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LIMITS ON JUMP INVERSION FOR STRONG REDUCIBILITIES

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Abstract. We show that Sacks' and Shoenfield's analogs of jump inversion fail for both tt- and wtt-reducibilities in a strong way. In particular we show that there is a Δ^0_2 set $B >_{tt} \emptyset'$ such that there is no c.e. set A with $A' \equiv_{wtt} B$. We also show that there is a Σ^0_2 set $C >_{tt} \emptyset'$ such that there is no Δ^0_2 set D with $D' \equiv_{wtt} C$.

§1. Introduction.

The concern of this paper is the interaction of two basic notions from computability theory. These are the jump operator and reducibilities stronger than Turing reducibility which are of the tabular type. We answer a question of Anderson [And08] by showing that there are no analogs of Sacks Jump Inversion Theorem [Sac63] and Shoenfield's Jump Inversion Theorem [Sho59] for these strong reducibilties.

The study of strong reducibilities has been part of computability since the dawn of the subject, as witnessed by Post's paper [Pos44]. A is Turing reducible to B, $A \leq_T B$, means that A can be computed by B via any oracle access mechanism. It is clearly natural to ask what happens when we restrict the access mechanism in the reduction from A to B. Tabular reducibilies such as weak truth table (wtt-) and truth table (tt-) reduciblities do not allow the reduciblity to be adaptive. Thus, as is well known, $A \leq_{tt} B$, is defined as $x \in A$ iff $B \models \sigma_{f(x)}$ where f is a computable function and $\sigma_{f(x)}$ is the $f(x)^{th}$ truth table. As is also well known a truth table reduction is simply a Turing reduction Φ which is total for all oracles. Weak truth table reducibility simply has the truth table being partial, or $\Phi^B = A$ where the use of the computation $\varphi(x)$ is a computable function. Thus in either case we are not allowed to *adapt* the size of the reduction as the oracle B varies. $A \leq_{tt} B$ implies $A \leq_{wtt} B$ but it is easy to construct examples where the converse fails.

These reducibilities also arise very naturally when we consider reducibilities coming from reductions in mathematical structures. For example, the reduction of the word problem to the conjucacy problem in combinatorial group theory is a tt-reduction and the degrees of bases of c.e. vector spaces are naturally represented by weak truth table degrees (Downey and Remmel [DR89]).

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In recent times, truth table reducibility has become a central area of interest as it has been shown to be a natural reducibility to study in *algorithmic randomness*, a fact first realized by Demuth [Dem88]. The point here is that if $A \leq_{tt} B$, via $\Phi^B = A$, with Φ total on all oracles, then we can use Φ to translate between measures effectively. For instance if B is random with respect to uniform measure, and A is noncomputable, A will be random wih respect to the measure generated by the inverse of Φ. Thus, for instance, truth table degrees are absolutely central to the deep investigations of Reimann and Slaman [RS08a, RS08b] on sets never continuously random. They are also deeply connected with things like the Cantor-Bendixson rank of sets for a similar reason.

All of this recent work has highlighted our lack of understanding as to how the finer structure of the (w)tt-degrees relates to the jump operator. The halting problem is a fundamental object of computability theory, and the jump $A' = \{e : \Phi_e^A(e)\downarrow\}$ is the relativized form of the halting problem.

For Turing reducibility, we know a lot about how the jump operator behaves. The most basic theorem is Friedberg's Jump Inversion Theorem [Fri58], that if $X \geq_T \emptyset'$ then there is a set A with $A' \equiv_T X \equiv_T A \oplus \emptyset'$. Early on, Mohrherr [Moh84] proved that if $X \geq_{tt} \emptyset'$, then there is a set A with $A' \equiv_{tt} X$. Mohrherr's proof came from an analysis of Friedberg's Theorem, and resulted in a 1-generic set A. It was only much later that Anderson [And08] proved that indeed the full analog of Friedberg's Theorem held; if $X \geq_{tt} \emptyset'$ then there is a set A with $A' \equiv_{tt} X \equiv_{tt} A \oplus \emptyset'$. Anderson's theorem was more difficult than Mohrherr's, and the method employed by Friedberg (which will give generics sets) provably *fails*, so that arguments akin to those from information theory were necessary.

All of this led to the present paper. The most important sets in computability theory are the c.e. sets as well as those computable from the halting problem, the Δ_2^0 sets. Shoenfield [Sho59] proved a jump theorem for such sets. Namely for any Σ_2^0 set $X \geq_T \emptyset'$ there is a Δ_2^0 set A with $A' \equiv_T X$. Famously, Sacks used the infinite injury method to show that the same result held with A a computably enumerable set, and after that many other intricate jump theorems were found culminating in Robinson's Jump Interpolation Theorem [Rob71]. (See Soare [Soa87] for more details.)

