_2 - 1]) + \cos(\pi x_2) \ \text{ for } 0 < x_1 < 1,\ 0 < x_2 < 1, \\
&\left.\begin{array}{l} T(x_1, 0, t) = q(t) \\ H(x_1, 1, t) = 0 \end{array}\right\} \text{ for } 0 < x_1 < 1,\ t \geq 0, \\
&\left.\begin{array}{l} H(0, x_2, t) = 0 \\ H(1, x_2, t) = 0 \end{array}\right\} \text{ for } 0 < x_2 < 1,\ t \geq 0, \\
&\iint\limits_R T(x_1, x_2, t) dx_1 dx_2 = \frac{3\sqrt{3}}{2\pi} \exp(-\frac{\pi^2 t}{9}) \text{ for } t \geq 0.
\end{aligned} \tag{42}$$

It may be verified that the solution of (41) satisfying (42) is given by

$$\begin{aligned}
T(x_1, x_2, t) = \cos(\frac{\pi}{3}[x_2 - 1]) \exp(-\frac{\pi^2 t}{9}) + \cos(\pi x_2) \exp(-\pi^2 t) \\
\text{with } q(t) = \frac{1}{2} \exp(-\frac{\pi^2 t}{9}) + \exp(-\pi^2 t).
\end{aligned} \tag{43}$$

To apply the dual-reciprocity boundary element method to recover the unknown control function $q(t)$, each side of the square is discretized into M equal length elements, each of equal length $\ell = 1/M$. The total number of boundary elements is $4M$. As in the first example, the collocation points on the sides of the square are chosen using (12) with $r = 1/4$. The time-step is given by $\Delta t = \ell$. Taking M to be an even integer, we choose $M^2/4$ interior collocation points to be given by $([2m+1]\ell, [2n+1]\ell]$ for m, $n = 0, 1, 2, \cdots,$ $M/2 - 1$. The numerical values of $q(t)$ obtained using $M = 10$ and $M = 20$ are compared with the exact ones at selected time t in Table 3.

Table 3. Numerical and exact values of the temperature control function q at selected time t (Problem 2).

Time t	$M = 10$	$M = 20$	Exact
0.1	0.7877	0.8132	0.8208
0.2	0.5170	0.5349	0.5404
0.3	0.3994	0.4086	0.4116
0.4	0.3362	0.3403	0.3417
0.5	0.2940	0.2956	0.2961
0.6	0.2609	0.2614	0.2616
0.7	0.2332	0.2330	0.2331
0.8	0.2085	0.2084	0.2083
0.9	0.1869	0.1866	0.1865
1.0	0.1674	0.1671	0.1670

From Table 3, it is clear that we have succeeded in recovering the control function $q(t)$ with reasonably good accuracy. The numerical values obtained improve significantly when M is increased from 10 to 20.

6 Conclusion

A time-stepping dual-reciprocity boundary element method is proposed for solving numerically a parabolic partial differential that governs the anisotropic diffusion of heat in a two-dimensional body containing a specified amount of heat energy. The method reduces the problem under consideration into a system of linear algebraic equations to be solved at consecutive time levels.

The coefficients of the unknowns in the system of linear algebraic equations are independent of time. Consequently, for the solution of the linear algebraic equations, the square matrix containing the coefficients of the unknowns needs to be evaluated and processed only once, provided that the size of the time-step used is always the same. For example, if the LU decomposition technique together with backward substitutions is used to solve the linear algebraic equations, then the square matrix has to be decomposed only once.

To check its validity, the proposed method is applied to solve some specific problems with known exact solutions. The numerical results obtained are in good agreement with the exact solutions. Convergence in the numerical values is observed when the calculation is refined by increasing the number of boundary elements and collocation points and by reducing the size of the time-step.

