ω-CHANGE RANDOMNESS AND WEAK DEMUTH RANDOMNESS

JOHANNA N.Y. FRANKLIN AND KENG MENG NG

Abstract. We extend our work on difference randomness. Each component of a difference test is a Boolean combination of two r.e. open sets; here we consider tests in which the k^{th} component is a Boolean combination of $g(k)$ r.e. open sets for a given recursive function g. We use this method to produce an alternate characterization of weak Demuth randomness in terms of these tests and further show that a real is weakly Demuth random if and only if it is Martin-Löf random and cannot compute a strongly prompt r.e. set. We conclude with a study of related lowness notions and obtain as a corollary that lowness for balanced randomness is equivalent to being recursive.

1. Introduction

In [8], we introduced a new kind of randomness based on the difference hierarchy. We defined this notion, difference randomness, by altering the definition of a Martin-Löf test so each component of the test was the difference of two r.e. open sets in the Cantor space instead of simply a single r.e. open set. We showed that the difference random reals are precisely the Turing incomplete Martin-Löf random reals; since every Martin-Löf random real is Turing incomplete if and only if it cannot compute a complete extension of Peano arithmetic [15], this is a natural class of random reals to study. In particular, this class satisfies our intuition that a random real should not have high computational strength.

In this paper, we extend our analysis to a form of randomness defined by tests where the k^{th} component is not a difference of two r.e. open sets but is instead the Boolean combination of $g(k)$ many r.e. open sets for some recursive function g. We also discuss variants on weak Demuth randomness in which particular recursive bounds on the mind-change functions are used and describe the relationship of these variants to the g-change randomness notions. Then we give an alternate characterization of weak Demuth randomness in terms of ω-change randomness and show that weak Demuth randomness is equivalent to Martin-Löf randomness combined with the inability to compute a strongly prompt r.e. set. Strongly prompt r.e. sets were introduced by Diamondstone and Ng [4] as a natural way to strengthen the classical notion of prompt simplicity. They showed that strongly prompt r.e. sets are intimately related to cupping in the r.e. degrees, extending the study of promptness and cupping in the well-known paper of Ambos-Spies, Jockusch, Shore, and Soare [1]. The result we obtain in this paper is of the same sort as many other theorems on randomness notions, such as weak 2-randomness: a real is weakly 2-random if and only if it is Martin-Löf random and does not compute any promptly simple r.e. set (this follows from [10]). In contrast, each incomplete Δ^0_2 Martin-Löf random real computes a promptly simple r.e. set.

Date: October 4, 2012.

Both authors acknowledge the support of the Institute for Mathematical Sciences and Department of Mathematics of the National University of Singapore and the John Templeton Foundation.

Finally, we describe the lowness classes for these variants on weak Demuth randomness. We obtain as a corollary that there is no nonrecursive set which is low for balanced randomness.

1.1. **Notation and terminology.** In general, our notation follows that of [14]; for a basic discussion of randomness, we refer the reader to [6, 12]. We will work within the Cantor space equipped with the usual clopen topology. Given a finite binary string σ, we will write $[\sigma]$ for the basic open subset of the Cantor space formed by all the infinite extensions of σ. We will extend this notation to a subset U of $2^{<\omega}$ and write $[U]$ for $\cup_{\sigma \in U}[\sigma]$. We use the Lebesgue measure μ, where $\mu([\sigma]) = 2^{-|\sigma|}$. For ease of notation we will often write σ and U instead of $[\sigma]$ and $[U]$. All functions mentioned in this paper, unless otherwise stated, will be recursive functions from ω to ω.

We will be interested in the Borel subsets of 2^ω that are g-change for some recursive function g. A set $X \subseteq \omega$ is *g-change* if there is a recursive approximation to X whose mind-change function is bounded by g. More formally, we say that X is g-change if there is a recursive approximation $X(n, s)$ to X such that $\#\{s \mid X(n, s) \neq X(n, s+1)\} \leq g(n)$ for all n. We will write n-change for g-change when $g(n) = n$ for every n. We wish to extend this notion to subsets of 2^ω and consider sets of the form $\cap_i V_i$ where each V_i is $g(i)$-r.e. The most obvious way to define a set $V_i \subseteq 2^\omega$ to be k-r.e. is to require that $V_i = [W]$ where W is a k-r.e. set presenting V_i. Unfortunately, it is easy to see that for every $k > 1$ and every k-r.e. set W, there is a set $\widehat{W} \leq_T \emptyset'$ such that $[W] = [\widehat{W}]$ and vice versa. Therefore the randomness notion generated by looking at either "k-r.e. tests" or "g-r.e. tests" defined in this way coincides with 2-randomness.

To get around this problem, we observe that in our naive approach to k-r.e. tests, we had allowed for a neighbourhood $[\sigma]$ to be put into and removed from V_i unboundedly (even infinitely) many times. It is therefore necessary to consider the enumerability of neighbourhoods rather than the enumerability of the presenting set. As in [8], we will write $D(U, V)$ for $[U] - [V]$ when U and V are subsets of $2^{<\omega}$. For an n-element collection $U^1, U^2, \ldots, U^n$ of subsets of $2^{<\omega}$, we will write $D(U^1, U^2, \ldots, U^n)$ for $([U^1] - [U^2]) \cup ([U^3] - [U^4]) \cup \ldots \cup ([U^{n-1}] - [U^n])$ when n is even and $([U^1] - [U^2]) \cup ([U^3] - [U^4]) \cup \ldots \cup ([U^{n-2}] - [U^{n-1}]) \cup [U^n]$ when n is odd. We will sometimes simplify this definition for the purpose of our proofs and consider only the "even" case, padding with $U^{n+1} = \emptyset$ if n is actually odd.

This allows us to extend the notion of a Martin-Löf test to that of a test whose components are Boolean combinations of open sets. Most randomness notions are generated by tests whose components are open subsets of 2^ω; it is by varying the effectivity of the presentation of the tests that we get varying randomness notions. Here we consider a randomness notion where the test components are not open sets, but are instead Δ^0_2 subsets of 2^ω with recursively bounded mind-change functions.

Definition 1.1. Let f be a recursive function. We say that an *f-change test* is a sequence $\left\langle D(U_i^1, \ldots, U_i^{f(i)}) \right\rangle_{i \in \omega}$ such that $\mu(D(U_i^1, \ldots, U_i^{f(i)})) \leq 2^{-i}$ for all i and that a real A is *f-change random* if for all f-change tests $\left\langle D(U_i^1, \ldots, U_i^{f(i)}) \right\rangle_{i \in \omega}$, $A \notin \cap_i D(U_i^1, \ldots, U_i^{f(i)})$.

In [8], we considered n-change tests for a fixed n and found that n-change randomness was equivalent to difference randomness for every $n > 1$ (clearly, 1-change randomness is equivalent to Martin-Löf randomness). f-change randomness is a natural extension of difference randomness. We also consider an even stronger notion:

Definition 1.2. A real A is *ω-change random* if it is f-change random for every recursive function f.

