DIFFERENCE RANDOMNESS

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ABSTRACT. In this paper, we define new notions of randomness based on the difference hierarchy. We consider various ways in which a real can avoid all effectively given tests consisting of n-r.e. sets for some given n. In each case, the n-r.e. randomness hierarchy collapses for $n \geq 2$. In one case, we call the resulting notion difference randomness and show that it results in a class of random reals that is a strict subclass of the Martin-Löf random reals and a proper superclass of both the Demuth random and weakly 2-random reals. In particular, we are able to characterize the difference random reals as the Turing incomplete Martin-Löf random reals. We also provide a martingale characterization for difference randomness.

1. INTRODUCTION

The most commonly studied randomness notion, Martin-Löf randomness, is defined in terms of avoidance of null sets created by a uniformly r.e. sequence of sets of finite binary strings $\langle V_i \rangle_{i \in \omega}$ such that the Lebesgue measure of $[V_i]$ is no more than 2^{-i} for each i. In this paper, we consider some possible ways of defining a randomness notion in which each set in the sequence is defined in an n-r.e. way for some fixed n instead of simply an r.e. way.

The first part of this paper is devoted to an investigation of the ways in which n-r.e. randomness can be defined. In Section 2, we discuss two ways in which an element of such a sequence can be considered to be n-r.e. In one case, the "n-r.e. random" reals are precisely those that are random in another, established sense; in the other, a new class of reals is produced. In fact, in the latter case, the hierarchy collapses for $n \geq 2$, and these reals form a proper subclass of the Martin-Löf random reals and a proper superclass of both the Demuth random reals and the weakly 2-random reals.

In Section 3, we identify the precise subclass of Martin-Löf random reals that corresponds to this class. It turns out to be a very natural class of the Martin-Löf random reals: those that do not Turing compute $\emptyset'$. We then discuss weakness with respect to this notion of randomness (and, thus, with respect to that subclass of the Martin-Löf random reals) in Section 4. In Section 5, we investigate possible characterizations of the n-r.e. random reals in terms of martingales.

Our notation is standard and generally follows Soare [19] and Odifreddi [15, 16]. For a general overview of randomness, we refer the reader to Downey and Hirschfeldt

[3] and Nies [13]. We will work within the Cantor space, 2^ω, and refer to its elements as reals. When we discuss the measure of a set A in this space, we will always mean the Lebesgue measure, and we will denote it by $\mu(A)$. The basic open set generated by a finite binary string σ is denoted by $[\sigma] = \{X \in 2^\omega \mid X \supset \sigma\}$. If S is a set of finite binary strings, we define $[S] = \cup\{[\sigma] : \sigma \in S\}$. When the context is clear, we will not distinguish between a set of strings S and the open set of reals $[S]$ that it generates. The length of a finite binary string σ will be denoted by $|\sigma|$.

We begin by recalling Martin-Löf's original definition of Martin-Löf randomness from [10].

Definition 1.1. A Martin-Löf test is a uniformly r.e. sequence $\langle V_i \rangle_{i \in \omega}$ of subsets of $2^{<\omega}$ such that $\mu(V_i) \le 2^{-i}$ for every i, and we say that a real A passes a test $\langle V_i \rangle_{i \in \omega}$ if $A \notin \cap_i V_i$. A real is said to be Martin-Löf random if it passes every Martin-Löf test.

When we define Martin-Löf randomness in this way, we are presenting the Martin-Löf random reals as the reals that pass all reasonable statistical tests in the form of effectively presented null sets. Here, the "effective presentation" is the uniform sequence of r.e. sets defining the null set.

Other, stronger randomness notions have been studied. To get such a notion, we place weaker effective conditions on the presentation of the tests. In particular, we will consider Demuth randomness and weak 2-randomness. The reals that are Demuth random and the reals that are weakly 2-random are all Martin-Löf random, though they form proper superclasses of the 2-random reals.

Demuth randomness, like Martin-Löf randomness, is defined in terms of tests. We maintain the Martin-Löf requirements on the measure and composition of the elements of the test, but we relax the uniformity condition on the indices of the elements of the tests and the conditions for passing a Demuth test. We say that a real *Solovay-passes* a test $\langle V_i \rangle_{i \in \omega}$ if it is contained in V_i for only finitely many i.

Definition 1.2. [1] A Demuth test is a sequence $\langle V_i \rangle_{i \in \omega}$ of r.e. open sets such that $V_i = W_{f(i)}$ for some ω-r.e. function f and $\mu(V_i) \le 2^{-i}$ for every i. A real A is Demuth random if it Solovay-passes every Demuth test.

Now we consider the notion of weak 2-randomness, introduced by Kurtz. In [9], Kurtz originally defined this notion in terms of Σ_2^0 classes: a real is weakly 2-random if it is not contained in any Σ_2^0 class of measure 1. The definition below, which is much more similar to the standard definitions of Demuth and Martin-Löf randomness, was demonstrated to be equivalent to the original definition by Wang in [22].

Definition 1.3. A generalized Martin-Löf test is a recursive sequence of r.e. open sets $\langle V_i \rangle_{i \in \omega}$ such that $V_i \supseteq V_{i+1}$ for all i and $\lim_i \mu(V_i) = 0$. A real is weakly 2-random if it passes all generalized Martin-Löf tests.

In short, a generalized Martin-Löf test is a Martin-Löf test whose rate of convergence is not fixed. By an observation of Hirschfeldt and Miller, a weakly 2-random string Z forms a minimal pair with $\emptyset'$ [13]. Hence a weakly 2-random real cannot be approximated in a Δ_2^0 way (in fact, not even in a Σ_2^0 way).

In this paper, we will introduce and explore a new randomness notion defined by weakening the Martin-Löf condition that each component of a test be r.e. However, our requirements will still be stronger than the requirements for Demuth and

weak 2-randomness. To do this, we will consider the higher levels of the difference hierarchy, introduced by Ershov in [5]. This can also be traced back to Putnam [17].

Definition 1.4. Let $n \in \omega$. A set X is n-r.e. if there is a recursive function $f : \omega^2 \to \{0, 1\}$ such that the following three conditions hold for all k.

(1) $f(k, 0) = 0$.
(2) $X(k) = \lim_s f(k, s)$.
(3) $|\{s \mid f(k, s + 1) \neq f(k, s)\}| \leq n$.

In other words, a set X is n-r.e. if it is defined by a recursive function that may change its mind about any element's membership in X up to and including n times. This is a very natural approach to take to consider variations on Martin-Löf randomness. The randomness class defined using this approach has turned out to be connected to classical Martin-Löf randomness in interesting ways that capture precisely the interactions between Martin-Löf randomness and computational strength, and the corresponding lowness notions have also turned out to be related to certain well-studied subclasses of the K-trivials defined in terms of cupping and coverability, which we will mention below.

Several recent results have indicated that the Turing incomplete Martin-Löf random reals is the right class of random reals to study. Stephan proved that any Martin-Löf random degree that is also a PA degree must Turing compute $\emptyset'$ [21]. (Recall that a degree is PA if it computes a complete extension of Peano arithmetic.) This result shows that there are only two kinds of Martin-Löf random reals. The first kind is computationally powerful enough to compute the halting problem, and the second kind is computationally weak in that these reals fail to compute a complete extension of PA. Since we expect randomness to be antithetical to computational strength, we would like to have a randomness notion in which the random reals are not very useful when used as oracles. For instance, weakly 2-random reals contain no common information with the halting problem. Stephan's result showed a certain dichotomy in Martin-Löf randomness: if we eliminate the Martin-Löf random reals of the first kind, then we get a subclass of the Martin-Löf random reals which obey our intuition above. In Theorem 3.1, we show that a real is difference random if and only if it is Turing incomplete and Martin-Löf random. In particular, our result shows that the Turing incomplete Martin-Löf random reals are precisely the reals that are random with respect to a natural notion.