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Appendix

To implement the dual-reciprocity boundary element method for the heat diffusion problem under consideration, one has to compute the line integrals

$$
\begin{aligned}
I_1^{(m)}(\xi_1, \xi_2) &= \int\limits_{C^{(m)}} \Phi(x_1, x_2, \xi_1, \xi_2) ds(x_1, x_2) \\
I_2^{(m)}(\xi_1, \xi_2) &= \int\limits_{C^{(m)}} d^{(m)}(x_1, x_2) \Phi(x_1, x_2, \xi_1, \xi_2) ds(x_1, x_2) \\
I_3^{(m)}(\xi_1, \xi_2) &= \int\limits_{C^{(m)}} \Gamma(x_1, x_2, \xi_1, \xi_2) ds(x_1, x_2) \\
I_4^{(m)}(\xi_1, \xi_2) &= \int\limits_{C^{(m)}} d^{(m)}(x_1, x_2) \Gamma(x_1, x_2, \xi_1, \xi_2) ds(x_1, x_2).
\end{aligned}
\tag{A1}
$$

If the boundary element $C^{(m)}$ is described by the parametric equations

$$
\left.
\begin{array}{l}
x_1 = a_1^{(m)} - t\ell^{(m)} n_2^{(m)} \\
x_2 = a_2^{(m)} + t\ell^{(m)} n_1^{(m)}
\end{array}
\right\} \text{ from } t = 0 \text{ to } t = 1,
$$

where $[n_1^{(m)}, n_2^{(m)}] = [b_2^{(m)} - a_2^{(m)}, a_1^{(m)} - b_1^{(m)}]/\ell^{(m)}$, then the line integral given by $I_1^{(m)}(\xi_1, \xi_2)$ becomes

$$
I_1^{(m)}(\xi_1, \xi_2) = \frac{\ell^{(m)}}{4\pi\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}} \int\limits_0^1 \ln[A^{(m)} t^2 + B^{(m)}(\xi_1, \xi_2) t + D^{(m)}(\xi_1, \xi_2)] dt,
$$

where

$$
\begin{aligned}
A^{(m)} &= [\ell^{(m)}]^2[(\mathrm{Re}\{\tau\}n_1^{(m)} - n_2^{(m)})^2 + (\mathrm{Im}\{\tau\}n_1^{(m)})^2] \\
B^{(m)}(\xi_1,\xi_2) &= \{[\mathrm{Re}\{\tau\}n_1^{(m)} - n_2^{(m)}](a_1^{(m)} - \xi_1 + \mathrm{Re}\{\tau\}(a_2^{(m)} - \xi_2)) \\
&\quad + (\mathrm{Im}\{\tau\})^2(a_2^{(m)} - \xi_2)n_1^{(m)}\}(2\ell^{(m)}) \\
D^{(m)}(\xi_1,\xi_2) &= [a_1^{(m)} - \xi_1 + \mathrm{Re}\{\tau\}(a_2^{(m)} - \xi_2)]^2 + (\mathrm{Im}\{\tau\})^2(a_2^{(m)} - \xi_2)^2.
\end{aligned}
$$

It can be shown that $Q^{(m)}(\xi_1,\xi_2) = 4A^{(m)}D^{(m)}(\xi_1,\xi_2) - [B^{(m)}(\xi_1,\xi_2)]^2 \geq 0$. In the evaluation of $I_1^{(m)}(\xi_1,\xi_2)$, one has to consider two separate cases, namely $Q^{(m)}(\xi_1,\xi_2) = 0$ and $Q^{(m)}(\xi_1,\xi_2) > 0$. The case $Q^{(m)}(\xi_1,\xi_2) = 0$ occurs when (ξ_1,ξ_2) is collinear with $(a_1^{(m)}, a_2^{(m)})$ and $(b_1^{(m)}, b_2^{(m)})$ (the extreme points of the boundary element $C^{(m)}$).

It follows that $I_1^{(m)}(\xi_1,\xi_2)$ is given by

$$
\begin{aligned}
I_1^{(m)}(\xi_1,\xi_2) = &\frac{\ell^{(m)}}{4\pi\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}} \\
&\times [\ln(A^{(m)}) + F^{(m)}(\xi_1,\xi_2,1) - F^{(m)}(\xi_1,\xi_2,0)] \qquad \text{(A2)}
\end{aligned}
$$

where

$$
\begin{aligned}
F^{(m)}(\xi_1,\xi_2,t) = &(t + \frac{B^{(m)}(\xi_1,\xi_2)}{2A^{(m)}})\ln(t^2 + \frac{B^{(m)}(\xi_1,\xi_2)}{A^{(m)}}t + \frac{D^{(m)}(\xi_1,\xi_2)}{A^{(m)}}) - 2t \\
&+ \frac{1}{A^{(m)}}\sqrt{Q^{(m)}(\xi_1,\xi_2)}\arctan\left(\frac{2A^{(m)}t + B^{(m)}(\xi_1,\xi_2)}{\sqrt{Q^{(m)}(\xi_1,\xi_2)}}\right) \\
&\text{if } Q^{(m)}(\xi_1,\xi_2) > 0 \qquad \text{(A3)}
\end{aligned}
$$