In Section 2, we discuss f-change randomness for a fixed recursive f, and in Section 3, we discuss ω-change randomness and its relationships to other strong randomness notions, in particular, weak Demuth randomness. We also give a characterization of weakly Demuth random reals based on their low computational strength. In Section 4, we consider one of the corresponding lowness notions as well as lowness for balanced randomness.

2. f-change randomness

Throughout this section, we will take f to be recursive. We begin by observing that if the range of f is bounded, then f-change randomness is clearly equivalent to difference randomness. Therefore, it makes sense to only consider f-change randomness where f is an order function (i.e., a recursive nondecreasing function with unbounded range).

Remark 2.1. In [8], we noted that there is a standard form for n-change test:s we called an n-change test $\langle D(U_i^1, U_i^2, \dots, U_i^n)\rangle_{i\in\omega}$ *canonical* if each U_i^k was prefix-free and for every i, σ, and k such that $1 < k \leq n$ and $\sigma \in U_i^k$, there was a τ in U_i^{k-1} that is an initial segment of σ. This means that after the first element of a test component, we only "remove" (or "add") neighborhoods that we "added" (or "removed") in the previous element. We note without ceremony that the same form can be found for an f-change test for any f (see Lemma 2.5 of [8]).

There is one very important way in which f-change randomness differs from Martin-Löf randomness. When a Martin-Löf test is defined, the rate of decrease of the measure of the test components is always recursively bounded, usually by 2^{-k}; however, the precise recursive bound does not matter because we can always take a subsequence of a test if we would like this rate of decrease to be faster. However, the function bounding the rate of decrease of the measure of an f-change test is as important as the function f itself because the different components of an f-change test may have to satisfy different requirements. We can no longer be sure, for instance, that the seventeenth component of an f-change test can be the fifth component of an f-change test because it may be that the seventeenth component may have more than $f(5)$ mind changes. Therefore, we will restrict our attention to the tests whose k^{th} components have a measure bounded by 2^{-k} (the precise bound will not matter, but we fix this for convenience). This issue was also considered in Figueira, Hirschfeldt, Miller, Nies, and Ng [7] and is one of their main motivations for considering balanced randomness.

We begin by recalling the definition of weak Demuth randomness and then presenting our variant of it.

Definition 2.2. [3, 11] A *Demuth test* is a sequence $\langle W_{g(i))}\rangle_{i\in\omega}$ of r.e. open sets where g is an ω-r.e. function and $\mu(W_{g(i)}) \leq 2^{-i}$ for every i. A real A is *weakly Demuth random* (WDR) if for every Demuth test $\langle W_{g(i)}\rangle_{i\in\omega}$, $A \notin \cap_i W_{g(i)}$.

Definition 2.3. If h is a recursive function, an *h-Demuth test* is a sequence $\langle W_{g(i)}\rangle_{i\in\omega}$ of r.e. open sets where g is an ω-r.e. function with mind-change function bounded by h and $\mu(W_{g(i)}) \leq 2^{-i}$ for every i. A real A is *h-weakly Demuth random* (h-WDR) if for every h-Demuth test $\langle W_{g(i)}\rangle_{i\in\omega}$, $A \notin \cap_i W_{g(i)}$.

We note that A is weakly Demuth random if A is h-WDR for every recursive function h.

We consider generalizing weak Demuth randomness by considering f-WDR for various recursive functions f. If $f = o(2^n)$ is a recursive function, then it is easy to see that f-WDR is equivalent to Martin-Löf randomness: Each f-WDR test $\left\langle W_{g(i)} \right\rangle_{i\in\omega}$ is covered by the Martin-Löf test $\left\langle \cup_s W_{g(\hat{f}(i),s)} \right\rangle_{i\in\omega}$, where $\hat{f}(i)$ is the least number j such that $f(j) < 2^{j-i}$. Thus it makes sense to consider $2^n f(n)$-WDR for an arbitrary recursive (not necessarily unbounded) function f. We note that 2^n-WDR is the same as balanced randomness. We first show that different choices of f can give rise to different randomness notions. In fact, we can specify exactly how far apart f and g have to be in order for $2^n f(n)$-WDR and $2^n g(n)$-WDR to be different.

Theorem 2.4. *Suppose that f and g are recursive nondecreasing functions.*

(i) *If $\limsup_n |f(n) - g(n)| < \infty$, then $2^n f(n)$-WDR and $2^n g(n)$-WDR are the same.*
(ii) *If $\limsup_n (f(n) - g(n)) = \infty$, then there is an A so that A is $2^n g(n)$-WDR but not $2^n f(n)$-WDR.*

Proof. (i): We first observe that for any constant M, any recursive subsequence of an $M2^n$-Demuth test is covered by an $M2^n$-Demuth test. To see this, fix M, a strictly increasing recursive function u, and an $M2^n$-Demuth test $\left\langle W_{k(n)} \right\rangle_{n\in\omega}$. Define $g(n,s)$ by letting $W_{g(n,s)}$ copy (all the different versions of) $W_{k(u(n),s)}$ until a stage is found such that $k(u(n),s)$ has changed its mind $2^{u(n)-n}$ many times. We then change $g(n,s)$ to a new index and copy the next $2^{u(n)-n}$ many different versions of $W_{k(u(n),s)}$, and so on. Clearly $\left\langle W_{\lim_s g(n,s)} \right\rangle_{n\in\omega}$ is an $M2^n$-Demuth test, and for every n, $W_{k(u(n))} \subseteq W_{\lim_s g(n,s)}$.

Now assume that $\left\langle W_{k(n)} \right\rangle_{n\in\omega}$ is a $2^n f(n)$-Demuth test. Define the partial recursive function u by letting $u(n+1)$ be the first number u greater than $u(n)$ found such that $k(u)$ has $2^u(f(u)-1)$ many mind changes. Either u is partial, in which case $\left\langle W_{k(n)} \right\rangle_{n\in\omega}$ is covered by a $2^n(f(n)-1)$-Demuth test, or else u is total, in which case $\cap_n [W_{k(n)}] \subseteq \cap_n [W_{k(u(n))}]$. Furthermore, the latter can be viewed as a subsequence of some 2^n-Demuth test, and by the comments in the preceding paragraph, it is covered by a 2^n-Demuth test. The statement follows after sufficiently many iterations. We note that the statement holds even if f and g are bounded.

(ii): We fix some uniform enumeration of all $2^n g(n)$-Demuth tests. Let $\left\langle W_{k_e(n)} \right\rangle_{n\in\omega}$ be the e^{th} test in this enumeration. We begin by fixing a recursive sequence $\langle n_k \rangle_{k\in\omega}$ as follows. Assume we have defined n_k such that $f(n_k) > \sum_{i\leq k} g(n_i)$. Find the least number $m > n_k$ such that $f(m) > 4\sum_{i\leq k} g(n_i)$. Now choose $n_{k+1} > m$ large enough so that for every $m \geq i \geq n_k$,

$$f(n_k) > \frac{1}{1 - \frac{3}{2} 2^{i-n_{k+1}}} \sum_{i\leq k} g(n_i),$$

and that $f(n_{k+1}) > \sum_{i\leq k+1} g(n_i)$. The reason for our choice of $\langle n_k \rangle_{k\in\omega}$ will become clear later.