We will also consider the lowness notions associated with difference randomness. Surprisingly, we can find yet another relationship with Martin-Löf randomness—this time from the point of view of K-triviality and lowness for Martin-Löf randomness. Recall that a set A is K-trivial if for some constant c, we have $K(A \restriction n) \leq n + c$ for every n, where $K(\sigma)$ denotes the prefix-free Kolmogorov complexity of σ for the binary string σ. There has been an extensive study of K-triviality and various subclass of K-triviality in the literature; we refer the reader to [13] for more details. We mention two related classes. Recall that a set A is *Martin-Löf coverable* if there is a Martin-Löf random real $Z \geq_T A$ such that $\emptyset' \nleq_T Z$ and that a set A is *weakly Martin-Löf cuppable* if there is a Martin-Löf random real Z such that $\emptyset' \nleq_T Z$ and $A \oplus Z \geq_T \emptyset'$. Hirschfeldt, Nies and Stephan showed that every Martin-Löf coverable r.e. set is K-trivial [6]. It also follows from the work of Downey, Hirschfeldt, Miller and Nies that every r.e. set that is not weakly Martin-Löf cuppable is K-trivial [2].

The question of whether either of these two notions is equivalent to K-triviality is still open [11] and seems to be a difficult problem.

In this paper, we contribute to the understanding of these two subclasses of the r.e. K-trivial reals. In Theorem 4.1, we show that an r.e. set A is Martin-Löf coverable if and only if it is a base for difference randomness (that is, $A \leq_T Z$ for some Z that is difference random relative to A). In Theorem 4.2, we show that an r.e. set A is weakly Martin-Löf cuppable if and only if it is low for difference randomness (that is, every difference random real is difference random relative to A). Hence the two aforementioned subclasses of the r.e. K-trivials defined using degree-theoretic notions can actually be expressed in terms of lowness properties for difference randomness.

Finally, in Theorem 5.1, we provide a characterization of difference randomness in terms of the reals which fail to win against a certain class of martingales. We call this class of martingales *difference martingales*. This points to a certain robustness in the definition of difference randomness.

2. Formalizing n-r.e. randomness

We want to formalize the intuition that an n-r.e. test is a sequence of sets $V_i \subset 2^{<\omega}$ with measure effectively vanishing to 0 such that the sequence $\langle V_i \rangle_{i \in \omega}$ is uniformly n-r.e. The issue is this: should the level of complexity of the set V_i refer to the number of times that a given *string* can be admitted to or removed from V_i or to the number of times that a given clopen *neighborhood* be admitted to or removed from $[V_i]$? In the case of a Martin-Löf test, this distinction is not necessary since strings (and therefore the corresponding clopen sets) are enumerated but never removed.

However, these notions differ for n-r.e. tests whenever $n \geq 2$. For example, suppose that the string 01 enters and then exits V_1, where $\langle V_i \rangle_{i \in \omega}$ is a d.r.e. test. If V_1 is d.r.e. with respect to neighborhoods, no subneighborhood of [01] will be contained in V_1. However, if V_1 is d.r.e. with respect to strings, this is possible: perhaps 0100, 0101, and 011 will also enter V_1 and never exit, so $[0100] \cup [0101] \cup [011] = [01]$ will be contained in V_1 after all.

We first show that the naive approach, that of simply requiring the set of strings defining each V_i to be n-r.e., does not produce a new class of random reals. We define this approach formally as follows.

Definition 2.1. For $n \geq 1$, a *naive n-r.e. test* is a uniform sequence $\langle V_i \rangle_{i \in \omega}$ of sets of finite binary strings such that for every i, $\mu(V_i) \leq 2^{-i}$ and (the set of code numbers for the strings in) V_i is n-r.e.

We note that the $(n+1)$-r.e. random reals are a subclass of the n-r.e. random reals for every n by definition. We now show that it is not a proper subclass unless $n = 1$; that is, that the hierarchy of naive n-r.e. randomness collapses for $n \geq 2$. In fact, it gives rise to a well-known notion. Recall that a real is 2-random if and only if it is Martin-Löf random with respect to $\emptyset'$.

Proposition 2.2. *The following statements are equivalent for a real A when $n \geq 2$.*

(1) *For every naive n-r.e. test $\langle V_i \rangle_{i \in \omega}$, $A \notin \cap_i V_i$.*
(2) *For every naive n-r.e. test $\langle V_i \rangle_{i \in \omega}$, $A \in V_i$ for only finitely many i.*
(3) *A is 2-random.*

Proof. Since every n-r.e. set is recursive in $\emptyset'$, the only non-trivial direction is (1) implies (3). We let $\langle U_i^{\emptyset'} \rangle_{i \in \omega}$ be a universal Martin-Löf test relative to $\emptyset'$ and construct a naive d.r.e. test $\langle V_i \rangle_{i \in \omega}$ such that for every i, $[V_i] = [U_i^{\emptyset'}]$. We fix an enumeration $\langle \emptyset'_s \rangle_{s \in \omega}$ of $\emptyset'$ and let $\langle U_{i,s}^{\emptyset'_s} \rangle_{s \in \omega}$ be a Σ_2^0 enumeration of $U_i^{\emptyset'}$. We may assume that $U_i^{\emptyset'}$ is prefix free. By the hat trick and by speeding up the approximation, we may also assume that $U_{i,s}^{\emptyset'_s}$ is prefix free for each s.

We define $V_{i,0}$ to be $\emptyset$. At a stage $s > 0$, for each $\sigma \in U_{i,s}^{\emptyset'_s}$, we let $t \leq s$ be the least such that $t \geq |\sigma|$ and for every $t \leq t' \leq s$, $\sigma \in U_{i,t'}^{\emptyset'_{t'}}$. Add every extension $\tau \supset \sigma$ where $|\tau| = t$ to $V_{i,s}$. That is, once we observe that $\sigma \in U_{i,s}^{\emptyset'_s}$, we add all extensions of σ of length s to V_i. These extensions stay in V_i until some stage $t > s$ in which σ leaves $U_{i,t}^{\emptyset'_t}$, at which point we remove all the extensions of σ of length s from $V_{i,t}$. It is easy to see, using the fact that each $U_{i,s}^{\emptyset'_s}$ is prefix-free, that $\langle V_{i,s} \rangle_{i \in \omega}$ is a d.r.e. approximation to the set $\cup_s V_{i,s}$ (as sets of strings). It is also easy to verify that $[V_i] = [U_i^{\emptyset'}]$ for every i. $\square$

Therefore, the same theorems hold for the naively d.r.e. random reals (and thus the naively n-r.e. random reals for every $n \geq 2$) that hold for the 2-random reals. For instance, every real A that is naively d.r.e. random is in GL_1 [7] (i.e., $A' \leq_T A \oplus \emptyset'$) and is even low for Ω [14] (i.e., Ω is Martin-Löf random relative to A, where Ω is the halting probability of some fixed universal prefix-free machine).

