# Understanding trigonometric functions and complex numbers

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## ABSTRACT

Two topics, trigonometric functions and complex numbers, are explained for students who feel that they are scary. This article aims to show how trigonometric functions can be naturally incorporated in our daily lives; and how consistent, beautiful and useful complex numbers are.

**Keywords:** Trigonometric functions, complex numbers

## 1. INTRODUCTION

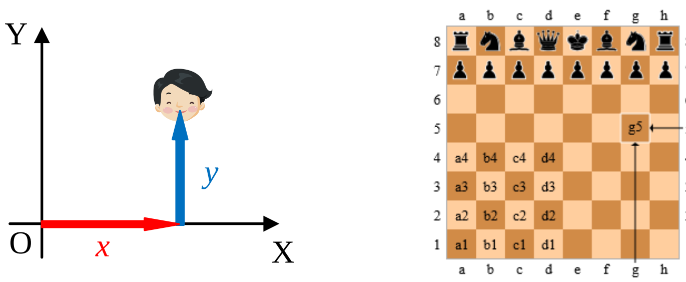
Trigonometric functions and complex numbers are two fundamental topics that university students are expected to know very well. They scatter here and there in various courses. When some students want to enhance their knowledge on these two topics, I would usually refer them to wiki sites where we could find almost everything needed. However, referring them to these sites would often make it seem that my answer was not serious and helpful enough. Therefore, as a response, I wrote this article. Basically, the words in this article are original; but the explanations would not be, as these topics are immensely common and fundamental. Nevertheless, during the writing, only wiki sites such as [1] and [2] were peeked at, and if similar explanations are found later, they will be linked in the reference list.

## 2. TRIGONOMETRIC FUNCTIONS

### 2.1 Describe where you are

Let's start with a simple but important problem: How should you describe your location so that I can easily find you? (Think for two minutes and then continue reading).

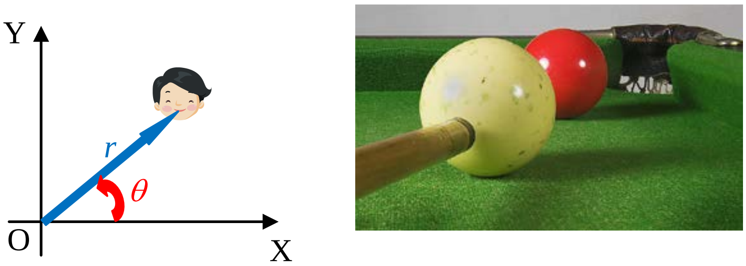
*Cartesian representation*: One possible approach is to construct an OXY coordinate system (called a Cartesian coordinate system) so that we have a reference point O and two perpendicular reference directions X and Y. We can move along X-axis by an amount of $x$ and then along Y-axis by an amount of $y$. If you describe your location as $(x, y)$, I can definitely find you. This idea is illustrated in Fig. 1(a). We indeed use this representation in our daily life. For example, the row number and the seat number on a movie ticket serve the same purpose. Another example is the naming of the chess squares (Fig. 1(b)). We constrain ourselves to a two-dimensional (2D) space in this article for simplicity. You can extend it to higher dimensions.




(a) (b)

Figure 1. Cartesian representation of locations
(a) location of a person; (b) location of squares (image source: [3])

*Polar representation*: Another possible approach, as illustrated in Fig. 2(a), is to rotate the X-axis by an angle $\theta$, followed by a move of a distance $r$. In other words, if you describe your location as $(r, \theta)$, we can again easily reach you. Such a presentation is called polar representation. In fact, one good example is the game of billiards, where we first have to point at an angle, and then fire a shot to a certain distance (Fig. 2(b)).




(a) (b)

Figure 2. Polar representation of locations
(a) location of a person; (b) location of a ball (image source: [4])

### 2.2 Conversion between the Cartesian and the polar

It can be seen that these two approaches have their own advantages and applications; and there are many times where $(x, y)$ is given, but $(r, \theta)$ is expected, and vice versa. But how can one obtain the desired representation quickly? (Think for two minutes and then continue reading).

One possible way is to construct a table, with $x$ and $y$ representing the rows and the columns respectively, and the values of $(r, \theta)$ at the point where each $x$ and $y$ value intersects; namely, to manually build up the relationship: $(x, y) \to (r, \theta)$. Similarly, we can construct another table to elucidate this relationship: $(r, \theta) \to (x, y)$ and solve the conversion problem by conducting a visual search on these two tables.

