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A Hypersingular Boundary Integral Formulation for Heat Conduction Across an Imperfect Interface

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Abstract. A hypersingular boundary integral formulation is derived for a steady-state two-dimensional heat conduction problem involving a bimaterial with a microscopically imperfect interface. To describe the interfacial condition, a macroscopic model which allows for a temperature jump which is proportional in magnitude to the thermal heat flux at the interface is used. For a specific problem, the integral equations in the formulation are solved numerically to determine the temperature field in the bimaterial.

Introduction

Composites which are made up of two or more dissimilar materials play an important role in modern technology. In many studies, the dissimilar materials are assumed to be perfectly joined or bonded to one another along their common boundaries (see e.g. Ang [1], Berger and Karageorghis [2], Clements [3] and Lee and Kim [4]).

However, a perfect bond does not exist in reality, as microscopic imperfections are bound to be present along the interfaces of the materials. Thus, in recent years, there is a growing interest among researchers in the investigation of microscopically imperfect interfaces in layered and composite materials (see e.g. Benveniste and Miloh [5], Fan and Sze [6], Benveniste [7], Torquato and Rintoul [8] and other references therein). In heat conduction problems, a macroscopic model for studying such an imperfect interface allows for a temperature jump which is proportional in magnitude to the thermal heat flux at the interface.

In the present paper, the two-dimensional problem of determining the steady-state temperature distribution in a thermally isotropic bimaterial with a homogeneously imperfect planar interface is considered. On the exterior boundary of the bimaterial, either the temperature or the heat flux (not both) is known at each and every point on the exterior boundary of the bimaterial.

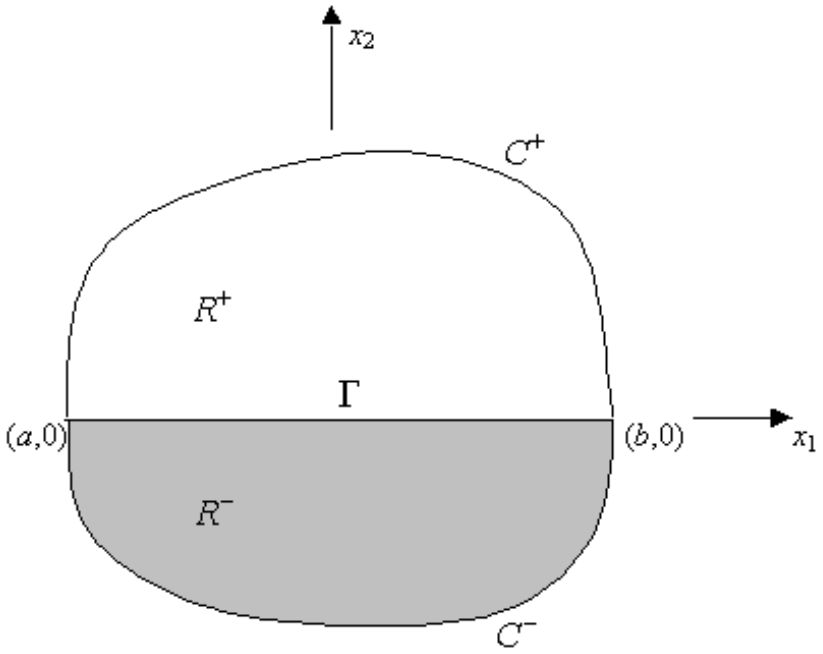

Figure 1. A sketch of the geometry of the problem.

The temperature is expressed in terms of a boundary integral over the exterior boundary of the bimaterial and the imperfect planar interface. With the use of a suitable Green's function for the corresponding perfect interface, the only unknown function that appears in the integral over the imperfect interface is the temperature jump. The boundary integral expression for the temperature can be suitably differentiated to obtain the heat flux for formulating the condition on the imperfect interface. This gives rise to hypersingular boundary integral equations for the problem under consideration. The proposed approach is similar to that of Chen and Hong [9] for solving a heat conduction problem with a degenerate bound-

ary. For a specific problem, the equations are solved numerically in order to determine the temperature field in the bimaterial.

The Problem

Referring to an $0x_1x_2x_3$ Cartesian co-ordinate system, consider a body comprising two homogeneous materials with possibly different thermal properties. The geometry of the body is independent of the x_3 co-ordinate. On the $0x_1x_2$ plane, the interface separating the two materials is the straight line segment Γ which lies on part of the x_1-axis between the points $(a, 0)$ and $(b, 0)$ (where a and b are given real numbers such that $a < b$), while the exterior boundary of the body is the simple closed curve C. The curve C consists of two parts: C^+ which lies above the x_1-axis, and C^- below the axis. A sketch of the geometry is given in Figure 1. The regions enclosed by $C^+ \cup \Gamma$ and $C^- \cup \Gamma$ are denoted by R^+ and R^- respectively.