Since the randomness notions based on n-r.e. sets of strings do not generate a new class, we now define a type of randomness in which we restrict the number of times a clopen neighborhood may enter or exit a test instead of a string. To avoid confusion, for sets $U, V \subseteq 2^{<\omega}$, we will write $D(U, V)$ to denote the set $[U] - [V]$. That is, a real $X \in D(U, V)$ iff $X \supset \sigma$ for some $\sigma \in U$ and $X \not\supset \tau$ for every $\tau \in V$. For a finite sequence of sets $U_1, U_2, \cdots, U_n \subseteq 2^{<\omega}$, we will simplify our notation and write $D(U_1, U_2, \cdots, U_n)$ for $D(U_1, U_2) \cup D(U_3, U_4) \cup \cdots \cup D(U_{n-2}, U_{n-1}) \cup [U_n]$ if n is odd and $D(U_1, U_2, \cdots, U_n)$ for $D(U_1, U_2) \cup D(U_3, U_4) \cup \cdots \cup D(U_{n-1}, U_n)$ if n is even.

Definition 2.3. Let $n \geq 1$. An *n-r.e. test* is a sequence $\langle D(W_{g_1(i)}, \cdots, W_{g_n(i)}) \rangle_{i \in \omega}$ where $g_1, \cdots, g_n$ are recursive functions and $\mu(D(W_{g_1(i)}, \cdots, W_{g_n(i)})) \leq 2^{-i}$ for every i. We will say that a real $A \in 2^\omega$ is *n-r.e. random* if for every n-r.e. test $\langle U_i \rangle_{i \in \omega}$, $A \notin \cap_i U_i$. If $n = 2$, we will call these reals *d.r.e. random*.

For instance, if $D(V_0, V_1)$ is a component of a d.r.e. test, then V_0 represents the class of clopen sets that we wish to put into our test and V_1 represents the class of clopen sets that we wish to remove from our test. If σ enters $V_{0,s}$ and some $\tau \supseteq \sigma$ enters $V_{1,t}$ for some $t > s$, then we can think of $[\tau]$ as being first enumerated at stage s and then removed at stage t. In a 3-r.e. test $D(V_0, V_1) \cup [V_2]$, we will allow extensions η of τ to be enumerated into V_2. This means that $[\eta]$ is first enumerated into our test, then removed, and then finally put back into our test.

We note that a component of a d.r.e. test is of the form $\mathcal{C} \cap \mathcal{D}$ for a Σ_1^0 class $\mathcal{C}$ and a Π_1^0 class $\mathcal{D}$. In the case of Martin-Löf tests, the components are Σ_1^0 classes. Thus, if we consider the descriptional complexity of the test components, this is the most natural way of generalizing Martin-Löf tests.

We note that replacing "r.e." with "co-r.e." gives us nothing new. In the case where $n = 1$, a 1-co-r.e. test is simply a uniform sequence of Π_1^0-classes with measure effectively shrinking to 0, so the resulting randomness notion is simply weak randomness. For $n > 1$, note that if D is a co-r.e. set of strings, then $[D] = 2^\omega - [U]$ for some r.e. set of strings U (and vice versa). Hence if X is a member of a n-co-r.e. test, then X is a member of an $(n + 1)$-r.e. test, which can in turn be covered by a d.r.e. test (by Theorem 2.8, where we show that the n-r.e. randomness hierarchy collapses). The latter can be expressed as an n-co-r.e. test, so we will be able to see that n-co-r.e. randomness is equivalent to d.r.e. randomness for every $n > 1$.

We now describe a normal form for an n-r.e. test.

Definition 2.4. We will call an n-r.e. test $D(U_i^1, U_i^2, \cdots, U_i^n)$ *canonical* if each U_i^k is prefix-free and for every i, σ, and k such that $1 < k \leq n$ and $\sigma \in U_i^k$, there is a τ in U_i^{k-1} such that $\tau \subseteq \sigma$.

For instance, in a canonical d.r.e. test, we only "remove" neighborhoods that we have previously put in. In a canonical 3-r.e. test, if we add a neighborhood $[\tau]$ to $[U_i^3]$, we require that it have been enumerated in U_i^2 previously. That is, $[U_i^3]$ will only contain clopen neighborhoods which have been "removed" by $[U_i^2]$. The following lemma says that every n-r.e. test can be represented in a canonical way.

Lemma 2.5. *Let $\langle D(U_i^1, U_i^2, \cdots, U_i^n)\rangle_{i \in \omega}$ be an n-r.e. test. Then there is a canonical n-r.e. test $\langle D(V_i^1, V_i^2, \cdots, V_i^n)\rangle_{i \in \omega}$ such that*

$$D(U_i^1, U_i^2, \cdots, U_i^n) = D(V_i^1, V_i^2, \cdots, V_i^n)$$

for every i.

Proof. First, we consider the case of d.r.e. tests. We may assume that U_i^1 and U_i^2 are prefix-free for every i. We then let $V_i^1 = U_i^1$, and whenever some σ enters U_i^2, we wait until some comparable τ is enumerated into V_i^1. We then enumerate the longer of the two strings σ and τ into V_i^2. This clearly produces a canonical d.r.e. test. Now we fix $m \geq 2$ and consider n-r.e. tests for $n = 2m$ (if n is odd, the proof follows similarly). We can assume that for every i and j and every $\sigma \in U_i^{2j}$, there is a τ in U_i^{2j-1} such that $\tau \subseteq \sigma$. We also assume that each U_i^k is prefix free.

Fix i. Since the process is uniform in i, we will drop the subscript. We let $V^1 = \cup_{k<m} U^{2k+1}$; that is, everything that ever enters $D(U_i^1, U_i^2, \cdots, U_i^n)$. For $1 \leq j \leq m$, we define V^{2j} to be the set of all σ such that

(1) $\sigma \supseteq \tau$ for some $\tau \in V^{2j-1}$ and
(2) $[\sigma] \subseteq [U^{2k}]$ for at least j many different values for k.

Similarly, for $1 \leq j < m$, we let V^{2j+1} be the set of all σ such that

(1) $\sigma \supseteq \tau$ for some $\tau \in V^{2j}$ and
(2) $[\sigma] \subseteq [U^{2k+1}]$ for at least $j + 1$ many different values for k.

Clearly, each of these sets is r.e. It is not hard to see that we can replace each V^j with an equivalent prefix-free set of strings while maintaining canonicity. Now we must verify that $D(U^1, U^2, \cdots, U^n) = D(V^1, V^2, \cdots, V^n)$.