or

$$
\begin{aligned}
F^{(m)}(\xi_1,\xi_2,t) = 2\left(t + \frac{B^{(m)}(\xi_1,\xi_2)}{2A^{(m)}}\right)\ln\left|t + \frac{B^{(m)}(\xi_1,\xi_2)}{2A^{(m)}}\right| - 2t \\
\text{if } Q^{(m)}(\xi_1,\xi_2) = 0. \qquad \text{(A4)}
\end{aligned}
$$

In a similar manner, the second line integral in (A1) is evaluated as

$$
\begin{aligned}
I_2^{(m)}(\xi_1,\xi_2) = &-\frac{r}{(1-2r)} I_1^{(m)}(\xi_1,\xi_2) \\
&+\frac{\ell^{(m)}}{4(1-2r)\pi\sqrt{\lambda_{11}\lambda_{22}-\lambda_{12}^2}} \\
&\times [\frac{1}{2}\ln(A^{(m)}) + G^{(m)}(\xi_1,\xi_2,1) - G^{(m)}(\xi_1,\xi_2,0)] \qquad \text{(A5)}
\end{aligned}
$$

where

$$
\begin{aligned}
G^{(m)}(\xi_1,\xi_2,t) = &\frac{1}{4}[2t^2 - (\frac{B^{(m)}(\xi_1,\xi_2)}{A^{(m)}})^2 + \frac{2D^{(m)}(\xi_1,\xi_2)}{A^{(m)}}] \\
&\times \ln\left(t^2 + \frac{B^{(m)}(\xi_1,\xi_2)}{A^{(m)}}t + \frac{D^{(m)}(\xi_1,\xi_2)}{A^{(m)}}\right) \\
&+\frac{t}{2}\left(\frac{B^{(m)}(\xi_1,\xi_2)}{A^{(m)}} - t\right) \\
&-\frac{B^{(m)}(\xi_1,\xi_2)}{2[A^{(m)}]^2}\sqrt{Q^{(m)}(\xi_1,\xi_2)}\arctan\left(\frac{2A^{(m)}t + B^{(m)}(\xi_1,\xi_2)}{\sqrt{Q^{(m)}(\xi_1,\xi_2)}}\right) \\
&\qquad \text{if } Q^{(m)}(\xi_1,\xi_2) > 0 \qquad \text{(A6)}
\end{aligned}
$$

or

$$
\begin{aligned}
G^{(m)}(\xi_1,\xi_2,t) = &[t^2 - \left(\frac{B^{(m)}(\xi_1,\xi_2)}{2A^{(m)}}\right)^2]\ln\left|t + \frac{B^{(m)}(\xi_1,\xi_2)}{2A^{(m)}}\right| \\
&-\frac{t}{2}(t - \frac{B^{(m)}(\xi_1,\xi_2)}{A^{(m)}}) \\
&\qquad \text{if } Q^{(m)}(\xi_1,\xi_2) = 0. \qquad \text{(A7)}
\end{aligned}
$$

To evaluate the remaining two line integrals in (A1), $\Gamma(x_1,x_2,\xi_1,\xi_2)$ is written as

$$
\Gamma(x_1,x_2,\xi_1,\xi_2) = \frac{W^{(m)}(\xi_1,\xi_2)}{2\pi\sqrt{\lambda_{11}\lambda_{22}-\lambda_{12}^2}} \cdot \frac{1}{|x_1-\xi_1+\tau(x_2-\xi_2)|^2}
$$
$$
\text{for } (x_1,x_2) \in C^{(m)}
$$

where

$$W^{(m)}(\xi_1, \xi_2) = n_1^{(m)}[\lambda_{11} + \mathrm{Re}\{\tau\}\lambda_{12}][a_1^{(m)} - \xi_1 + \mathrm{Re}\{\tau\}(a_2^{(m)} - \xi_2)] + [\mathrm{Im}\{\tau\}]^2(a_2^{(m)} - \xi_2)[\lambda_{12}n_1^{(m)} + \lambda_{22}n_2^{(m)}].$$