We will build the Demuth test $\langle U_k \rangle_{k\in\omega}$ and argue at the end that this is a $2^n f(n)$-Demuth test. Henceforth, we will write $G_e[s]$ for $W_{k_e(n_e,s)}[s]$ and say that G_e *switches version* at s if $k_e(n_e, s-1) \neq k_e(n_e, s)$. For each i, we let $k(i)$ be the largest k such that $n_k \leq i$.

During the construction, when we *update* U_i at stage s, we enumerate into U_i every string σ of length s extending some string in $\cap_{j<i}[U_j]$ such that $[\sigma] \cap \cup_{j\leq k(i)} [G_j[s]] = \emptyset$. If the measure of all such σ is greater than 2^{-i}, we put in the first 2^{-i} much of these σ (in the lexicographic ordering).

Construction of $\langle U_k \rangle_{k\in\omega}$. At stage $s = 0$, we update U_1. At a stage $s > 0$, we search for the least $i < s$ such that $\cap_{j\leq i}[U_j] \subseteq \cup_{j<s}[G_j[s]]$ and $\mu(U_i) = 2^{-i}$. Now we switch version for U_i and enumerate into the new version of U_i all the $[\sigma]$ contained in the old version such that $[\sigma] \subseteq \cap_{j<i}[U_j]$ and $[\sigma] \cap \cup_{j\leq k(i)}[G_j[s]] = \emptyset$. (The reason we have to do this is that these strings may be lexicographically very far to the right and may not be enumerated into the new version of U_i by the updating procedure.) Update $U_0, U_1, \cdots, U_s$. This ends the construction.

Verification. We assume each U_i changes version finitely often (this will be verified later). Hence for each i, $\mu(U_i) \leq 2^{-i}$. It is easy to see that each $[U_i]$ is clopen: when we build U_i, we are only considering finitely many G_js, which will each have a final, stable version. At the point when each of these G_js are stable, U_i will be stable as well by the nature of the construction. We now argue inductively that for each i, $\cap_{j\leq i}[U_j] \not\subseteq \cup_{j\in\omega}[G_j]$. This is certainly true for $i = 1$ because $U_1 = [0]$ which can never be covered by $\cup_{j<s}[G_j[s]]$ at any stage s. Assume this is true for i. If $\mu(U_i)$ is ever 2^{-i}, then $\cap_{j\leq i}[U_j] \not\subseteq \cup_{j\in\omega}[G_j]$ by compactness. Therefore, we may assume that $\mu(U_i) < 2^{-i}$ at almost every stage. Then $\cap_{j<i}[U_j] - [U_i]$ must be covered by the final version of $\cup_{j\leq k(i)}[G_j]$: otherwise, the construction would enumerate some suitably long extension of $\cap_{j<i}[U_j] - [U_i]$ into U_i. Thus by the induction hypothesis, $\cap_{j\leq i}[U_j] \not\subseteq \cup_{j\in\omega}[G_j]$, so for every i, $\cap_{j\leq i}[U_j] \not\subseteq \cup_{j\in\omega}[G_j]$. Then each $\cap_{j\leq i}[U_j]$ is clopen and nested, and by König's Lemma, $\cap_{i\in\omega}[U_i] \not\subseteq \cup_{j\in\omega}[G_j]$.

Now we only have to bound the number of version changes to each U_i. We argue that each U_i changes version at most $\varepsilon_i 2^i \sum_{j\leq k(i)} g(n_j)$ times, where $\varepsilon_i = \frac{1}{1-\frac{3}{2}2^{i-n_{k(i)+1}}}$ (it is easy to see that for every i, $\varepsilon_i \leq 4$). Fix $i \in \omega$, and let $t_0 < t_1$ be two consecutive stages where U_i has a version switch. Between t_0 and t_1, the strings enumerated in U_i have measure 2^{-i}. Let σ be enumerated in U_i at some stage t between t_0 and t_1. Certainly $[\sigma] \cap \cup_{j\leq k(i)}[G_j[t]] = \emptyset$. We argue that $[\sigma] \subseteq \chi_1 \cup \chi_2$, where

$$\chi_1 = \bigcup_{j\leq k(i), t<t'\leq t_1} [G_j[t']] \quad \text{and} \quad \chi_2 = \bigcup_{j<t_1}[G_j[t_1]].$$

Let $X \supset \sigma$. We work towards a contradiction and assume that $X \notin \chi_2$. Then between t and t_1 some U_j for $j < i$ must switch version, since otherwise $[\sigma] \subseteq \cap_{j\leq i}[U_j[t_1]] \subseteq \chi_2$. We now assume that $X \notin \chi_1$. Let $t' > t$ be the first stage where some $j' < i$ switches version. We have $X \in \cap_{j\leq j'}[U_j[t']] \subseteq \cup_{j<t'}[G_j[t']]$. This means that we must have $[X\restriction t'] \cap \cup_{j\leq k(j')}[G_j[t']] = \emptyset$, and so by construction we immediately enumerate $X\restriction t'$ in $U_{j'}$. This argument shows that $X \in \cap_{j\leq i}[U_j[t_1]] \subseteq \chi_2$, and we are done.

Now we estimate the measure of $\chi_2 - \chi_1$ from above. Clearly

$$\chi_2 - \chi_1 \subseteq \bigcup_{k(i)<j<t'} [G_j[t_1]] < 2^{-n_{k(i)+1}} + 2\cdot 2^{-n_{k(i)+2}}$$

Since by our choice of the sequence $\langle n_x \rangle_{x\in\omega}$, $n_{x+1} > n_x + 2$ holds for any x, it now follows by an easy calculation that the measure of $\chi_2 - \chi_1$ is at most $\frac{3}{2}2^{-n_{k(i)+1}}$. Therefore the measure of the reals in χ_1 is at least $2^{-i} - \frac{3}{2}2^{-n_{k(i)+1}} = 2^{-i}(1 - \frac{3}{2}2^{i-n_{k(i)+1}}) = \frac{1}{\varepsilon_i}2^{-i}$. But when X was enumerated in U_i, $X \notin \cup_{j\leq k(i)}[G_j[t]]$. Hence the total number of version changes for U_i is bounded by $\varepsilon_i 2^i \sum_{j\leq k(i)} g(n_j)$.