Suppose $A \in D(V^1, V^2, \cdots, V^n)$. Then $A \in [V^{2j-1}] - [V^{2j}]$ for exactly one $j \leq m$. Since $A \in [V^{2j-1}]$, this means that there are distinct $k_1, \cdots, k_j$ such that $A \in [U^{2k_1-1}] \cap \cdots \cap [U^{2k_j-1}]$. If we also have $A \in [U^{2k_1}] \cap \cdots \cap [U^{2k_j}]$, then some initial segment of A witnesses that $A \in [V^{2j}]$. Since $A \notin [V^{2j}]$, this means that $A \in D(U^{2k_1-1}, U^{2k_1}, \cdots, U^{2k_j-1}, U^{2k_j}) \subseteq D(U^1, U^2, \cdots, U^n)$. Suppose now that $A \in$

$D(U^1, U^2, \cdots, U^n)$. Let $k_1, \cdots, k_j$ be precisely the k such that $A \in [U^{2k_1}], \cdots, A \in [U^{2k_j}]$. Since we have assumed that each $D(U^{2k-1}, U^{2k})$ is canonical, we have $A \in [U^{2k_1-1}] \cap \cdots \cap [U^{2k_j-1}]$. In fact, we must also have $A \in [U^{2k_{j+1}-1}]$ for some k_{j+1} distinct from the others. There will be some initial segment of A that will witness that $A \in [V^1], A \in [V^2], \cdots, A \in [V^{2j+1}]$. However we cannot have $A \in [V^{2j+2}]$ by the maximality of j, so $A \in D(V^{2j+1}, V^{2j+2})$. $\qquad\square$

It is clear that every m-r.e. random real is n-r.e. random if $m \geq n$. Once again, we must address the question of whether this hierarchy collapses. It turns out that the answer is yes: the n-r.e. tests are no more powerful than the d.r.e. tests for any $n > 2$. This demonstrates a certain amount of robustness in the class of d.r.e. random reals. To prove this, we will use the notion of a Solovay test and the characterization of Martin-Löf randomness in terms of Solovay tests.

Definition 2.6. [20] A Solovay test is a uniformly r.e. sequence $\langle S_i \rangle_{i \in \omega}$ of subsets of $2^{<\omega}$ such that $\sum_i \mu(S_i) < \infty$. A real A is Solovay random if for every such test, there are only finitely many i such that some initial segment of A is an element of S_i.

Theorem 2.7. [20] *A real is Martin-Löf random if and only if it is Solovay random.*

Theorem 2.8. *If $n > 1$, then the n-r.e. random reals are precisely the d.r.e. random reals.*

Proof. If A is not n-r.e. random, there is an n-r.e. test $\langle D(U_i^1, U_i^2, \cdots, U_i^n) \rangle_{i \in \omega}$ such that $A \in \cap_i D(U_i^1, U_i^2, \cdots, U_i^n)$. Note that $\mu(D(U_i^{2k-1}, U_i^{2k})) < 2^{-i}$ for every $k \leq \frac{n}{2}$ and every i. We consider the cases of odd n and even n separately.

Suppose that $n > 1$ is odd. Then $\langle D(U_i^1, U_i^2, \cdots, U_i^{n-1}) \rangle_{i \in \omega}$ is an $(n-1)$-r.e. test and $\cup_i U_i^n$ is a Solovay test. Either $A \in D(U_i^1, U_i^2, \cdots, U_i^{n-1})$ for almost every i or A extends infinitely many strings in $\cup_i U_i^n$. Therefore, A is either not $(n-1)$-r.e. random or not Martin-Löf random.

Now suppose that $n > 2$ is even. By Lemma 2.5, we can suppose that the U_j^k are in canonical form. Observe that for any i, $\langle D(\cup_{j>i+1} U_j^{n-1}, \cup_{j>i+1} U_j^n) \rangle_{i \in \omega}$ is a d.r.e. test since $D(\cup_{j>i+1} U_j^{n-1}, \cup_{j>i+1} U_j^n) \subseteq \cup_{j>i+1} D(U_j^{n-1}, U_j^n)$, and the measure is bounded by 2^{-i}. By canonicity, $A \notin [\cup_{j>i} U_j^n]$ for any i. Therefore, either $A \in D(U_i^1, U_i^2, \cdots, U_i^{n-3}, U_i^{n-2})$ for almost every i or $A \in D(\cup_{j>i} U_j^{n-1}, \cup_{j>i} U_j^n)$ for almost every i. In this case, A is either not $(n-2)$-r.e. random or not d.r.e. random. $\qquad\square$

At this point, we observe that calling these reals "n-r.e. random" is somewhat misleading since one has an automatic tendency to assume that the choice of n is significant. Since this hierarchy of randomness notions collapses for $n \geq 2$, henceforth we will refer to this notion as *difference randomness* instead, which emphasizes the main point of contrast between it and Martin-Löf randomness very clearly. We will however, still refer to the tests described in Definition 2.3 as n-r.e. tests, and we will primarily use the characterization of this class as that of d.r.e. randomness in proofs.

We next give an alternative way of characterizing difference randomness. In Lemma 2.9, we show that we can essentially restrict our attention to tests which are Σ_1^0 classes at the cost of increasing the complexity of the indices. A Demuth test $\langle W_{g(i)} \rangle_{i \in \omega}$ is *strict* if there is an ω-r.e. approximation $g(i, s)$ of g such that for

every i and s such that $g(i,s) \neq g(i, s+1)$, we have $[W_{g(i,s+1)}] \cap [\cup_{t \leq s} W_{g(i,t)}] = \emptyset$. A strict Demuth test can be thought of as a d.r.e. test in which we remove strings only finitely often and, each time we effect such a removal, we remove every string enumerated into the test so far.

Proposition 2.9. *A real A is difference random if and only if for every strict Demuth test $\{W_{g(i)}\}_{i \in \omega}$, $A \notin \cap_i W_{g(i)}$.*

Proof. We begin by supposing that $A \in \cap_i W_{g(i)}$ for some strict Demuth test $\langle W_{g(i)} \rangle_{i \in \omega}$. Let $U_i = \cup_s W_{g(i,s)}$ and $V_i = \cup \{W_{g(i,s)} : g(i,s) \neq g(i)\}$. Then $\langle D(U_i, V_i) \rangle_{i \in \omega}$ is a d.r.e. test, and in fact $D(U_i, V_i) = [W_{g(i)}]$ for every i, so A is not difference random.

Now suppose that $\langle D(U_i, V_i) \rangle_{i \in \omega}$ is a d.r.e. test and that $A \in \cap_i D(U_i, V_i)$. We will build a strict Demuth test $\{W_{g(i)}\}_{i \in \omega}$ and a Solovay test Z such that A must fail to pass one of them. We build an approximation $g(i,s)$ for g and assume by the Recursion Theorem that we are building W_m for an infinite recursive set of indices for m. By speeding up the enumeration for V_i, we can assume that for every i and s, $\mu(D(U_{i,s}, V_{i,s})) \leq 2^{-i}$. For each i, we reserve $2^i + 1$ indices $m_1, \cdots, m_{2^i+1}$ for building $W_{g(i)}$. We start by letting $g(i, 0)$ equal the first index m_1. We keep $g(i,s) = m_1$ and let W_{m_1} copy $U_{i+1,s}$ until we find some s_1 such that $\mu(W_{m_1,s_1}) > 2^{-i}$. If this happens, we then enumerate the clopen set $D(U_{i+1,s_1}, V_{i+1,s_1})$ into Z, move on to the next index m_2, and stop building W_{m_1}. In general, if we are making our k^{th} attempt at constructing $W_{g(i)}$, we assume that we already have reached stages $s_1, \cdots, s_{k-1}$ and that we have stopped building $W_{m_1}, \cdots, W_{m_{k-1}}$. We keep $g(i,s) = m_k$ and let W_{m_k} copy the clopen set $[U_{i+k,s}] - [W_{m_1} \cup \cdots \cup W_{m_{k-1}}]$ until the first s_k is found such that $\mu(W_{m_k,s_k}) \geq 2^{-i}$. We then enumerate a prefix-free set of strings representing the clopen set $D(U_{i+k,s_k}, V_{i+k,s_k})$ into Z, move on to the next index m_{k+1}, and stop building W_{m_k}.