### 2.3 Simplifications of the conversion and the rise of the tangent

How can we make these two tables more concise? (Think for two minutes and then continue reading).

*Simplification 1*: We utilize the Pythagorean theorem for a right triangle (Fig. 3(a)), which we all know,

$$r^2 = x^2 + y^2 . \qquad (1)$$

It means that the $r$ value can be computed analytically and easily without the table. The table $(x, y) \to (r, \theta)$, which is 2D → 2D can now be simplified to $(x, y) \to \theta$ which is 2D → 1D.

*Simplification 2*: By examining Fig. 3(b), we immediately find that $(x_1, y_1)$ and $(x_2, y_2)$ maps to the same angle $\theta$. What really affects the angle is the ratio of the coordinators rather than the coordinates themselves! This further simplifies the relationship from $(x, y) \to \theta$ to $y / x \to \theta$ which is 1D → 1D. This mapping is essential to the problem we are considering and is worth a special name - we call it tangent:

$$\tan \theta = y / x . \qquad (2)$$

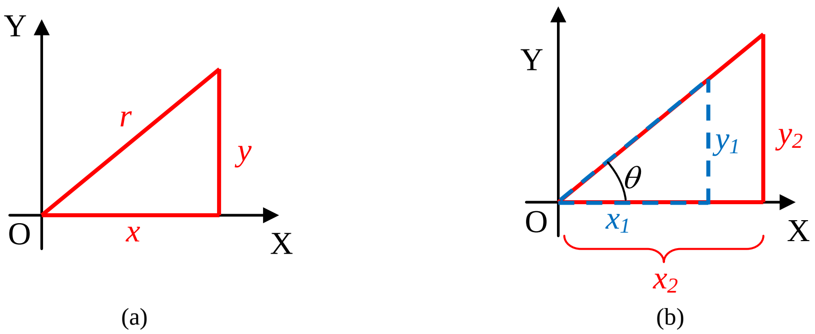



Figure 3. Facts for simplifications
(a) a right triangle; (b) relationship between the angle and the sides of the triangle

Before our further exploration, it is time for a break. Please try the following questions:

*Homework 1*: In Eq. (2), $x$ cannot be zero, but in our real life, this should not be a constraint. How can you solve this problem?

*Homework 2*: For $x = y = 1$, what are the angle and its tangent value? How about for $x = y = -1$? What do you find from these results?

*Homework 3*: Can you also try to simplify the relationship $(r, \theta) \rightarrow (x, y)$?

*Homework 4*: (optional) I once interestingly saw the word “tangent” in one of my son’s music books. Explore its meaning if you have interest.

### 2.4 More trigonometric functions

In the previous section, the ratio, $y/x$, is adopted to map to the angle $\theta$. There are five more such possibilities: $x/y$, $x/r$, $r/x$, $y/r$ and $r/y$. We thus define a total of six functions as listed in Table 1. The basic relationships among themselves are given in square brackets. Three functions, $\text{ctan}\,\theta$, $\csc\theta$ and $\sec\theta$, are reciprocals of $\tan\theta$, $\sin\theta$ and $\cos\theta$, respectively, and thus apparently redundant. Also, it can be seen that the remaining two functions, $\sin\theta$ and $\cos\theta$, are redundant as well, as their relationships to $\tan\theta$ can be easily derived from Eqs. (1) and (2).

But why do we need so many trigonometric functions? The answer is that the redundancy gives the conciseness in representation, and you will benefit from it. To give a simple example, instead of saying, “I am extremely angry”, sometimes we say, “I am furious” instead. Interestingly, obtaining a concise representation through redundancy to enable better processing is currently one of the hottest topics in signal and image processing [5].