We now argue that our choice of $\langle n_k \rangle_{k\in\omega}$ guarantees that for every i, $f(i) > \varepsilon_i \sum_{j\leq k(i)} g(n_j)$, which will complete the proof of the theorem. To see this, fix k and an i such that $n_k \leq i < n_{k+1}$

(hence $k = k(i)$). If $i \leq m$ (in the choice of n_k), then $f(i) \geq f(n_k) > \varepsilon_i \sum_{i \leq k} g(n_i)$. If $i > m$, then $f(i) \geq f(m) > 4 \sum_{i \leq k} g(n_i) \geq \varepsilon_i \sum_{i \leq k} g(n_i)$. □

Theorem 2.4 says that the constant M does not matter in $M2^n$-WDR. That is, varying the constant in the definition for balanced randomness does not matter, a fact already pointed out in [7]. However for $O(2^n f(n))$-WDR, where f is an order function, the constant factor makes a difference:

Corollary 2.5. *For any order function f and constant $M \in \omega$, $M2^n f(n)$-WDR is strictly weaker than $(M+1)2^n f(n)$-WDR.*

Corollary 2.6. *For any order function f, 2^n-WDR is strictly weaker than $2^n f(n)$-WDR.*

Corollary 2.7. *There is no single order function f such that $2^n f(n)$-WDR randomness is the same as weak Demuth randomness.*

We now show a similar result for f-change randomness. In contrast to the situation for n-change randomness, different functions f may give rise to different classes of random reals.

Theorem 2.8. *Suppose that f and g are recursive nondecreasing functions.*

(i) *If $\limsup_n |f(n) - g(n)| < \infty$, then f-change randomness is the same as g-change randomness.*
(ii) *If $\limsup_n (f(n) - g(n)) = \infty$, then there is an A such that A is g-change but not f-change random.*

Proof. The proof of (i) is similar to that of Theorem 2.8 in [8]. Suppose there is a f-change test $\left\langle D(U_i^1, \ldots, U_i^{f(i)}) \right\rangle_i$ in canonical form such that $A \in \cap_i D(U_i^1, \ldots, U_i^{f(i)})$. Since we can pad the test with $\emptyset$, we may assume that for every i, $f(i)$ is an even number larger than 2. Let $\limsup_n |f(n) - g(n)| = k$. Let $V_i = \cup_{j>i+1} U_j^{f(j)-1}$ and $W_i = \cup_{j>i+1} U_j^{f(j)}$. Since $\mu(D(V_i, W_i)) \leq \mu\left(\cup_{j>i+1} D(U_j^{f(j)-1}, U_j^{f(j)})\right) < 2^{-i}$, it follows that $\langle D(V_i, W_i) \rangle_{i \in \omega}$ is a difference test. By canonicity, $A \notin W_i$ for every i. Therefore either A is in $D(U_i^1, \ldots, U_i^{f(i)-3}, U_i^{f(i)-2})$ for almost every i or $A \in D(V_i, W_i)$ for almost every i. Repeating this $\frac{k}{2}$ times yields that A is either not g-change random or not difference random.

The proof of (ii) is similar to that of Theorem 2.4(ii) and contains no new ideas, so we will simply sketch the proof. Suppose that we have an enumeration of the g-change tests. We will denote the k^{th} g-change test by $\mathcal{U}^k = \langle U_i^k \rangle_{i \in \omega}$. We will construct an f-change test $\langle V_i \rangle_{i \in \omega}$ and a real A such that $A \in \cap_i V_i$ but $A \notin U_j^k$ for some j for every k.

We begin by choosing a recursive sequence $\langle n_k \rangle_{k \in \omega}$ such that for every k, $f(n_k) > \sum_{i \leq k} g(n_k)$ and $n_k \geq 2k$. The general idea behind our construction is this: for every i and s, we will approximate an initial segment $a_{i,s}$ of A at stage s such that $a_{i,s} \subseteq a_{i+1,s}$ for every i and let $A = \cup_i \lim_s a_{i,s}$. We must keep $a_{i,s}$ out of $U_{n_k}^k$ for every k (henceforth, we will drop the superscript). At the same time, we will construct our V_is so for every i and s, $a_{i,s} \in V_{i,s}$.

Consider $i \geq n_0$. (For $i < n_0$, we will simply "hardwire" a_i to be $A \upharpoonright i$ and V_i to be $\{a_i\}$.) For each such i, we will choose a very large length ℓ_i that $a_{i,s}$ will have at every stage s. We initialize by defining $a_{i,0}$ to be 0^{ℓ_i} and $V_{i,0} = \{a_{i,0}\}$ for every i.

At each stage $s > 0$, we find the smallest $i \leq s$ such that $[a_{i,s}] \subseteq \cup_{n_k \leq s} [U_{n_k,s}]$. If there is no such i, we say that $a_{i,s} = a_{i,s-1}$ for every i and go on to stage $s+1$. We begin by identifying the set of strings σ such that $|\sigma| = \ell_i$ and $[\sigma] \notin \cup[U_{n_k,s}]$, that is, the set of candidates for $a_{i,s}$. We know this set must be nonempty because we have required that $n_k \geq 2k$ for every k, so $\mu(\cup_k[U_{n_k,s}]) \leq \frac{1}{2}$ (we can assume that at any stage s, $\mu([U_{n_k,s}]) \leq \frac{1}{2^{n_k}}$ by speeding up the enumerations of the sets that count "removals" from U_{n_k}). Now we choose the element of that set that has been chosen least often as $a_{i,s}$. If there is more than one that has been chosen the minimum number of times, we choose the lexicographically least such string.

Now that we have found $a_{i,s}$ such that $[a_{i,s}] \notin [U_{n_k,s}]$ for all $n_k \leq s$, we must ensure that $a_{i,s} \in V_{i,s}$ by adding $a_{i,s}$ to $V_{i,s}$. However, we must also ensure that the measure of $V_{i,s}$ is appropriately small.

If the smallest n_k such that $[a_{i,s-1}] \subseteq [U_{n_k,s}]$ is no bigger than i, we remove $a_{i,s-1}$ from V_i and add $a_{i,s}$ to it. Since there can only be $\sum_{i \leq k} g(n_k) < f(n_k) \leq f(i)$ stages where this happens, this will not prevent $\langle V_i \rangle_{i\in\omega}$ from being an f-change test.

On the other hand, if the smallest n_k such that $[a_{i,s-1}] \in [U_{n_k,s}]$ is larger than i, we cannot remove $a_{i,s-1}$ from V_i and be certain that $\langle V_i \rangle_{i\in\omega}$ will be an f-change test in the end. However, we can arrange for ℓ_i to be large enough that we can have as many different "versions" of a_i in V_i as we need to make sure that $a_i \notin U_{n_k}$ for any k.

Now we repeat this procedure for each $i \leq s$ such that $[a_{i,s}] \in \cup_{n_k \leq s}[U_{n_k,s}]$ in increasing order, making sure that $a_{j,s} \subseteq a_{j+1,s}$ for all j. For $i > s$, we simply let $a_{i,s} = a_{s,s} * 0^{\ell_i - \ell_s}$. □

The following two corollaries are immediate.

Corollary 2.9. *For any order function f, difference randomness is strictly weaker than f-change randomness.*

Corollary 2.10. *There is no single order function f such that f-change randomness is the same as ω-change randomness.*

We observe that although these notions are distinguishable, this does not result in a linear hierarchy: it is possible to have a real that is f-change random but not g-change random and another real that is g-change random but not f-change random.