It is not hard to see that for $k \neq k'$, $[W_{m_k}] \cap [W_{m_{k'}}] = \emptyset$. Therefore, we will have at most 2^i changes to $g(i, -)$, which means that g is ω-r.e. and that we have allocated enough indices for the construction of $W_{g(i)}$. It is clear that $\mu(W_{g(i)}) \leq 2^{-i}$ for every i, so $\langle W_{g(i)} \rangle_{i \in \omega}$ is a strict Demuth test. Z is a Solovay test because for each $W_{g(i)}$, we enumerate strings representing the clopen sets $D(U_{i+1,s_1}, V_{i+1,s_1}) \cup \cdots$ into Z. The weight of these strings is at most 2^{-i}, so $\sum_{\sigma \in Z} 2^{-|\sigma|} \leq 1$.

If $A \in [W_{g(i)}]$ for almost every i, then there is a strict Demuth test that A does not pass. Suppose that this is not the case and there are infinitely many i such that $A \notin [W_{g(i)}]$. We fix such an i and let m_k be the index for the final version of $W_{g(i)}$. Since this is the final version, we must have $[U_{i+k}] - [W_{m_1} \cup \cdots \cup W_{m_{k-1}}] = [W_{g(i)}]$. However, $A \in [U_{i+k}]$, which means that $A \in [W_{m_j}] \subseteq [U_{i+j,s_j}]$ for some $j < k$. Since $A \notin [V_{i+j}]$, it follows that $A \in D(U_{i+j,s_j}, V_{i+j,s_j})$. Therefore, A extends some string in Z enumerated for the sake of $W_{g(i)}$. This means that A is not Martin-Löf random, so the universal Martin-Löf test is a strict Demuth test which A does not pass. □

The above proposition lets us see that every Demuth random real is difference random. However, the converse is not true.

Proposition 2.10. *If A is weakly 2-random or Demuth random, then A is difference random. If A is difference random, then A is Martin-Löf random. All inclusions are proper.*

$$\begin{array}{c} \text{Demuth} \\ \subset \qquad\qquad \supset \\ \text{2R} \qquad\qquad\qquad \text{DiffR} \subset \text{ML} \\ \supset \qquad\qquad \subset \\ \text{W2R} \end{array}$$
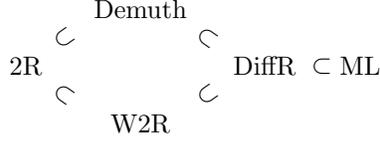
FIGURE 1. Relationships between classes of random reals.

Proof. Let $\langle D(U_i, V_i)\rangle_{i\in\omega}$ be a canonical d.r.e. test. Then $\cap_i D(U_i, V_i) = (\cap_i [U_i]) \cap (2^\omega - \cup_i [V_i])$. $\cap_i [U_i]$ is a Π_2^0 class and $2^\omega - \cup_i [V_i]$ is a Π_1^0 class, so $\cap_i D(U_i, V_i)$ is a Π_2^0 null class. This tells us that every weakly 2 random is difference random. By Proposition 2.9, every Demuth random is difference random since every strict Demuth test is a Demuth test. These inclusions are proper because weak 2-randomness and Demuth randomness are incomparable notions. There are Δ_2^0 Demuth random reals but not Δ_2^0 weakly 2-random reals, so not every Demuth random real is weakly 2-random. Furthermore, there is a weakly 2-random real that is not GL_1, but every Demuth random real is GL_1 (see [13]), so not every weakly 2-random real is Demuth random.

Each difference random real is clearly Martin-Löf random. It is easy to see that no left r.e. real is d.r.e. random: suppose that $\langle \alpha_s \rangle_{s\in\omega}$ is a left r.e. approximation to a left r.e. real α. Let $U_i = \{\alpha_s \restriction (i+1) : s \in \omega\}$ and $V_i = \{\alpha_s \restriction (i+1) : s \in \omega$ and $\alpha_s \restriction (i+1) \neq \alpha_{s+1} \restriction (i+1)\}$. Then $\alpha \in \cap_i D(U_i, V_i)$, which is a d.r.e. test, so α is not difference random. Perhaps the most obvious example of such an α is Chaitin's Ω. $\qquad\square$

This gives us the following diagram of relationships between these classes. All subsets are proper.

We finally show that, as might be expected, there is no universal d.r.e. test.

Proposition 2.11. *There is no d.r.e. test that is universal for the class of difference random reals.*

Proof. Suppose that $\langle D(U_i, V_i)\rangle_{i\in\omega}$ is a universal d.r.e. test. If every $V_i = \emptyset$, then Ω passes $\langle D(U_i, V_i)\rangle_{i\in\omega}$ but is not difference random. Hence there are some σ and i such that $\sigma \in V_i$. This means that every real extending σ passes the test $\langle D(U_i, V_i)\rangle_{i\in\omega}$. Since $\sigma^\frown 0^\omega$ is not difference random, we get a contradiction. $\qquad\square$

3. Characterizing the difference random reals within the Martin-Löf random reals

We now characterize the Turing incomplete Martin-Löf random reals in terms of difference randomness.

Theorem 3.1. *The difference random reals are precisely the Martin-Löf random reals that are not Turing complete.*

Proof. By Lemma 2.9, if A is not difference random, then there is a strict Demuth test $\langle W_{g(i)} \rangle_{i\in\omega}$ such that $A \in \cap_i W_{g(i)}$. We set $\langle \emptyset'_s \rangle_{s\in\omega}$ as a recursive approximation to $\emptyset'$ and construct a Solovay test E as follows. For each i and s such that $i \in \emptyset'_s - \emptyset'_{s-1}$, we add $W_{g(i,s),s}$ to E. For each i there is at most one s for which $W_{g(i,s),s}$ is added to E, so $\sum_{\sigma \in E} 2^{-|\sigma|} < 1$. If A is not Martin-Löf random, then we are done. Otherwise, $A \supset \sigma$ for finitely many $\sigma \in E$. Let n_0 be the length of the longest

initial segment of A contained in E (so $A{\restriction}n_0 \in E$). We now A-recursively compute whether $i \in \emptyset'$. We begin by searching for the first stage s such that $A{\restriction}n \in W_{g(i,s),s}$ for some n. We know that such an s must exist since $A \in [W_{g(i)}]$. Then we claim that for every $i > n_0$, $i \in \emptyset'$ if and only if $i \in \emptyset'_s$. Otherwise, we would have $i \in \emptyset'_t - \emptyset'_{t-1}$ for some $t > s$. By construction, $E \supseteq W_{g(i,t),t} = W_{g(i,s),t} \supseteq W_{g(i,s),s}$ since $g(i,s) = g(i,t) = g(i)$. Since $\mu(W_{g(i,s),s}) < 2^{-i}$, it follows that $n > i > n_0$. However, this means that $A{\restriction}n \in E$, which contradicts our choice of n_0.