If the steady-state temperature field in the body is independent of x_3 and given by $T(x_1, x_2)$, then together with the classical Fourier's law of heat conduction the energy equation gives rise to the two-dimensional Laplace's equation

$$\frac{\partial^2 T}{\partial x_k \partial x_k} = 0 \text{ in } R^{\pm}. \tag{1}$$

Note that the Einsteinian convention of summing over a repeated index is adopted for latin subscripts running from 1 to 2.

The bond between the materials in R^+ and R^- at the interface Γ is microscopically damaged. The microscopic damage is assumed to be uniformly distributed over the interface. A macroscopic model for the heat conduction across the imperfect interface is given by

$$k^+ \left.\frac{\partial T}{\partial x_2}\right|_{x_2=0^+} = k^- \left.\frac{\partial T}{\partial x_2}\right|_{x_2=0^-} = \lambda \Delta T(x_1) \quad \text{for } x_1 \in [a, b], \tag{2}$$

where k^+ and k^- are the (constant) thermal conductivities of the materials in R^+ and R^- respectively, λ is a given positive coefficient, and $\Delta T(x_1) = T(x_1, 0^+) - T(x_1, 0^-)$ is the temperature jump across the interface. If the microscopic damage is uniformly distributed over the interface then λ is a constant.

At each and every point on the exterior boundary $C = C^+ \cup C^-$, either the temperature T or its normal flux $-k(x_1, x_2) n_p \partial T / \partial x_p$ (but not both) is specified. (Note that $k(x_1, x_2)$ denotes the thermal conductivity at the

point (x_1, x_2) in the bimaterial and $[n_1(x_1, x_2), n_2(x_1, x_2)]$ is the unit normal vector to C at (x_1, x_2) which points out of the region R enclosed by C.) The problem is to determine the temperature in the body by solving (1) subject to the boundary condition on C and the interface condition as given by (2).

Hypersingular boundary integral formulation

For $\xi_2 \neq 0$, guided by the analysis in Clements [10], one may derive a boundary integral solution for (1) in the form

$$\begin{aligned}\gamma(\xi_1,\xi_2)T(\xi_1,\xi_2) = \oint_C k(x_1,x_2)[T(x_1,x_2)n_p(x_1,x_2)\frac{\partial}{\partial x_p}\Phi(x_1,x_2,\xi_1,\xi_2) \\ - \Phi(x_1,x_2,\xi_1,\xi_2)n_p(x_1,x_2)\frac{\partial}{\partial x_p}T(x_1,x_2)]ds(x_1,x_2) \\ - k^+\int_a^b \Delta T(x_1)\,\frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\Big|_{x_2=0^+} dx_1, \qquad (3)\end{aligned}$$

where $\gamma(\xi_1,\xi_2) = 1$ if (ξ_1,ξ_2) lies inside R^+ or R^-, $0 < \gamma(\xi_1,\xi_2) < 1$ if (ξ_1,ξ_2) lies on C^+ or C^- [$\gamma(\xi_1,\xi_2) = 1/2$ if (ξ_1,ξ_2) lies on a smooth part of C^+ or C^-] and

$$\begin{aligned}\Phi(x_1,x_2,\xi_1,\xi_2) &= \frac{1}{2\pi}[\frac{1}{k^+}H(x_2) + \frac{1}{k^-}H(-x_2)]\,\mathrm{Re}\{\ln(z-c)\} \\ &\quad +\Psi(x_1,x_2,\xi_1,\xi_2), \qquad (4)\end{aligned}$$

with $z = x_1 + ix_2$, $c = \xi_1 + i\xi_2$, $i = \sqrt{-1}$, $H(x)$ being the Heaviside unit-step function and $\Psi(x_1,x_2,\xi_1,\xi_2)$ being given by

$$\begin{aligned}\Psi(x_1,x_2,\xi_1,\xi_2) &= -\frac{\mu}{k^+}H(x_2)\,\mathrm{Re}\{H(-\xi_2)\ln(z-c) + H(\xi_2)\ln(z-\overline{c})\} \\ &\quad +\frac{\mu}{k^-}H(-x_2)\,\mathrm{Re}\{H(-\xi_2)\ln(\overline{z}-c) + H(\xi_2)\ln(\overline{z}-\overline{c})\}. \qquad (5)\end{aligned}$$

where $\mu = (k^- - k^+)/[2\pi(k^- + k^+)]$ and the bar denotes the complex conjugate of a complex number.