3. ω-CHANGE RANDOMNESS

We begin by observing that for ω-change randomness, the rate of convergence of the measures of the components of the tests no longer matters: since failing to be ω-change random is equivalent to failing an f-change test for some f, we can simply convert our f-change test with rate of convergence p to a g-change test with some other rate of convergence q if necessary. However, if we do not specify a rate of convergence, it should be assumed that it is the standard 2^{-k} rate.

We now consider the way the class of ω-change random reals is related to other classes of random reals. We show that the natural extension of difference randomness to ω-change randomness coincides with a well-known existing notion of randomness—weak Demuth randomness. That is, we can interpret each f-change test where the test components are Δ^0_2 sets of reals as a g-Demuth test where the test components are open sets of reals and vice versa.

For each order function f, $O(f)$-change tests are significantly more powerful than $O(f)$-Demuth tests. We show that each f-Demuth test can be covered by a $2f$-change test, while each f-change test can only be covered by a $2^{i+1}f(i)$-Demuth test. This is because each $D(U,V)$ can pretend to cover all of 2^ω before finally settling down on a small subset.

Theorem 3.1. *Let f be an order function.*

(i) *Each $2f$-change random real is f-WDR.*
(ii) *Each $2^i f(i+1)$-WDR real is f-change random.*

Proof. (i): Let $\langle W_{g(i)} \rangle_{i \in \omega}$ be a f-Demuth test. Let

$$U_i^{2k} = \bigcup \{W_{g(i,s)} \mid \#\{t < s \mid g(i,t) \neq g(i,t+1)\} = k\},$$
$$U_i^{2k+1} = W_{g(i,s)}, \text{ where } \#\{t < s \mid g(i,t) \neq g(i,t+1)\} = k \text{ and } g(i,s) \neq g(i,s+1).$$

Then $\left\langle D(U_i^0, \ldots, U_i^{2f(i)}) \right\rangle_{i \in \omega}$ is a $2f$-change test covering $\langle W_{g(i)} \rangle_{i \in \omega}$.

(ii): Now we consider a canonical f-change-test $\left\langle D(U_i^1, \ldots, U_i^{f(i)}) \right\rangle_{i \in \omega}$. By padding, we may assume that $f(n)$ is even for every n. We fix i and describe how to get $W_{\lim_s g(i,s)}$ covering $G_i = D(U_i^1, \ldots, U_i^{f(i)})$. For $\sigma \in 2^{<\omega}$ and $s \in \omega$, we say that $\sigma \in G_{i,s}$ if there exists some odd k such that $[\sigma] \subseteq [U_{i,s}^k]$ and $[\sigma] \cap [U_{i,s}^{k+1}] = \emptyset$. We assume by the *s-m-n* Theorem that we are building W_m for an infinite recursive set of indices for m. By speeding up the enumerations for the U_i^ks, we can assume that for every i and s, $\mu(G_{i,s}) < 2^{-i}$. For each i, we reserve $2^i f(i+1)$ many indices $m_1, \ldots, m_{2^i f(i+1)}$ for building $W_{g(i)}$.

We start by letting $g(i,s)$ equal the first index m_1 and call this the first version. For the k^{th} version, we keep $g(i,s) = m_k$ and enumerate into W_{m_k} every string σ found such that $\sigma \in G_{i+1,s}$ until a stage $s_k > s_{k-1}$ is found such that $\mu(W_{m_k,s_k}) > 2^{-i}$. When this happens we move to the next index m_{k+1} and repeat the process.

It is clear that $[G_{i+1}] \subseteq [W_{\lim_s g(i,s)}]$ if the limit $\lim_s g(i,s)$ exists because $U_{i+1,s}^k$ is a finite set of neighborhoods for each k and s. Now we argue that we will not run out of indices m_k. We claim that for each k, if we find s_k, then $\mu\left(\cup_j \left(U_{i+1,s_k}^{2j+1} - U_{i+1,s_{k-1}}^{2j+1}\right)\right) \geq 2^{-i-1}$. To see this, we suppose not for a contradiction and fix a counterexample k. Then it is easy to see that at least 2^{-i-1} much measure of the strings in W_{m_k,s_k} must be in G_{i+1,s_k}, since any σ with $\sigma \in W_{m_k,s_k}$ and $\sigma \notin G_{i+1,s_k}$ must have some extension in $U_{i+1,s_k}^{2j+1} - U_{i+1,s_{k-1}}^{2j+1}$ for some j. This is a contradiction to our assumption that $\mu(G_{i+1,s}) < 2^{-i-1}$ for every s.

Now it is easy to see that the number of different indices we need is at most $2^i f(i+1)$. Once more, we suppose not. By a simple combinatorial argument, we see that there must be some σ where σ appears in $\cup_j \left(U_{i+1,s_k}^{2j+1} - U_{i+1,s_{k-1}}^{2j+1}\right)$ for at least $2^{-1} f(i+1)$ many different k. This is a contradiction. Hence $\langle W_{\lim_s g(i,s)} \rangle_{i \in \omega}$ is a $2^i f(i+1)$-Demuth test covering $\left\langle D(U_i^1, \ldots, U_i^{f(i)}) \right\rangle_{i \in \omega}$. □

From this we immediately get the equivalence of ω-change randomness and weak Demuth randomness.

Corollary 3.2. *For any real A, A is ω-change random if and only if A is weakly Demuth random.*

Next, we investigate the hypothesis that stronger randomness notions correlate with lower computational power. Many results, which we summarize in Table 1, support this hypothesis: several strong randomness notions have been characterized as Martin-Löf randomness together with a property asserting computational feebleness. The property we consider here is strong promptness.

Definition 3.3 (Diamondstond, Ng [4])**.** An r.e. set B is *strongly prompt* if there is an enumeration $\langle B_s \rangle_{s \in \omega}$ of B, an increasing recursive function $p : \omega \to \omega$, called the "promptness function," and an

Randomness notion	**Martin-Löf random and cannot compute...**
difference randomness	• $0'$ [8] • any PA-degree [15]
weak Demuth randomness	• any strongly prompt r.e. set (this paper)
Demuth randomness	• any r.e. set that is not strongly jump traceable [11]
weak 2-randomness	• any nonrecursive r.e. set [5] • any promptly simple r.e. set [10]
2-randomness	• any set that is low for Ω [13]

Table 1. Strong randomness notions and computational weakness

ω-r.e. function $g : \omega \to \omega$, such that the following holds:

$$|W_e| \geq g(e) \to (\exists x)(\exists s)[x \in W_{e,\text{ at } s} \wedge B_s{\upharpoonright}x \neq B_{p(s)}{\upharpoonright}x]. \tag{1}$$

Here each time some large x enters B, either B must permit promptly below x or $g(e, s)$ must increase. Hence the intuition behind Definition 3.3 is that a strongly prompt set has an r.e. enumeration where there is a recursive bound on the number of times a request for a prompt change can be denied. In contrast, an r.e. set of promptly simple degree can be viewed as having an r.e. enumeration where the number of times a request for a prompt change can be denied is finite. For more information we refer the reader to [4].