Now we prove the other direction. Since every difference random real is Martin-Löf random, we assume that $A \geq_T \emptyset'$ and build a strict Demuth test $\langle W_{g(i)} \rangle_{i \in \omega}$ such that $A \in \cap_i W_{g(i)}$. We will build an r.e. set F and use the Recursion Theorem to assume that $F = \Phi(A)$ for some Φ. We partition ω into intervals I_i such that I_i has 2^i members. We fix i and define $g(i,s)$ and F on I_i. For each $0 \leq k \leq 2^i$, we let $Q_k = \{\sigma : \Phi^\sigma {\restriction} I_i \downarrow = 1^k 0^{2^i - k}\}$. As in Proposition 2.9, we assume that we are building $W_{m_1}, \cdots, W_{m_{2^i+1}}$. We let $W_{m_k} = Q_{k-1}$ for every $1 \leq k \leq 2^i + 1$ and note that the Q_ks are all r.e. sets of finite strings. We let $g(i) = m_k$ for the least k such that $\mu(Q_{k-1}) \leq 2^{-i}$. For each $k \in I_i$, we enumerate k into F whenever we find that $\mu(Q_{k-\min(I_i)}) > 2^{-i}$ (and all smaller $k' \in I_i$ are also in F).

Now F is clearly an r.e. set. The function g is ω-r.e. via the obvious approximation $g(i,s)$ because for each i, the sets $[Q_0], \ldots, [Q_{2^i}]$ are pairwise disjoint. In fact $g(i,-)$ changes no more than 2^i times. We certainly have $\mu(W_{g(i)}) \leq 2^{-i}$ by the definition of $g(i)$. Again, by the fact that the Q_k are pairwise disjoint, $\langle W_{g(i+1)} \rangle_{i \in \omega}$ is a strict Demuth test. Finally, we verify that for every i, $A \in [W_{g(i)}]$. Let $g(i) = m_k$. This means that for every $1 \leq j < k$, we have $\mu(Q_{j-1}) \geq 2^{-i}$ and consequently $\min I_i + j - 1 \in F$. In fact, $1^{k-1}0^{2^i-k+1} = F {\restriction} I_i = \Phi(A) {\restriction} I_i$, so $A \in [Q_{k-1}] = [W_{g(i)}]$. □

We can relativize the notion of difference randomness to a real X in the natural way: an X-d.r.e. test is a sequence $\langle D(W^X_{g(i)}, W^X_{h(i)}) \rangle_{i \in \omega}$ where g and h are recursive functions such that $\mu(D(W^X_{g(i)}, W^X_{h(i)})) \leq 2^{-i}$ for every i. A real A is difference random relative to X if for every X-d.r.e. test $\langle D(U^X_i, V^X_i) \rangle_{i \in \omega}$, $A \notin \cap_i D(U^X_i, V^X_i)$. We can relativize the above theorem as follows.

Corollary 3.2. *Let A and X be reals. Then A is difference random relative to X if and only if A is Martin-Löf random relative to X and $A \oplus X \ngeq_T X'$.*

Proof. If we relativize the proof of Proposition 2.9, we can see that A is difference random relative to X if and only if it passes every test of the form $\langle W^X_{g(i)} \rangle_{i \in \omega}$, where the following conditions hold.

(1) There is some $\tilde{g} \leq_T X$ such that for every i, $\lim_s \tilde{g}(i,s) = g(i)$ and the number of times $\tilde{g}(i,-)$ changes is recursively bounded.
(2) $\mu(W^X_{g(i)}) < 2^{-i}$ for every i.
(3) For every i and s such that $\tilde{g}(i,s) \neq \tilde{g}(i,s+1)$, we have $[W^X_{\tilde{g}(i,s+1)}] \cap [\cup_{t \leq s} W^X_{\tilde{g}(i,t)}] = \emptyset$.

Now we simply relativize the proof of Theorem 3.1, paying attention to the applications of the Recursion Theorem. □

4. Relationships with lowness classes

After a notion of randomness is well defined and explored, it is standard to consider properties of reals that are antithetical to this randomness notion. These properties usually indicate weakness in different ways and are sometimes called lowness properties. For instance, if R is the class of reals that are random with respect to a certain notion, we say that a set A is *low for the class* R if $\mathsf{R}^A = \mathsf{R}$. This means that A is so feeble in terms of its derandomization power that it does not help to make random reals appear to be nonrandom when it is used as an oracle. Another example of a lowness property frequently considered in algorithmic randomness is that of being a *base for* R: a set A is a base for R if there is some $Z \geq_T A$ such that $Z \in \mathsf{R}^A$. As is well known, if R is the class of Martin-Löf random reals, then these two properties coincide with K-triviality [6, 12] . In fact, Nies showed in [12] that every K-trivial real is actually superlow.s

After establishing difference randomness to be a very natural randomness notion, our attention naturally turns towards the corresponding lowness properties. At this point, the reader might be tempted to conjecture that the corresponding lowness notions for difference randomness might coincide with K-triviality as they do in the case of lowness for weak 2-randomness [4, 8]. We show a connection between the corresponding lowness notions for difference randomness and two other well-known lowness classes arising in the study of K-triviality.

Theorem 4.1. *Suppose that A is an r.e. set. Then A is a base for difference randomness if and only if A is Martin-Löf coverable.*

Proof. Suppose that A is a base for difference randomness, and let $Z \geq_T A$ be such that Z is difference random relative to A. Then Z is difference random and therefore Martin-Löf random and Turing incomplete by Theorem 3.1, so A is Martin-Löf coverable.

Now suppose that A is an r.e. set that is Martin-Löf coverable. Then, by Theorem 3.1, there is some difference random $Z \geq_T A$. Since every Martin-Löf coverable r.e. set is low for Martin-Löf randomness, Z is Martin-Löf random relative to A. Furthermore, $Z \oplus A \not\geq_T A'$, since otherwise, we would have that $Z \equiv_T Z \oplus A \geq_T A' \geq_T \emptyset'$. By Corollary 3.2, Z is difference random relative to A, and A must be a base for difference randomness. $\square$

Theorem 4.2. *Suppose A is an r.e. set. Then A is low for difference randomness if and only if A is not weakly Martin-Löf cuppable.*

Proof. Suppose that A is an r.e. set that is low for difference randomness. We recall that if R and S are classes of random reals, then $\mathrm{Low}(\mathsf{R},\mathsf{S})$ is the class of reals A such that R is a subset of S^A. Then our A is in $\mathrm{Low}(\mathsf{W2R},\mathsf{ML})$ and thus it is K-trivial [4]. In particular, A must be low. If A were weakly Martin-Löf cuppable, there would be a difference random real Z such that $Z \oplus A \geq_T \emptyset' \equiv_T A'$. Therefore, Z is not difference random relative to A and A is not low for difference randomness, which gives us a contradiction.

Now suppose that A is an r.e. set that is not weakly Martin-Löf cuppable. Then A must be low for Martin-Löf randomness. To see that A is low for difference randomness, we consider an arbitrary difference random real Z. Since A is not weakly Martin-Löf cuppable, $Z \oplus A \not\geq_T \emptyset'$. Since A is low for Martin-Löf randomness, Z

must be Martin-Löf random relative to A. Furthermore since $Z \oplus A \not\geq_T A'$, by Corollary 3.2, Z is difference random relative to A. $\qquad\square$

5. A martingale characterization of difference randomness

We should also consider the robustness of this randomness notion. Martin-Löf randomness can be defined in three different ways: that of measure theory, that of computational complexity, and that of unpredictability [10, 18]. We have only addressed the notion of n-r.e. randomness from the perspective of measure theory. It is natural to ask whether equivalent definitions can be given in terms of the other two perspectives. Here, we present an equivalent definition in terms of unpredictability. We study a class of martingales—the *difference martingales*—which we will use to characterize difference randomness.