Anderson asked: do the analogs of any of these basic theorems hold for tt- or perhaps wtt-reducibilities? We prove that the analogs fail to hold and in fact that they fail in more or less the strongest way that they can. Our first result shows that Sacks' Jump Inversion Theorem fails for both the tt- and wtt-reducibilities, by constructing a Δ_2^0 counter-example. We will in fact prove something stronger:

THEOREM 3.3. *For any computable sequence of Δ_2^0 sets $\{V_e\}_{e\in\mathbb{N}}$ (given by their Δ_2^0 indices), there exists a Δ_2^0 set $S \geq_{tt} \emptyset'$ such that for every e, $V_e' \not\equiv_{wtt} S$.*

Using Theorem 3.3 we can easily show the failure of Sacks' Jump Inversion for both tt- and wtt-reducibilities:

THEOREM 3.4. *There exists an $\omega + 1$-c.e. set $S >_{tt} \emptyset'$ such that there is no c.e. set A with $A' \equiv_{wtt} S$.*

We remind the reader what an $\omega + 1$-c.e. set is. $S \leq_T \emptyset'$ is said to be $\omega + 1$-*c.e.*, if there exists a computable approximation $g : \omega^2 \mapsto \{0, 1\}$ of S and a partial computable function $h : \omega \mapsto \omega$ so that for every x, $\lim_{s\to\infty} g(x, s) = S(x)$ and the number of mind changes of $g(x, -)$ is either 0, or else $h(x) \downarrow$ and bounds the number of mind changes of $g(x, -)$.

That is, the set S belongs to the first level of the Ershov hierarchy for which such a counter-example is not immediately ruled out by existing results. The result also gives an interesting fact about the Δ^0_2 wtt-degrees which are realized as the jump of a low c.e. set. Clearly there are such wtt-degrees $\boldsymbol{a} > \boldsymbol{0}'_{wtt}$, namely the wtt-degrees which are the jump of a low but not superlow c.e. set. Our result shows that *not every* Δ^0_2 wtt-degree $\boldsymbol{a} > \boldsymbol{0}'_{wtt}$ can be realized by the jump of a (low) c.e. set.

Our second result shows that the analogue of Shoenfield's Jump Inversion Theorem fails for both the tt- and wtt-reducibilities. By Mohrherr's result, the counter-example S has to be strictly Σ^0_2:

THEOREM 3.5. *There exists a Σ^0_2 set $S >_{tt} \emptyset'$ such that there is no Δ^0_2 set A with $A' \equiv_{wtt} S$.*

1.1. Notation. We follow standard notation for Computability Theory, as found in Cooper [Coo04] and Soare [Soa87].

§2. The basic module.

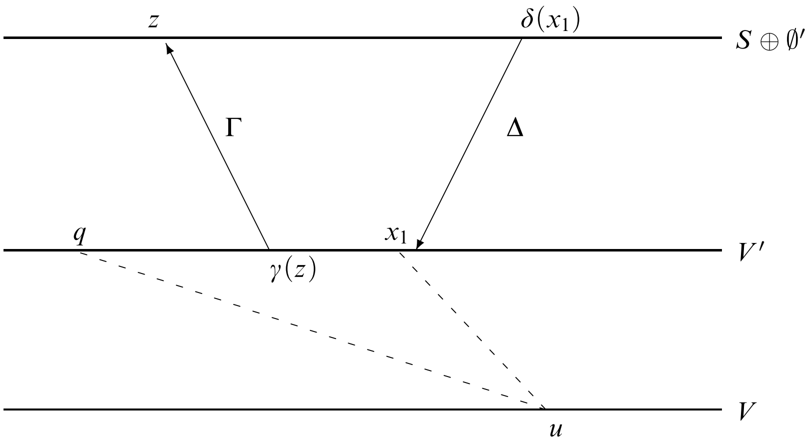
2.1. The plan for the c.e. case. Let $(\Gamma_e, \Delta_e, \gamma_e, \delta_e)_{e \in \omega}$ run through all possible 4-tuples where Γ_e and Δ_e are Turing functionals, and γ_e and δ_e are partial computable functions. Let us suppose we wanted to prove Theorem 3.4 directly by constructing S. We must then meet for all $e \in \omega$ the requirements:

$$R_e \colon \Gamma_e^{V_e'} \neq S \text{ or } \Delta_e^{S \oplus \emptyset'} \neq V_e',$$

where V_e is the e^{th} c.e. set, and γ_e and δ_e bound the uses of the computations of Γ_e and Δ_e, respectively. Then $S \oplus \emptyset'$ will be the desired set. Note that the requirements automatically ensure that $S \oplus \emptyset' \not\equiv_{wtt} \emptyset'$.