Table 1. Definitions of trigonometric functions.

| | | Numerator | | |
|---|---|---|---|---|
| | | $y$ | $x$ | $r$ |
| Denominator | $y$ | 1 | Cotangent: $\text{ctan}\,\theta\left[=1/\tan\theta\right]$ | Cosecant: $\csc\theta\left[=1/\sin\theta\right]$ |
| | $x$ | Tangent: $\tan\theta$ | 1 | Secant: $\sec\theta\left[=1/\cos\theta\right]$ |
| | $r$ | Sine: $\sin\theta\left[=\pm\tan\theta/\sqrt{1+\tan^2\theta}\right]$ | Cosine: $\cos\theta\left[=\pm1/\sqrt{1+\tan^2\theta}\right]$ | 1 |

Sine and cosine functions have been emphasized in many books. There are a few reasons:

- Using sine and cosine functions, the conversion of $(r,\theta)\rightarrow(x,y)$ is simpler (shown in subsection 2.5);
- From sine and cosine functions, we obtain tangent easily:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \tag{3}$$

- (Sketch these two functions with respect to the angle $\theta$ according to the definitions now, or do it after reading subsection 2.6.). The sine and cosine functions have a very good property: they wave periodically. A sine or cosine curve is similar to, for example, our daily atmospheric temperature. The following figure was taken from "BBC Weather" [6], for a particular Sunday in Singapore. I then traced a sine curve in green and overlaid it onto the weather data, which then gives a very nice, although not perfect fitting. To make it perfect, we would need to use more sines and cosines [7]. In fact, the sine and cosine are important functions for modeling many phenomena, some of which can be found in our courses.

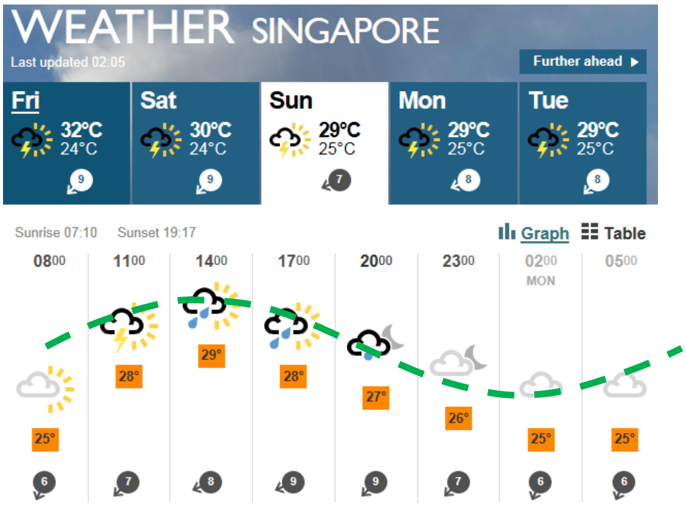



Figure 4. The weather can be approximately represented by a sine curve in green (image source: [6])

### 2.5 Return to the conversion problem

With the above definitions and discussions, we are able to concisely write the conversion problem as follows:

$$(x,y)\rightarrow(r,\theta):\begin{cases} r=\sqrt{x^2+y^2} \\ \theta=\operatorname{atan}(y/x) \end{cases}, \tag{4}$$

$$(r,\theta)\rightarrow(x,y):\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}. \tag{5}$$

where $\operatorname{atan}$ is the inverse tangent function [8].

### 2.6 Properties and identities of trigonometric functions

Due to the importance of trigonometric functions, it is necessary to know the properties and identities of trigonometric functions. We start with some basic properties, which I have already completed half of them in Table 2, and the remaining half are to be completed by you. Try not to refer to any materials and work it out by yourself.

Table 2. Basic properties of trigonometric functions.

| $\sin(\pm\theta)$ | $\cos(\pm\theta)$ | $\sin(\pi/2\pm\theta)$ | $\cos(\pi/2\pm\theta)$ | $\sin(\pi\pm\theta)$ | $\cos(\pi\pm\theta)$ | $\sin(2\pi\pm\theta)$ | $\cos(2\pi\pm\theta)$ |
|---|---|---|---|---|---|---|---|
| = | = | = | = | = | = | = | = |
| $\pm\sin\theta$ | | $\cos\theta$ | | $\mp\sin\theta$ | | $\pm\sin\theta$ | |

Once you have completed your work, evaluate $\sin\theta$ and $\cos\theta$ when $\theta$ takes these special values: $\frac{0\pi}{12},\frac{2\pi}{12},\frac{3\pi}{12},\frac{6\pi}{12}$. Then using the relationships you have discovered in Table 2, find $\sin\theta$ and $\cos\theta$ for $\theta=\frac{k\pi}{12}, k=0,1,\cdots 24$. Verify your result with Fig. 5 below, where in the round brackets are sine and cosine values. (If you can figure all these out, you will love trigonometric functions!)