Definition 5.1. A *difference martingale* is a martingale $m : 2^{<\omega} \mapsto \mathbb{R}^{\geq 0}$ such that there are recursive functions m_1, m_2, and b which map $2^{<\omega} \times \omega \mapsto \mathbb{Q}^{\geq 0}$ such that the following hold:

 (i) For every σ and s, $m_1(\sigma, s+1) \geq m_1(\sigma, s)$. The same holds for m_2 and b.
 (ii) For every σ, $m(\sigma) = \lim_s m(\sigma, s)$, where $m(\sigma, s) = m_1(\sigma, s) - m_2(\sigma, s)$.
 (iii) For every σ and s, if $m(\sigma, s) < b(\sigma, s)$, then $m(\tau, t) \leq m(\sigma, s)$ for every $t \geq s$ and $\tau \supseteq \sigma$.
 (iv) For every $X \in 2^{\omega}$, there is a constant $c \in \omega$ such that $\limsup_n b(X {\restriction} n) \leq \limsup_n m(X {\restriction} n) + c$. Here $b(\sigma) = \lim_s b(\sigma, s)$.

We call the triple (m_1, m_2, b) an (effective) presentation of m. We say that a difference martingale m *succeeds* on a real $X \in 2^{\omega}$ if there is a presentation (m_1, m_2, b) of m such that $\limsup_n b(X {\restriction} n) = \infty$.

We would expect that if d.r.e. randomness can be characterized in terms of martingales, then an obvious candidate is the class of martingales which are "d.r.e." in some sense. For instance, we could consider martingales which are the difference of two r.e. martingales, or we could consider martingales where the value of $m(\sigma)$ is uniformly a d.r.e. real. Neither of these candidates is good enough because if we were only given a martingale where $m(\sigma)$ can be approximated in a "d.r.e." fashion, then $m(\sigma)$ may rise above and fall below a threshold arbitrarily many times. If we were trying to build a d.r.e. test to capture the set of reals on which m succeeds, then we would not be able to correctly determine when to remove σ from the test. For this reason, we need a martingale with a "d.r.e." approximation and an "r.e." lower bound for the value of m.

Conditions (i) and (ii) in Definition 5.1 say that $m(\sigma)$ can be approximated as the difference of two increasing sequences (though $m_1(\sigma, s)$ and $m_2(\sigma, s)$ can be unbounded as $s \to \infty$.) Similarly, we can define $b(\sigma)$ as $\lim_s b(\sigma, s)$ for every σ and approximate it effectively from below. Condition (iii) says that $b(\sigma)$ serves as a partial lower bound for $m(\sigma)$ in the following sense. Once $m(\sigma, s)$ drops below $b(\sigma, s)$, then $m(\tau)$ has to stay uniformly bounded in the cone above σ. In other words, once the lower bound is violated at some node σ, then the martingale stops betting in the cone above σ. This tells us that we cannot have success on any real extending σ, so we may remove σ from our d.r.e. test. Finally, condition (iv) says that b serves as a lower bound for m when the values along infinite strings are considered. In other words, if an infinite string X wins in the sense of having

unbounded capital as calculated using $b(X \restriction n)$, then X has to win against the martingale, too.

Theorem 5.2. *A real A is difference random if and only if no difference martingale succeeds on it.*

Proof. We first prove the easy direction. Assume that there is a difference martingale that succeeds on A, and fix such a martingale m and its presentation (m_1, m_2, b). Then $\limsup_n b(A \restriction n) = \infty$. For each $x \in \omega$, let $U_x = \{\sigma \in 2^{<\omega} \mid \exists s(b(\sigma, s) \geq 2^x m(\emptyset))\}$ and $V_x = \{\sigma \in U_x \mid \exists s(m(\sigma, s) < b(\sigma, s))\}$. For each x, let T be the set of all $\sigma \in U_x - V_x$ such that no prefix of σ is in $U_x - V_x$. Clearly, $[T] \supseteq D(U_x, V_x)$, and T is a prefix-free set of strings. For each $\sigma \in T$, we have $m(\sigma) \geq 2^x m(\emptyset)$, and so $\mu(D(U_x, V_x)) \leq \sum_{\sigma \in T} 2^{-|\sigma|} \leq 2^{-x}$. The last inequality follows by the usual counting argument. Furthermore, $A \in D(U_x, V_x)$ for every x because of conditions (ii) and (iv) in Definition 5.1, so A is not difference random.

Now suppose $A \in \cap_i W_{g(i)}$ for a strict Demuth test $\langle W_{g(i)} \rangle_{i \in \omega}$. We may assume that we have a presentation $\langle W_{g(i,s)} \rangle$ in which for every i and s, $W_{g(i,s)}$ is prefix free and the test is nested: for all i and s and for all $\sigma \in W_{g(i+1,s)}$, there is $\tau \subseteq \sigma$ such that $\tau \in W_{g(i,s)}$. In fact, by the Recursion Theorem, we can see that for all i, s, and t and for all $\sigma \in W_{g(i+1,s),t}$, there is $\tau \subseteq \sigma$ such that $\tau \in W_{g(i,s),t}$. We define the functions m_1^x, m_2^x, and b^x uniformly in x as follows. For each σ and s,

$$
\begin{aligned}
m_1^x(\sigma, s) &= \mu(\cup_{t \leq s}[W_{g(x,t),s}] \mid \sigma), \\
m_2^x(\sigma, s) &= \mu(\cup_{t \leq s^-}[W_{g(x,t),s}] \mid \sigma), \\
b^x(\sigma, s) &= 1 \text{ if } \sigma \supseteq \tau \text{ for some } \tau \in \cup_{x < t \leq s} W_{g(x,t),s} \text{ and } 0 \text{ otherwise}
\end{aligned}
$$

where $\mu(\mathcal{C} \mid \sigma)$ is the conditional probability $\mu(\mathcal{C} \cap [\sigma]) \cdot 2^{|\sigma|}$ and s^- is the largest stage less than s where $g(x, s^-) \neq g(x, s^- + 1)$. If this does not exist, then let s^- be -1. Clearly $m^x(\sigma) = \lim_s m^x(\sigma, s)$ exists, where $m^x(\sigma, s) := m_1^x(\sigma, s) - m_2^x(\sigma, s)$, since the values of $\langle m_1^x(\sigma, s) \rangle_s$ and $\langle m_2^x(\sigma, s) \rangle_s$ are all bounded above by 1. It is easy to verify that m^x is a martingale by rearranging the limits. Finally we claim the following:

(†) For every σ, s, and x, if $m^x(\sigma, s) < b^x(\sigma, s)$, then $m^y(\eta, t) = 0$

for every $t \geq s$, $\eta \supseteq \sigma$, and $y \geq x$.

Furthermore, $b^y(\eta, t) = 0$ for every $t \geq s$, $\eta \supseteq \sigma$, and $y \geq s$.