We noted above that being weakly 2-random can be characterized as being Martin-Löf random and computing no promptly simple r.e. set. Our next result is a pleasing analogue of this result. Since the proof makes heavy use of cost functions, we recall their definition for the reader:

Definition 3.4. [9, 12] A *monotone cost function* is a computable function $c : \omega \times \omega \to \mathbb{Q}^{\geq 0}$ such that for every n, the sequence $c(n, 0), c(n, 1), \ldots$ is nondecreasing and converges to a limit and for every s, the sequence $c(0, s), c(1, s), \ldots$ is nonincreasing. A monotone cost function c is *benign* if there is a computable function $g : \mathbb{Q}^{>0} \to \omega$ such that whenever $q \in \mathbb{Q}^{>0}$ and I is a set of pairwise disjoint intervals of ω such that $c(n, s) \geq q$ for all $[n, s) \in I$, $\#I \leq g(q)$.

Theorem 3.5. *A is weakly Demuth random if and only if A is Martin-Löf random and A does not compute a strongly prompt r.e. set.*

Proof. In this proof, to avoid confusion, we let V_e be the e^{th} r.e. set of strings and W_e be the e^{th} r.e. set. We first prove the easier direction. Assume that A is Martin-Löf random and not weakly Demuth random. We claim that there is a benign cost function $c(x, s)$ such that if B is a r.e. set obeying c, then $B \leq_T A$. Let $g(i, s)$ be a recursive function such that $A \in \cap_i [V_{g(i)}]$, where $g(i) = \lim_s g(i, s)$ with a recursively bounded number of mind changes. We define the recursive sequence b^i_s as follows. Initially, we set $b^i_0 = i$ for every i. At stage s, find the least $i < s$ such that $b^i_s < s$ and $g(i, s-1) \neq g(i, s)$. Set $b^{i+j}_s = s + j$ for every $j \geq 0$. Now let $c(x, s) = 2^{-i}$ for the largest i where $b^i_s \leq x$. It is easy to see that c is monotonic and benign.

Now take an r.e. set B obeying c. We claim that $B \leq_T A$. First define Z to contain $V_{g(j,s)}$ for every j and s such that x is enumerated into B at stage s and $c(x, s) = 2^{-j}$. Then, since B obeys c, Z is a Solovay test. We fix x and wait for a stage s such that $c(x, s) = 2^{-i}$ and $A \in \cap_{j \leq i} [V_{g(j,s)}]$. We claim that for almost every x, $x \in B$ if and only if $x \in B_s$. If this fails for x, then there must be a t such that $x \in B_t - B_s$. Let $j \leq i$ be such that $c(x,t) = 2^{-j}$. Hence we have $g(j,s) = g(j,t)$, which means that A is put into Z. Since A can extend only finitely many strings in Z, this means that our computation can fail for only finitely many x. Finally, if c is a benign cost function, then by Diamondstone and Ng there is a strongly prompt r.e. set obeying c [4]. This completes one direction.[1]

Now suppose that $B = \Gamma^A$ where B is strongly prompt via the enumeration $\langle B_s \rangle_{s\in\omega}$ as witnessed by the function $b(x) = \lim_s b(x,s)$ and the promptness function p. To utilize the strong promptness of B we will (uniformly) define an array of r.e. sets $U_{e,c}$. By the recursion theorem and the slowdown lemma, there is a recursive function q such that for all e and c, we have $W_{q(e,c)} = U_{e,c}$, and every element enumerated into $U_{e,c}$ appears strictly later in $W_{q(e,c)}$. For more information on the use of the recursion theorem here we refer the reader to [4].

Fix e. We describe a procedure that is uniform in e to build $V_{g(e)}$, which will be the e^{th} component of a Demuth test $\langle V_{g(e)} \rangle$ which catches A. To this end we assume that we are building $V_{m_1}, V_{m_2}, \ldots$. Let m be the current index. We define a nondecreasing sequence of numbers $\langle b_s \rangle$ and keep c as a parameter which initially starts off as $c = 1$. It will be incremented by one each time we get a prompt permission from $U_{e,c}$.

Initially we let $c = 1$ and $b_0 = b(q(e,c)) + 1$. At each stage s, we copy $\Gamma^{-1}(B_s \upharpoonright b_s) = \{\sigma \mid \Gamma^\sigma = B_s \upharpoonright b_s\}$ into V_m until we find that $\mu(V_m) \geq 2^{-e}$. If this happens, then we challenge $B \upharpoonright b_s$ to change by enumerating all elements less than b_s into $U_{e,c}$. We then wait for $b(q(e,c))$ to increase beyond b_s or for B to permit below b_s (one of the two must happen due to the recursion theorem and the fact that B is strongly prompt). If $b(q(e,c))$ increases, then we increase b_{s+1} to match $b(q(e,c)) + 1$ and go on to the next index for m. If B has permitted below b_s, we increment c by 1, set $b_{s+1} = b(q(e,c)) + 1$ for this new c, and go on to the next index for m.

Clearly, if we only use finitely many indices, then $\mu(V_{\lim_i m_i}) < 2^{-e}$ and $A \in [V_{\lim_i m_i}]$. It remains to verify that we use at most $\sum_{c \leq 2^e} \left[\tilde{b}(q(e,c)) + 1\right]$ many indices m, where $\tilde{b}(k)$ is the mind-change bound for $b(k)$. First observe that if V_m and $V_{m'}$ were assigned to copy Γ^{-1} under different values of c, then $[V_m] \cap [V_{m'}] = \emptyset$. Since we only abandon an index when $\mu(V_m) \geq 2^{-e}$, this means that c can be no larger than 2^e. Each time we abandon an index we either increment c or force an increase in $b(q(e,c))$ (since new values of b_s are picked to be larger than the current $b(q(e,c))$ value). Hence we get a recursive bound on the number of indices used. $\square$

Remark 3.6. We could have studied Δ^0_2-change randomness by requiring a real A to pass every f-change test for every total Δ^0_2 function f instead of only the recursive ones. To ensure that the tests are presentable by Boolean combinations of effective open sets instead of allowing the tests to be defined using access to an oracle f, we may consider each Δ^0_2-change test to be a recursive double sequence of r.e. open sets $\langle D(U^1_i, U^2_i, \ldots) \rangle_{i\in\omega}$ such that for every i and every $j > f(i)$, $U^j_i = \emptyset$. Of course, we also require the usual measure restriction $\mu(D(U^1_i, \ldots)) \leq 2^{-i}$ for all i. By the correspondence in Theorem 3.1 which can be easily generalized, we see that A is Δ^0_2-change random if and only if for every limit test $\langle W_{g(i)} \rangle_{i\in\omega}$, $A \notin \cap_i [W_{g(i)}]$. Here a limit test is identical to a Demuth test except that we allow $g \leq_T \emptyset'$. The latter notion is easily seen to be equivalent to weak 2-randomness. We note that a stronger notion called limit randomness was studied in

[1] The authors thank André Nies for pointing out that this proof can be presented using cost functions.