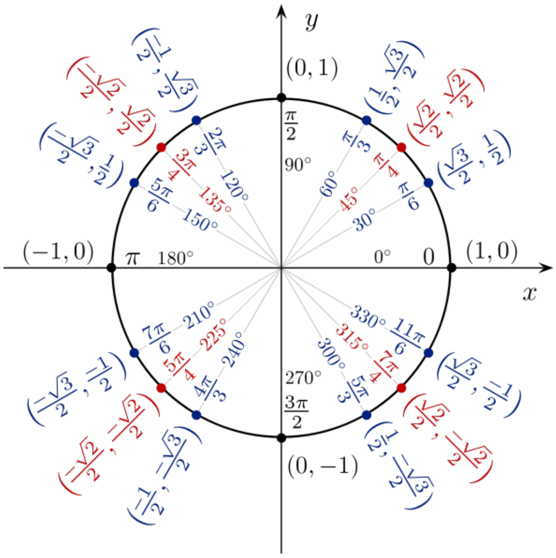



Figure 5. Angles and their sine and cosine values (image source: [1])

Now, continue your work by filling in Table 3. This time you can search the Internet and fill in the results. All of them can be found in [1]. You may wonder how does one obtain these results. If you are curious, try to prove the first identity in Table 3 using geometry and verify your work by searching the proof in [1]. Now we move to complex numbers. After we have learned it, I will introduce another simple proof.

Table 3. Useful identities of trigonometric functions.

| | | | |
|---|---|---|---|
| Angle sum and difference | $\sin(\theta \pm \varphi)$ | = | $\sin\theta\cos\varphi \pm \cos\theta\sin\varphi$ |
| | $\cos(\theta \pm \varphi)$ | = | |
| Double angle | $\sin(2\theta)$ | = | $2\sin\theta\cos\theta$ |
| | $\cos(2\theta)$ | = | |
| Sum-to-product | $\sin\theta + \sin\varphi$ | = | $2\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$ |
| | $\sin\theta - \sin\varphi$ | = | |
| | $\cos\theta + \cos\varphi$ | = | $2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$ |
| | $\cos\theta - \cos\varphi$ | = | |
| Product-to-sum | $\sin\theta\sin\varphi$ | = | $\frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$ |
| | $\sin\theta\cos\varphi$ | = | |
| | $\cos\theta\sin\varphi$ | = | $\frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$ |
| | $\cos\theta\cos\varphi$ | = | |

# 3. COMPLEX NUMBERS

## 3.1 Numbers, and more numbers

When we learnt subtraction in primary school, our teachers used to tell us that a small number could not be subtracted by a larger number. They convinced us by asking that, if you had one apple, how could you eat two, or if you had five dollars, how could you spend ten. And we nodded and accepted it. But later, we eventually learnt that, negative numbers in mathematics made subtracting a big number from a small number possible.

Assume $a > b$, we can represent the subtraction as follows

$$c = a - b\,. \tag{6}$$

Subsequently we define

$$-c = b - a\,, \tag{7}$$

where a minus sign "–" is used to indicate the new subtraction situation. In Eq. (6), $c$ can be replaced by $+c$ to increase the clarity such that "+" and "–" clearly indicates two subtractions: the former subtracts a small number from a larger one and the latter subtracts a larger number from a smaller one. The subtraction becomes more general by catering for both cases.

The story repeats on complex numbers. Recall that our secondary school teachers might have emphasized that a negative number does not have a square root. But now we want to break this constraint.

Assume $a > 0$. Its positive square root is

$$b = \sqrt{a} \,. \tag{8}$$

Subsequently, we define

$$i \cdot b = \sqrt{-a} \,, \tag{9}$$

where the '*i*' sign is used to indicate the new square root situation, and a '$\cdot$' sign indicates multiplication. In Eq. (8), *b* can be replaced by $1 \cdot b$ to increase the clarity so that '1' and '*i*' clearly indicates two square root situations: the former takes square root from a positive number, and the latter from a negative number. The square root operation becomes more general by catering for both cases.