It can be shown that if $Q^{(m)}(\xi_1, \xi_2) = 0$ then $W^{(m)}(\xi_1, \xi_2) = 0$. Thus,

$$\left.\begin{array}{l} I_3^{(m)}(\xi_1, \xi_2) = 0 \\ I_4^{(m)}(\xi_1, \xi_2) = 0 \end{array}\right\} \text{ if } Q^{(m)}(\xi_1, \xi_2) = 0.$$

In general, the analytical formulae for $I_3^{(m)}(\xi_1, \xi_2)$ and $I_4^{(m)}(\xi_1, \xi_2)$ are given by

$$I_3^{(m)}(\xi_1, \xi_2) = \frac{\ell^{(m)}}{2\pi\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}} \times [V^{(m)}(\xi_1, \xi_2, 1) - V^{(m)}(\xi_1, \xi_2, 0)] \tag{A8}$$

and

$$I_4^{(m)}(\xi_1, \xi_2) = -\frac{r}{(1-2r)} I_3^{(m)}(\xi_1, \xi_2) + \frac{\ell^{(m)}}{2A^{(m)}\pi(1-2r)\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}} \times [S^{(m)}(\xi_1, \xi_2, 1) - S^{(m)}(\xi_1, \xi_2, 0)] \tag{A9}$$

where

$$V^{(m)}(\xi_1, \xi_2, t) = \frac{2W^{(m)}(\xi_1, \xi_2)}{\sqrt{Q^{(m)}(\xi_1, \xi_2)}} \arctan\left(\frac{2A^{(m)}t + B^{(m)}(\xi_1, \xi_2)}{\sqrt{Q^{(m)}(\xi_1, \xi_2)}}\right) \text{ if } Q^{(m)}(\xi_1, \xi_2) > 0 \tag{A10}$$

or

$$V^{(m)}(\xi_1, \xi_2, t) = 0 \text{ if } Q^{(m)}(\xi_1, \xi_2) = 0 \tag{A11}$$

and

$$
\begin{aligned}
S^{(m)}(\xi_1,\xi_2,t) = {} & \frac{W^{(m)}(\xi_1,\xi_2)}{2}\ln\left(t^2+\frac{B^{(m)}(\xi_1,\xi_2)}{A^{(m)}}t+\frac{D^{(m)}(\xi_1,\xi_2)}{A^{(m)}}\right) \\
& -\frac{W^{(m)}(\xi_1,\xi_2)B^{(m)}(\xi_1,\xi_2)}{\sqrt{Q^{(m)}(\xi_1,\xi_2)}} \\
& \times \arctan\left(\frac{2A^{(m)}t+B^{(m)}(\xi_1,\xi_2)}{\sqrt{Q^{(m)}(\xi_1,\xi_2)}}\right) \\
& \qquad \text{if } Q^{(m)}(\xi_1,\xi_2)>0
\end{aligned} \tag{A12}
$$

or

$$
S^{(m)}(\xi_1,\xi_2,t)=0 \text{ if } Q^{(m)}(\xi_1,\xi_2)=0. \tag{A13}
$$

Formulae (A2), (A5), (A8) and (A9) are valid for any (ξ_1,ξ_2), even if (ξ_1,ξ_2) lies on $C^{(m)}$. (Note that $Q^{(m)}(\xi_1,\xi_2)=0$ if (ξ_1,ξ_2) is on $C^{(m)}$.) However, for $(\xi_1,\xi_2)=(a_1^{(m)},a_2^{(m)})$ or $(\xi_1,\xi_2)=(b_1^{(m)},b_2^{(m)})$, a slight modification is, strictly speaking, required in the formulae (A2) and (A5). For $(\xi_1,\xi_2)=(a_1^{(m)},a_2^{(m)})$, the terms $F^{(m)}(\xi_1,\xi_2,0)$ and $G^{(m)}(\xi_1,\xi_2,0)$ [in (A2) and (A5)] are respectively replaced by the limits of $F^{(m)}(\xi_1,\xi_2,t)$ and $G^{(m)}(\xi_1,\xi_2,t)$ as $t\to 0^+$, while for $(\xi_1,\xi_2)=(b_1^{(m)},b_2^{(m)})$, $F^{(m)}(\xi_1,\xi_2,1)$ and $G^{(m)}(\xi_1,\xi_2,1)$ are superceded respectively by the limits of $F^{(m)}(\xi_1,\xi_2,t)$ and $G^{(m)}(\xi_1,\xi_2,t)$ as $t\to 1^-$.