Barmpalias, Miller and Nies [2] and Kučera and Nies [11], where A is limit random if and only if for every limit test $\langle W_{g(i)} \rangle_{i\in\omega}$, $A \notin [W_{g(i)}]$ for almost every i.

4. Lowness

We now investigate the associated lowness notions. Recall that for randomness notions $\mathcal{C}$ and $\mathcal{D}$, the class $Low(\mathcal{C},\mathcal{D})$ is the class of all reals A such that every $\mathcal{C}$-random real is $\mathcal{D}$-random relative to A; that is, $\mathcal{C} \subseteq \mathcal{D}^A$. Every K-trivial is low for Martin-Löf randomness and hence in the class $Low(WDR, ML)$, while $Low(WDR, ML)$ is contained in the class $Low(W2R, ML)$. The work of Downey, Nies, Weber, and Yu shows that the class $Low(WDR, ML)$ is exactly the K-trivial sets [5].

We consider the corresponding lowness notions for f-WDR. For a fixed recursive nondecreasing function f, an f-Demuth test relative to A is a sequence $\left\langle W^A_{g^A(i)} \right\rangle_{i\in\omega}$ of A-r.e. open sets where g^A has an A-recursive approximation with mind-change function bounded by f and $\mu(W^A_{g^A(i)}) \leq 2^{-i}$ for every i. As usual, we define a real X to be f-WDR relative to A if it passes every f-Demuth test relative to A. A real A is *low for f-WDR* if every real that is f-WDR is f-WDR relative to A.

For each fixed recursive nondecreasing function f, every low for f-WDR is in $Low(WDR, ML)$ and hence K-trivial. If $f = o(2^n)$, then the sets that are low for f-WDR are exactly the K-trivial sets. We show that the class of sets that are low for $2^n f(n)$-WDR is the class of recursive sets.

Theorem 4.1. *Let f be a recursive nondecreasing (possibly bounded) function. If A is low for $2^n f(n)$-WDR, then A is recursive.*

Proof. By the remarks in the preceding paragraph, it is enough to show that A is of hyperimmune-free degree. We fix an arbitrary real A of hyperimmune degree and build a $2^n f(n)$-Demuth test relative to A which is not covered by any unrelativized $2^n f(n)$-Demuth test. We follow the proof of Theorem 2.4(ii). Fix an A-recursive function F which is not dominated by any recursive function and a uniform enumeration of all unrelativized $2^n f(n)$-Demuth tests. Let $\langle W_{k_e(n)} \rangle_{n\in\omega}$ be the e^{th} test in this enumeration. During the construction, we will approximate the sequence $\langle n_k \rangle_{k\in\omega}$ by $\langle n_{k,s} \rangle_{k,s\in\omega}$. We ensure that for every k and s, $n_{k+1,s} > n_{k,s}$ and $n_{k,s} \leq n_{k,s+1}$. At stage s, to *redefine* n_k means to reset the values of n_j for $j \geq k$. To do this, we assume that n_j has been (re)defined for $j \geq k-1$ and find the least number $m > n_j$ such that $f(m) > 5\sum_{i\leq j} f(n_i)$. Now we choose $n_{j+1} > \max\{m, s\}$ large enough so that for every $m \geq i \geq n_j$,

$$\frac{1}{1 - \frac{3}{2} 2^{i-n_{j+1}}} + 2^{-i+1} < \frac{3}{2}$$

and $f(n_{j+1}) > \frac{3}{4}\sum_{i\leq j+1} f(n_i)$. Hence this action moves (or lifts) the markers n_{k+j} beyond s for every $j \in \omega$ and spreads them out sufficiently sparsely. Finally, we speed up the construction until stage $F(s)$.

We will write $G_e[s]$ for $W_{k_e(n_{e,s},s)}[s]$ and say that G_e changes version at s if $k_e(n_{e,s}, s-1) \neq k_e(n_{e,s}, s)$ and $n_{e,s-1} = n_{e,s}$. For each i, we let $k(i,s)$ be the largest k such that $n_{k,s} \leq i$. We will not mention s where it causes no confusion. We build the A-relative Demuth test $\langle U_k \rangle_{k\in\omega}$ and argue at the end that this is a $2^n f(n)$-Demuth test relative to A.

As before, when we *update* U_i during the construction, we enumerate into U_i every string σ of length s extending some string in $\cap_{j<i} U_j$ such that $[\sigma] \cap \cup_{j\leq k(i)} [G_j] = \emptyset$. If the measure of all such σ is greater than 2^{-i} we put in the first 2^{-i} much σ in the lexicographic ordering.

Construction of $\langle U_k\rangle_{k\in\omega}$. At stage $s=0$, we update U_0. At a stage $s>0$, we find the least $j<s$ such that G_j has changed version exactly $2^{n_j-1}f(n_j)$ times and the final change took place strictly after n_j was last moved; that is, s is the least such that $\#\{t<s \mid k_j(n_{j,s},t-1)\neq k_j(n_{j,s},t)\} = 2^{n_j-1}f(n_j)$ and n_j was not moved at s. Redefine n_j and then search for the least $i<s$ such that $\cap_{j\leq i}[U_j]\subseteq \cup_{j<s}[G_j[s]]$ and $\mu(U_i)=2^{-i}$. Switch version for U_i and enumerate into the new version of U_i all the $[\sigma]$ contained in the old version such that $[\sigma]\subseteq\cap_{j<i}[U_j]$ and $[\sigma]\cap\cup_{j\leq k(i)}[G_j[s]]=\emptyset$. Update $U_0,U_1,\cdots,U_s$. This ends the construction.

Verification. First we argue that each n_j is moved finitely often. Suppose n_j is moved infinitely often and that this movement takes place at the stages $s_1<s_2<\cdots$. We may assume that $n_0,\cdots,n_{j-1}$ are never moved after s_1. For each i, after n_j is moved at s_i, we must have that $\#\{t<F(s_i)\mid k_j(n_{j,s_i},t-1)\neq k_j(n_{j,s_i},t)\}<2^{n_j-1}f(n_j)$ because otherwise n_j cannot be moved again. Since s_{i+1} has to be the first stage larger than s_i such that $k_j(n_j)$ changes its mind exactly $2^{n_j-1}f(n_j)$ times, from s_i we can compute s_{i+1} and hence the next value of n_j. This can be done without knowledge of F or the construction. Therefore $\langle s_i\rangle_{i\in\omega}$ is a recursive sequence dominating $F(s_i)$ and hence $F(i)$, which results in a contradiction.

We assume each U_i changes version finitely often (this will be verified later). By the same reasoning as in the proof of Theorem 2.4(ii), we have for each i, $\mu(U_i)\leq 2^{-i}$, $[U_i]$ is clopen and $\cap_{i\in\omega}[U_i]\not\subseteq\cup_{j\in\omega}[G_j]$.