If we set $a = 1$, we can immediately obtain $b = 1$ from Eq. (8) and $i = \sqrt{-1}$ from Eq. (9). We typically call '*i*' as an imaginary unit. We can therefore move from a real number to a complex number representation, which is defined as

$$z = x + iy \,, \tag{10}$$

where the '$\cdot$' sign between '*i*' and '*y*' has been omitted for simplicity.

Relating from the analogy of the positive-negative number example described before, I hope you could feel the analogy extension of the real to complex numbers acceptable. However, at this point, one might still be wondering about the necessity of taking the square root of a negative number: Why do we have to learn it? I will list three strong forces that drive us to learn them. But for now, let's learn the basic operations of the complex numbers.

### 3.2 Basic operations of complex numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Four basic operations are defined as follows,

$$\text{Addition: } z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \,. \tag{11}$$

$$\text{Subtraction: } z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2) \,. \tag{12}$$

$$\text{Multiplication: } z_1 z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \,. \tag{13}$$

$$\text{Division: } \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1 x_2 + y_1 y_2) + i(-x_1 y_2 + x_2 y_1)}{x_2^2 + y_2^2} \,. \tag{14}$$

Because a real number is just a special complex number without the imaginary part, the above operations should be consistent with what we have learned earlier for real numbers. Let's test whether it is true. We set $y_1 = y_2 = 0$ to null the imaginary parts and make both $z_1$ and $z_2$ real. Equation (11) then becomes $x_1 + x_2 = x_1 + x_2$, which is trivially true, and so do all other three equations (try it out). I now present the first driving force for complex numbers.

*Driving force 1*: A real number is a special complex number. The knowledge about complex numbers we are learning is consistent with the knowledge about real numbers we have learned before. It is not harmful to learn complex numbers.

### 3.3 Linking to trigonometric functions

A complex number has two parts, real and imaginary. It gives us an opportunity to use it to describe the coordinates of a point. Thus *x* and *y* in Eq. (10) can be referred to as the coordinates in Fig. 1(a). A natural question is, how about the coordinates in Fig. 2(a)?

Due to Eq. (5), Eq. (10) can be re-written as

$$z = x + iy = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta) \,. \tag{15}$$

We now introduce the Euler's formula without proof (which means that you do not ask how we get it but you can search the internet if you are curious)

$$e^{i\theta} = \cos\theta + j\sin\theta . \qquad (16)$$

The beauty of Euler's formula is that it leads to the polar form of a complex number,

$$z = re^{i\theta} . \qquad (17)$$

We call this the polar form. The interesting thing is that the rules for exponential functions that we have learned before can be applied, thus contributing to more basic operations.

### 3.4 Basic operations of complex numbers: continued

Rewrite $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, and let $z = re^{i\theta}$. We define the following operations,

$$\text{Multiplication: } z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} . \qquad (18)$$

$$\text{Division: } \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} . \qquad (19)$$

$$n^{\text{th}} \text{ power: } z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} . \qquad (20)$$

$$n^{\text{th}} \text{ roots: } z^{1/n} = \left(re^{i\theta}\right)^{1/n} = r^{1/n} e^{i(\theta/n+2k\pi/n)}, k = 0,1,\cdots,n-1 . \qquad (21)$$

You can again verify that these operations are consistent with those for real numbers we have learned earlier.

We will now talk a little bit on whether a number set is closed under a certain operation. If we put all positive integers together in a set called $\mathbf{Z}^+$, then arbitrarily take two numbers out from $\mathbf{Z}^+$ and calculate its sum, and if the result is still in $\mathbf{Z}^+$, then mathematically, we can say that $\mathbf{Z}^+$ is closed under addition.

However, note that $\mathbf{Z}^+$ is not closed under subtraction. A smaller positive number minus a larger positive number returns a negative number, which is out of $\mathbf{Z}^+$! However, if we put all positive and negative numbers into a set and also include zero, this new set (called **Z**) would then also be closed under subtraction.

We already know that the real number set **R** is not closed on square root. Equation (21) says that, to take $n^{\text{th}}$ roots of a complex number, there are $n$ roots altogether which can be computed according to this equation. Thus we can say that the complex number set **C** is closed on $n^{\text{th}}$ roots.

*Driving force 2*: The complex number set **C** is closed under addition, subtraction, multiplication, division, $n^{\text{th}}$ power and $n^{\text{th}}$ roots. These operations are what we generally encounter and need.