Suppose we wanted to satisfy R_e. We can first try making $\Delta^{S \oplus \emptyset'} \neq V'$ (for the purpose of the discussion we drop subscript e). In particular we assume that the recursion theorem gives us infinitely many indices $x_1, x_2, \dots$ for which we can control $V'(x_i)$. The obvious plan is to keep $V'(x_1) = 0$ until $\Delta^{S \oplus \emptyset'}(x_1) \downarrow = 0$. We then make $V'(x_1) = 1$ by enumerating an axiom with some use $V \upharpoonright u$. The only way in which $\Delta^{S \oplus \emptyset'}(x_1)$ can later change to be 1, is for some number $< \delta(x_1)$ to enter $\emptyset'$. Our next step would be then to extract x_1 out from V'; if we could always do this then we would know what to do. We would alternate the value of $V'(x_1)$, and we will eventually succeed because $\emptyset'$ is c.e. and the use $\delta(x_1)$ is fixed. Unfortunately we only have partial control over V' and extraction can only be achieved by forcing a change in $V \upharpoonright u$.

We can start another line of attack by trying to make $\Gamma^{V'} \neq S$ true. We pick a follower z for S, and for simplicity let us first consider the case where Γ is an m-reduction; i.e., $z \in S$ iff $q \in V'$ for some q. We begin by making $S(z) = 1$, and wait for $V'(q) = 1$, i.e., $\Phi_q^V(q) \downarrow$ with some use u. Note that while the uses on Δ and Γ are bounded, this use u may be unbounded. We then begin the attack above by first waiting for $\Delta^{S \oplus \emptyset'}(x_1) \downarrow = 0$. We then enumerate an axiom $\Phi_{x_1}^V(x_1)$ with the same V-use u, and wait for a $\emptyset'$-change below $\delta(x_1)$.



If no $\emptyset'$-change occurs then it is clear that we would succeed at $\Delta^{S\oplus\emptyset'}(x_1) \neq V'(x_1)$. If on the other hand a $V\restriction u$ change occurs before a $\emptyset'$-change, then we would wait for $\Phi_q^V(q)\downarrow$ again with a new use u', and then make $\Phi_{x_1}^V(x_1)$ converge with the same use u'. The point is that if V changes infinitely often this way with no $\emptyset'$-change, then $V'(q) = 0$ and we would succeed via $\Gamma^{V'}(z_e) \neq S(z)$. Lastly if $\emptyset'$ changes then we would remove z from S, and wait for $\Phi_q^V(q)$ to become undefined again. This has to happen (unless already $\Gamma^{V'} \neq S$), and so at some point we will also get a divergence $\Phi_{x_1}^V(x_1)\uparrow$. We can then repeat by making $S(z) = 1$ again. Note that we only toggle z in S whenever $\emptyset'$ changes below $\delta(x_1)$, so requirement R_e can be satisfied with only finite action on S (although we might possibly enumerate infinitely many axioms for x_1).

The strategy for a general wtt-reduction Γ is as above, but we will run a separate copy of the strategy above for each possible configuration of the use $V_e'\restriction\gamma_e(z_e)$. We have $2^{\gamma_e(z_e)}$ many different x's corresponding to each different configuration of $V_e'\restriction\gamma_e(z_e)$. At each stage we look at the current approximation for $V_e'\restriction\gamma_e(z_e)$ (see section 2.2) and apply the above plan. As in the basic case we will only toggle z_e in S if $\emptyset'$ changes below some δ_e-use. Each time we toggle z_e we will force the configuration $V_e'\restriction\gamma_e(z_e)$ to change. This can only move lexicographically right finitely many times (consecutively), hence after finitely much toggling of z_e, the configuration for $V_e'\restriction\gamma_e(z_e)$ will return to an earlier one. This makes all the x_τ (for all the τ on the right of the current V_e'-configuration) undefined, so that if $\tau \subset V_e'[s]$ holds again later we can use x_τ to cause further $\emptyset'$-changes.

The above works when diagonalizing against all c.e. sets. However if V is Δ_2^0 then whenever the configuration of $V'\restriction\gamma_e(z_e)$ returns to an earlier one, there is no guarantee that all x_τ (for τ on the right of the current V'-configuration) become undefined. However we can show that some amount of progress has been made because in this case, V has to *return to a previous x_τ axiom*, and thus we will threaten V to be not Δ_2^0.

From the above discussion, the reader will notice that the different requirements act almost independently of one another. In fact all that a single requirement needs to know is the correct initial segment of S. When diagonalizing against all Δ_2^0 sets, it may be possible for a requirement to flip S infinitely often. This, however, does not necessitate re-picking the followers of lower priority requirements. The only reason why R_2 needs to pick a new z_2 is because R_1 has seen δ_1, γ_1 converge, and wants to protect now a certain segment of S. This initialization happens only finitely often (despite R_1 flipping $S(z_1)$ infinitely often). Therefore it will be straightforward to combine the requirements, and will not require a tree argument as one usually expects in full approximation arguments.