Again it remains to bound the number of version changes to each U_i. We argue that each U_i changes version at most $\left(\varepsilon_i 2^{i-1}+1\right)\sum_{j\leq k(i,i)}f(n_{j,i})$ times, where $\varepsilon_i=\frac{1}{1-\frac{3}{2}2^{i-n_{k(i,i)+1}}}$. Note that we only begin building U_i at stage i. Again we have $\varepsilon_i\leq 4$. Fix $i\in\omega$ and let $t_0<t_1$ be two consecutive stages where U_i has a version switch and assume that no n_k below i is moved between t_0 and t_1. The same argument as before (in the proof of Theorem 2.4(ii)) shows that the strings enumerated in U_i between t_0 and t_1 is covered by $\chi_1\cup\chi_2$, where χ_1 and χ_2 are defined exactly as before.

Now the measure of $\chi_2-\chi_1$ is at most $\frac{3}{2}2^{-n_{k(i,t_0)+1}}$. Note that $n_{k(i,t_0)+1}\geq n_{k(i,i)+1}$. Therefore the measure of the set of reals X in χ_1 is at least $2^{-i}-\frac{3}{2}2^{-n_{k(i,t_0)+1}}=2^{-i}(1-\frac{3}{2}2^{i-n_{k(i,t_0)+1}})\geq\frac{1}{\varepsilon_i}2^{-i}$. But when X was enumerated in U_i, $X\notin\cup_{j\leq k(i,t_0)}[G_j[t]]$. Each G_j can change version at most $2^{n_j-1}f(n_j)$ times before it is redefined and removed from the calculation. Hence the total number of version changes for U_i is bounded by $\varepsilon_i 2^{i-1}\sum_{j\leq k(i,i)}f(n_{j,i})$. This calculation did not include those stages $[t_0,t_1]$ where some n_k below i was moved. There are at most $k(i,i)\leq\sum_{j\leq k(i,i)}f(n_{j,i})$ many of these stages. Adding these, we get the promised upper bound of $\left(\varepsilon_i 2^{i-1}+1\right)\sum_{j\leq k(i,i)}f(n_{j,i})$.

We now argue that our choice of $\langle n_k\rangle$ guarantees that for almost every i, $f(i)>\left(\frac{\varepsilon_i}{2}+2^{-i}\right)\sum_{j\leq k(i,i)}f(n_{j,i})$, which will complete the proof of the theorem. To see this, fix k and i such that $n_k\leq i<n_{k+1}$ (hence $k=k(i,i)$) at the largest stage less than i where $k(i,i)$ was redefined. If $i\leq m$ (in the choice of n_k), then $f(i)\geq f(n_{k,i})>\frac{3}{4}\sum_{j\leq k}f(n_{j,i})$. It is easy to see that $\frac{3}{4}>\frac{\varepsilon_i}{2}+2^{-i}$. On the other hand, if $i>m$, then $f(i)\geq f(m)>5\sum_{j\leq k}f(n_{j,i})\geq\left(\frac{\varepsilon_i}{2}+2^{-i}\right)\sum_{j\leq k}f(n_{j,i})$. □

As a corollary we obtain that there no nonrecursive real that is low for balanced randomness. Recall that a real is balanced random if it passes every balanced test; i.e., every sequence $\langle W_{f(m)}\rangle_{m\in\omega}$ of r.e. sets such that f is a 2^n-change function and $\mu([W_{f(m)}])\leq 2^{-m}$ for every m [7].

Corollary 4.2. *Every real that is low for balanced randomness is recursive.*

5. Questions

At this point, we have only analyzed the differences between f-change randomness and g-change randomness at the level of individual reals. It is now natural to ask when, if at all, these notions can be separated at the level of degrees as well and, if so, for which type of degree.

Question 5.1. If a Turing degree $\mathbf{a}$ contains a difference random real, does it contain an f-change random real for every recursive function f? More generally, if there is an f-change random real that is not g-change random, is there a Turing degree $\mathbf{a}$ that contains an f-change random real but not a g-change random real?

Question 5.2. If it turns out that the answer to the second part of Question 5.1 is negative, is there a weak truth table degree or a truth table degree for which the answer is positive?

References

[1] K. Ambos-Spies, C. Jockusch, R. Shore, and R. Soare. An algebraic decomposition of recursively enumerable degrees and the coincidence of several degree classes with the promptly simple degrees. *Transactions of the American Mathematical Society*, 281:109–128, 1984.

[2] George Barmpalias, Joseph S. Miller, and André Nies. Randomness notions and partial relativization. *Israel J. Math.*, 191(2):791–816, 2012.

[3] Osvald Demuth. On some classes of arithmetical real numbers. *Comment. Math. Univ. Carolin.*, 23:453–465, 1982. In Russian.

[4] David Diamondstone and Keng Meng Ng. Strengthening prompt simplicity. *J. Symbolic Logic*, 76(3):946–972, 2011.

[5] Rod Downey, André Nies, Rebecca Weber, and Liang Yu. Lowness and Π^0_2 nullsets. *J. Symbolic Logic*, 71(3):1044–1052, 2006.

[6] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic Randomness and Complexity.* Springer, 2010.

[7] Santiago Figueira, Denis Hirschfeldt, Joseph S. Miller, Keng Meng Ng, and André Nies. Counting the changes of random Δ^0_2 sets. In *Programs, proofs, processes*, volume 6158 of *Lecture Notes in Comput. Sci.*, pages 162–171. Springer, Berlin, 2010.

[8] Johanna N.Y. Franklin and Keng Meng Ng. Difference randomness. *Proc. Amer. Math. Soc.*, 139(1):345–360, 2011.

[9] Noam Greenberg and André Nies. Benign cost functions and lowness properties. *J. Symbolic Logic*, 76(1):289–312, 2011.

[10] D. Hirschfeldt and J. Miller. Unpublished.

[11] Antonín Kučera and André Nies. Demuth randomness and computational complexity. *Ann. Pure Appl. Logic*, 162(7):504–513, 2011.

[12] André Nies. *Computability and Randomness.* Clarendon Press, Oxford, 2009.

[13] André Nies, Frank Stephan, and Sebastiaan A. Terwijn. Randomness, relativization and Turing degrees. *J. Symbolic Logic*, 70(2):515–535, 2005.

[14] Robert I. Soare. *Recursively Enumerable Sets and Degrees.* Perspectives in Mathematical Logic. Springer-Verlag, 1987.

[15] Frank Stephan. Martin-Löf Random and PA-complete Sets. Technical Report 58, Matematisches Institut, Universität Heidelberg, Heidelberg, 2002.

Department of Mathematics, 196 Auditorium Road, University of Connecticut, U-3009, Storrs, CT 06269-3009, USA

E-mail address: `johanna.franklin@uconn.edu`

School of Physical & Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore

E-mail address: `kmng@ntu.edu.sg`