If $m^x(\sigma, s) < b^x(\sigma, s)$ for some σ, x, and s, then $b^x(\sigma, s) = 1$, which means that $\sigma \supseteq \tau$ for some $\tau \in \cup_{t \leq s} W_{g(x,t),s}$. Now if $\tau \notin \cup_{t \leq s^-} W_{g(x,t),s}$, then we must have $[\sigma] \cap \cup_{t \leq s^-}[W_{g(x,t),s}] = \emptyset$. This means that $m_2^x(\sigma, s) = 0$ and $m_1^x(\sigma, s) = 1$, which is a contradiction. Therefore, we must have $\tau \in \cup_{t \leq s^-} W_{g(x,t),s}$, which means that $m^x(\eta, t) = 0$ for any $\eta \supseteq \sigma$ and every $t \geq s$.

Now generally for $y > x$ and $\eta \supseteq \sigma$, we must have $[\eta] \cap [W_{g(y,t)}] = \emptyset$ for every $t \geq s$ because of our assumption that the strict Demuth test is nested. This means that for every $t \geq s$, $[\eta] \cap \cup_{u \leq t}[W_{g(y,u)}] = [\eta] \cap \cup_{u \leq s^-}[W_{g(y,u)}]$ where s^- is now the last change prior to s in the version of $W_{g(y)}$. In fact we also have $[\eta] \cap \cup_{u \leq t}[W_{g(y,u),t}] = [\eta] \cap \cup_{u \leq s^-}[W_{g(y,u),t}]$, and this implies that $m^y(\eta, t) = 0$. To see the second part of (†), we note that if $b^y(\eta, t) = 1$ for $y, t \geq s$ and $\eta \supseteq \sigma$, then $[\eta] \subseteq [W_{g(x,u)}]$ for some $u > y \geq s$, which is impossible since a comparable $[\tau]$ is already contained in $\cup_{u \leq s^-}[W_{g(x,u)}]$. This proves (†).

Now let $m(\sigma) = \sum_{x \in \omega} m^x(\sigma)$. $m(\sigma)$ is defined because for every σ and x, $0 \leq m^x(\sigma) \leq 2^{\min\{|\sigma| - x, 0\}}$, and m is a martingale because each m^x is. Now we define an effective presentation of m as follows.

$$m_1(\sigma, s) = \sum_{x \leq s} m_1^x(\sigma, s)$$

$$m_2(\sigma, s) = \sum_{x \leq s} m_2^x(\sigma, s)$$

$$b(\sigma, s) = \sum_{x \leq s} b^x(\sigma, s)$$

We need to verify that m_1, m_2 and b represent m. Condition (i) is trivial. For condition (ii), we need to see that $m(\sigma) = \lim_s(m_1(\sigma, s) - m_2(\sigma, s))$. It suffices to show that $\sum_x m^x(\sigma) = \lim_s \sum_{x \leq s} m^x(\sigma, s)$. Given any $\varepsilon > 0$, fix an s_0 such that $\sum_{x \leq s_0} m^x(\sigma) > m(\sigma) - \varepsilon/2$. We then proceed to pick a $t_0 > s_0$ such that for each $x \leq s_0$ and $t \geq t_0$, we have $|m^x(\sigma, t) - m^x(\sigma)| < \varepsilon/2s_0$. This shows that $\sum_x m^x(\sigma) \leq \lim_s \sum_{x \leq s} m^x(\sigma, s)$. To prove the other direction, note that for every s, x, and σ, we have $m^x(\sigma, s) = \mu(W_{g(x,s),s} \mid \sigma) < 2^{|\sigma| - x}$. Now let $M = \lim_s \sum_{x \leq s} m^x(\sigma, s)$. Given $\varepsilon > 0$, we pick s_0 such that for every $s \geq s_0$, we have $\sum_{x \leq s} m^x(\sigma, s) > M - \varepsilon/3$. Fix $r > s_0$ such that $2^{|\sigma| - r + 1} < \varepsilon/3$ and fix $t_0 > r$ such that for every $x < r$ we have $m^x(\sigma, t_0) < m^x(\sigma) + \varepsilon/3r$. Now

$$\sum_{x < r} m^x(\sigma) \geq \left(\sum_{x < r} m^x(\sigma, t_0)\right) - \frac{\varepsilon}{3} \geq \left(\sum_{x \leq t_0} m^x(\sigma, t_0)\right) - \frac{2\varepsilon}{3} > M - \varepsilon,$$

so $\sum_x m^x(\sigma) \geq M$.

For condition (iii), we fix a σ and s such that $N + 1 = b(\sigma, s) > m(\sigma, s)$. It is not hard to see that if $b^{x+1}(\sigma, s) = 1$, then $b^x(\sigma, s) = 1$, again using the assumption that the strict Demuth test is nested. Hence $b^x(\sigma, s) = 1$ for all $x \leq N$. Fix the least $x \leq N$ such that $m^x(\sigma, s) < 1 = b^x(\sigma, s)$. Now, by the minimality of x, we must have $m^y(\sigma, s) = 1$ for every $y < x$. Applying (†), we get that for any $t \geq s$ and $\eta \supseteq \sigma$,

$$m(\eta, t) = \sum_{y \leq t} m^y(\eta, t) = \sum_{y < x} m^y(\eta, t) \leq x \leq \sum_{y < x} m^y(\sigma, s) = \sum_{y \leq s} m^y(\sigma, s) = m(\sigma, s).$$

Finally, we verify property (iv). Fix an $X \in 2^\omega$. We only need to show that if $\limsup_n b(X{\restriction}n) = \infty$, then $\limsup_n m(X{\restriction}n) = \infty$. In fact, it is easy to see that if $\limsup_n b(X{\restriction}n) = \infty$, then b serves as a true lower bound for m along X: If $b(\sigma) > m(\sigma)$ for some $\sigma \subset A$, then there are some s and x where $b^x(\sigma, s) > m^x(\sigma, s)$. By the second part of (†), for any $\eta \supseteq \sigma$, $b(\eta) \leq s + 1$ which is a contradiction. Now it is easy to see that $\limsup_n b(A{\restriction}n) = \infty$, so m succeeds on A. □

We note that we have not discussed the possibility of a characterization of difference randomness in terms of initial-segment complexity. While we have considered several possibilities, it seems that the class of Turing machines involved always either fails to work or is so far removed from any class previously considered that we cannot accept it as a reasonable characterization.

6. Conclusion

The study of n-r.e. randomness allows us to separate the ideas involved in measure-theoretic definitions of randomness in a way that Martin-Löf randomness cannot. Is the complexity of the way the tests are generated the most important thing (naive n-r.e. randomness), or is it the complexity of the neighborhoods they determine (n-r.e. randomness)?

In each case, the hierarchy of n-r.e. randomness notions collapses for $n \geq 2$. Therefore, once an element of a test can be removed even once, the particular number of additions and removals is irrelevant. In the case of naive n-r.e. randomness, we find that it is equivalent to the well-investigated notion of 2-randomness. This makes a certain amount of intuitive sense: we are allowed to change our minds about the presence of any basic open set in any element V_i of the test arbitrarily many times regardless of n, so we can approximate any Martin-Löf test recursive in $\emptyset'$ by a naive n-r.e. test for any $n \geq 2$.

However, when we restricted the number of times we can change our minds about the presence of a basic open set in an element V_i to some particular n, we obtained a characterization of the Turing incomplete Martin-Löf random reals. This natural class of reals had never before been characterized solely in terms of a randomness notion. We hope that our characterization will shed some light on the other properties of this class as well as further the study of the properties of the K-trivial reals.

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