Note that $\Phi(x_1,x_2,\xi_1,\xi_2)$ as given by (4) together with (5) is the Green's function for the perfect interface, i.e. it satisfies the conditions

$$\Phi(x_1,0^+,\xi_1,\xi_2) - \Phi(x_1,0^-,\xi_1,\xi_2) = 0 \text{ for } -\infty < x_1 < \infty, \qquad (6)$$

$$k^+ \frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\Big|_{x_2=0^+} = k^- \frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\Big|_{x_2=0^-}$$
$$\text{for } -\infty < x_1 < \infty. \tag{7}$$

Differentiating (3) partially with respect to ξ_2 and then letting (ξ_1,ξ_2) approach a point on the imperfect interface, we find that the interfacial condition (2) may be re-written as

$$k^+ \oint_C k(x_1,x_2)[T(x_1,x_2)n_p(x_1,x_2)\,[\frac{\partial^2}{\partial x_p \partial \xi_2}\Phi(x_1,x_2,\xi_1,\xi_2)]\Big|_{\xi_2=0^+}$$
$$-n_p(x_1,x_2)\frac{\partial}{\partial x_p}T(x_1,x_2)\,[\frac{\partial}{\partial \xi_2}\Phi(x_1,x_2,\xi_1,\xi_2)]\Big|_{\xi_2=0^+}]ds(x_1,x_2)$$
$$+\frac{k^+k^-}{\pi(k^+ + k^-)}\mathcal{H}\int_a^b \frac{\Delta T(x_1)}{(\xi_1 - x_1)^2}dx_1 = \lambda \Delta T(\xi_1) \text{ for } a < \xi_1 < b, \tag{8}$$

where $\mathcal{H}$ denotes the integral over the interval $[a,b]$ is to be interpreted in the Hadamard finite-part sense, i.e.

$$\mathcal{H}\int_a^b \frac{\Delta T(x)dx}{(\xi - x)^2} \stackrel{\text{def}}{=} \lim_{\sigma \to 0^+}[\int_a^b \frac{(\xi - x)^2 \Delta T(x)dx}{[(\xi - x)^2 + \sigma^2]^2} - \frac{\pi}{2\sigma}\Delta T(\xi)] \text{ for } a < \xi < b. \tag{9}$$

A simple numerical procedure

For (3) to give an explicit expression for the required temperature field, the unknown quantities on $C \cup \Gamma$, i.e. T and $kn_p\partial T/\partial x_p$ on C and ΔT on Γ, must be determined. A simple numerical procedure for finding these unknowns from (3) and (8) is as follows.

The boundary C is discretized into M straight line elements denoted by $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(M-1)}$ and $C^{(M)}$. Over the element $C^{(m)}$, T and $kn_p\partial T/\partial x_p$ are approximated as constants, i.e.

$$T(x_1,x_2) \simeq T^{(m)} \text{ and } k(x_1,x_2)n_p(x_1,x_2)\frac{\partial T}{\partial x_p} \simeq H^{(m)} \text{ for } (x_1,x_2) \in C^{(m)}, \tag{10}$$

where $T^{(m)}$ and $H^{(m)}$ are constants.

From the condition given on the exterior boundary C, either T or $kn_p\partial T/\partial x_p$ is known on a boundary element. Thus, there are M unknown constants in (10) to be determined.

The interval $[a, b]$ representing the interface Γ is divided into L subintervals denoted by $[x^{(0)}, x^{(1)}]$, $[x^{(1)}, x^{(2)}]$, $\cdots$, $[x^{(L-2)}, x^{(L-1)}]$ and $[x^{(L-1)}, x^{(L)}]$, with $x^{(0)} = a$ and $x^{(L)} = b$. The interfacial temperature jump ΔT is approximated using

$$\Delta T(x_1) \simeq J^{(\ell)} \text{ for } x_1 \in [x^{(\ell-1)}, x^{(\ell)}], \tag{11}$$

where $J^{(\ell)}$ are unknown constants to be determined.

With (10) and (11), if we let (ξ_1, ξ_2) in (3) be given by the midpoints of $C^{(r)}$ for $r = 1, 2, \cdots, M$, and if we take ξ_1 in (8) to be given by $(x^{(j-1)} + x^{(j)})/2$ for $j = 1, 2, \cdots, L$, we obtain a system of $M + L$ linear algebraic equations in $M + L$ unknowns. Once these unknowns are determined, (3) can be used to compute T approximately at any interior point in the bimaterial.