### 3.5 Application to the proof of a trigonometric identity

As the first application of complex numbers, we prove

$$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi . \qquad (22)$$

To do so, we consider a complex number $z = \mathrm{e}^{i(\theta+\varphi)}$. Using Euler's formula in Eq. (16), we immediately have

$$z = \cos(\theta+\varphi) + i\sin(\theta+\varphi) . \qquad (23)$$

We can derive that

$$
\begin{aligned}
z &= \mathrm{e}^{i(\theta+\varphi)} \\
&= \mathrm{e}^{i\theta}\mathrm{e}^{i\varphi} \leftarrow [\because \text{a property of exponential functions}] \\
&= (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) \leftarrow [\because \text{Eq. (16)}] \\
&= (\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi) \\
&\uparrow [\because \text{Multiplication of complex numbers}]
\end{aligned} \tag{24}
$$

Equating the imaginary parts in Eqs. (17) and (18) yields and proves Eq. (22). As a byproduct, equating the real parts in Eqs. (17) and (18) yields

$$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi. \tag{25}$$

This proof shows the power of complex numbers in their debut.

*Homework 5*: Using complex numbers to prove derive $\sin(3\theta)$ and $\cos(3\theta)$.

### 3.6 Application to Fourier transform

Complex numbers are essential in Fourier transform. The basic idea of the Fourier transform is to decompose a signal into many pieces with pre-defined and self-contented forms (mathematically, a basis). Two sets of functions, sine and cosine, are chosen, but using two sets of functions as one basis is not convenient. Due to Euler's formula, we are able to use just one set of complex exponential functions to build one basis. I will not discuss this in detail because I will have another article that solely focuses on Fourier transform [7].

### 3.7 Application to wave optics

Elemental optical waves can be written in a form of $a\cos(\omega t+\varphi)$ where the variables *a*, *ω*, *t* and *φ* represent the amplitude, frequency, time and phase, respectively. However, this form is less often used than the complex version, $ae^{i\varphi}e^{i\omega t} = Ae^{i\omega t}$. The main reason is for convenience. As an example, we calculate the energy of the wave during a certain period *T* using the cosine version. We assume *a*, *ω* and *φ* are constant, and *ω* is high enough such that $\omega T \gg 1$. We then have

$$
\begin{aligned}
I &= \int_0^T \left| a\cos(\omega t+\varphi) \right|^2 dt \\
&= a^2 \int_0^T \cos^2(\omega t+\varphi) dt \leftarrow [\because \text{assume } a \text{ is constant}] \\
&= \frac{a^2}{2}\int_0^T \left[1+\cos(2\omega t+2\varphi)\right] dt \leftarrow [\because \text{double angle}] \\
&= \frac{a^2}{2}\left[ t + \frac{\sin(2\omega t+2\varphi)}{2\omega} \right]\Bigg|_0^T \leftarrow [\because \text{evaluate the integral}] \\
&= \frac{a^2}{2}\left[ T + \frac{\sin(2\omega T+2\varphi)-\sin(2\varphi)}{2\omega} \right] \leftarrow [\because \text{further evaluation}] \\
&\approx \frac{1}{2}a^2 T \leftarrow [\because \omega T \gg 1]
\end{aligned} \tag{26}
$$

The derivation is not difficult but quite tedious, but if we use the complex version instead, immediately we have

$$I = \int_0^T \left| a e^{j(\omega t + \varphi)} \right|^2 dt = \int_0^T a^2 dt = a^2 T , \tag{27}$$

which is amazingly simple. The difference of 1/2 is not important if we stick to only one representation.

*Driving force 3*: The complex numbers are practical and convenient. Therefore, more and more fields gradually adopt them. And now, as complex numbers has become a fundamental tool, we have no choice but study and master them.

## 4. CONCLUSIONS

Since you have read to this section I would assume that you have read through this article. You can draw your own conclusions on these two topics. The following are mine. For trigonometric functions, I emphasize that the quest for them is natural, and consequently, the properties and identities of trigonometric functions become useful and worth spending time on them. As for complex numbers, I emphasized the three driving forces of learning them: consistence (the new knowledge on complex numbers is consistent with our previous knowledge and will not confuse us), beauty (the complex number set is closed on all basic operations), and convenience (people feel that they are convenient and use them as a well-established tool). Finally, the main relationship between these two topics is that they are basically two close relatives linked by Euler's formula.

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