A specific example

To test the proposed numerical procedure, let us take R^+ to be the region $0 < x_1 < 1$, $0 < x_2 < 1/2$, with $k^+ = 1/5$, and R^- to be $0 < x_1 < 1$, $-1/2 < x_2 < 0$, with $k^- = 1/2$. On the interface between R^+ and R^-, i.e $0 < x_1 < 1$, $x_2 = 0$, we impose the condition (2) with $\lambda = 1$.

A solution of (1) which satisfies the interface condition (2) with $k^+ = 1/5$, $k^- = 1/2$ and $\lambda = 1$ is given by

$$\begin{aligned} T(x_1, x_2) &= \{H(x_2)[2\cos(x_2) + 5\sin(x_2)] \\ &\quad + H(-x_2)[\cos(x_2) + 2\sin(x_2)]\} \exp(-x_1) \end{aligned} \tag{12}$$

To devise a test problem, let us use (12) to generate boundary values of the temperature T on the sides $x_2 = \pm 1/2$, $0 < x_1 < 1$, and boundary values of the normal derivative of T on $x_1 = 0$, $-1/2 < x_2 < 1/2$ and also on $x_1 = 1$, $-1/2 < x_2 < 1/2$. The proposed numerical procedure is then applied to solve (1) subject to the boundary data thus generated and the interface condition (2). If it really works, we should be able to recover the solution (12) and the corresponding interfacial temperature jump $\Delta T(x_1) = \exp(-x_1)$ approximately.

For the practical implementation of the numerical procedure, each of the 4 sides of the bimaterial is divided into N equal length boundary elements (so that $M = 4N$) and the interface $[0, 1]$ into L equal subintervals. To avoid ambiguity, we require a boundary element to be in either R^+ or R^- but not partly in both the regions. One of the endpoints of the element

is allowed to be on Γ, however. Thus, N must be chosen to be an even integer.

Table 1. A comparison of the numerical and exact values of T with the exact ones at various points in the interior of the bimaterial.

(x_1, x_2)	$(M, L) = (40, 5)$	$(M, L) = (120, 15)$	Exact
$(0.8000, 0.3000)$	1.5254	1.5229	1.5225
$(0.7000, -0.2000)$	0.2883	0.2892	0.2894
$(0.1000, 0.4000)$	3.4261	3.4283	3.4286
$(0.3000, -0.1000)$	0.5897	0.5895	0.5892
$(0.5000, 0.4950)$	2.2966	2.5074	2.5081
$(0.7500, 0.0050)$	1.0132	0.9713	0.9565

Table 2. A comparison of the numerical and exact values of the interfacial temperature jump ΔT at various points on the interface.

(x_1, x_2)	$(M, L) = (40, 5)$	$(M, L) = (120, 15)$	Exact
$(0.1000, 0)$	0.8918	0.9006	0.9048
$(0.3000, 0)$	0.7376	0.7400	0.7408
$(0.5000, 0)$	0.6060	0.6064	0.6065
$(0.7000, 0)$	0.4980	0.4970	0.4966
$(0.9000, 0)$	0.4130	0.4085	0.4066

In Table 1, we compare the numerical values of the temperature T at selected points in the interior of the bimaterial, as computed using (3) with $(M, L) = (40, 5)$ and $(M, L) = (120, 15)$, with the exact solution (12). The two sets of numerical values show good agreement with the exact solution, except at points that are very close to the exterior boundary or the interface, e.g. $(0.5000, 0.4950)$ and $(0.7500, 0.0050)$, where there is a relatively higher percentage of error. The adverse effect of the boundary on the accuracy of the numerical values at points that are at a distance much smaller than the lengths of nearby elements is a well known phenomenon in boundary element research. One way of improving the numerical values at those points is to refine the discretization of the nearby boundary. This is clearly shown in Table 1 by the fact that the numerical values of the temperature at $(0.5000, 0.4950)$ and $(0.7500, 0.0050)$ improve significantly

in accuracy when we treble the number of subintervals on the interface and also the boundary elements. The percentage errors of T at $(0.5000, 0.4950)$ and $(0.7500, 0.0050)$ are reduced further to about 0.0002% and 0.15% respectively when we use $(M, L) = (240, 30)$ in our computation.

In Table 2, the numerical values of the interfacial temperature jump ΔT, as obtained using $(M, L) = (40, 5)$ and $(M, L) = (120, 15)$, are compared with the exact values at selected points on the interface. There is a definite improvement in the numerical values when the discretization of the imperfect interface and the exterior boundary of the bimaterial is refined.

Conclusion

A hypersingular boundary integral formulation is derived for the plane steady-state heat conduction in a bimaterial with a microscopically imperfect interface. It is used to devise a boundary element method for computing approximately the temperature distribution in the bimaterial. The numerical result obtained for a specific test problem indicates that the method is capable of delivering accurate approximate solution for the problem under consideration.

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