2.2. The modular approach. We proceed in a general setting, and then obtain the main theorems as corollaries. We start by fixing a computable sequence $\{V_e\}_{e\in\mathbb{N}}$ of possible Δ^0_2-approximations. That is, $V_{e,s}(x)$ is a computable function of e, s, x. We say that V_e is Δ^0_2 if $\lim_s V_{e,s}(x)$ exists for all x and $V_e(x)$ is this limit.

Let the natural approximation of the jump of V_e be $V'_{e,s}(n) = 1$ iff $\Phi^{V_{e,s}}_{n,s}(n)\downarrow$ (as is customary we assume the hat trick, that there must be a divergence between consecutive convergences with different uses, see Soare [Soa87]). If V_e is Δ^0_2 then this serves as a natural Σ^0_2-approximation to the characteristic function of V'_e in the sense that $V'_e(n) = \liminf_s V'_{e,s}(n)$ for every n. However when approximating $V'_e\restriction x$ as a finite string, $V'_{e,s}\restriction x$ is obviously not ideal because $V'_e\restriction x$ might not be the lexicographically leftmost string specified by $V'_{e,s}\restriction x$ at infinitely many s. Here we think of 0 as being to the left of 1. It is easy to fix this by delaying any entry of n into the (approximation for the) jump in the following way.

We define another approximation $Q_e\restriction x[s]$ for $V'_e\restriction x$ this time by induction as follows: $0 \in Q_e[s]$ iff $\Phi^{V_{e,s}}_{0,s}(0)\downarrow$. For $n > 0$, let $t < s$ be maximal such that $Q_e\restriction n[t] = Q_e\restriction n[s]$. If $\Phi^{V_{e,r}}_{n,r}(n)\downarrow$ for all $t < r \le s$, and V_e has been stable below the use during this period, declare $n \in Q_e[s]$, and declare $n \notin Q_e[s]$ otherwise. Hence if V_e is Δ^0_2 then the lexicographically leftmost segment $Q_e\restriction x[s]$ specified infinitely often is the segment of the true jump $V'_e\restriction x$. The "delayed" approximation $\{Q_e[s]\}_s$ will be used when deciding whether or not to act for a module, since it is correct infinitely often. Furthermore the delayed approximation Q_e for V'_e is obtained effectively in e.

We have infinitely many modules $M_{\theta,e}$ indexed by a finite binary string θ and $e \in \mathbb{N}$. Here $M_{\theta,e}$ works in a similar way to requirement R_e above, and guesses that $\theta \subset S$. It outputs (effectively) an infinite binary sequence $m_{\theta,e}$ listing the stage by stage guesses as to whether our toggle point $z_{\theta,e}$ is in S, as well as a number $d_{\theta,e}$ such that if V_e is Δ^0_2, then

(P1) $m = \lim_s m_{\theta,e}(s)$ exists,

(P2) if γ_e and δ_e are total, then additionally $d_{\theta,e}\downarrow$ and we have either $\Gamma^{V'_e}_e(z_{\theta,e}) \neq m$ or $V'_e \neq \Delta_e^{(\theta^\frown m^\frown 0^\omega \oplus \emptyset')\restriction d_{\theta,e}}$.

Note that undefined counts as being not equal. For the rest of this section the reader should think of the construction of each $M_{\theta,e}$ as being run at every stage in isolation. That is, at stage s of the construction we evaluate the given parameters Q_e, γ_e, δ_e at stage s and output $m_{\theta,e}(s)$. There are no interactions between different modules. In section 3 we will then show how to combine the independent modules into a construction of a set S.

2.3. The construction for $M_{\theta,e}$. Now we give the actions of the module $M_{\theta,e}$.

Step 1: Let $z_{\theta,e} = |\theta|$.

Step 2: Wait for $\gamma_e(z_{\theta,e})\downarrow$. Using the recursion theorem, for each $\sigma \in 2^{\gamma_e(z_{\theta,e})}$, let $x_\sigma > \gamma_e(z_{\theta,e})$ be a number that we control for V'_e. That is, we may enumerate axioms for $\Phi^X_{x_\sigma}(x_\sigma)$ and also specify the use of the axioms. The recursion theorem allows us to obtain values for x_σ effectively. In short, we call these *axioms for* x_σ. Note that x_σ for different modules are different.

Wait for $\delta_e(\max_\sigma x_\sigma)\downarrow$. Let $d_{\theta,e} = \delta_e(\max_\sigma x_\sigma)$, and proceed to Step 3.

Step 3: We say that *s is a recovery stage* if $\Gamma^{Q_e}_e(z_{\theta,e})[s]\downarrow = m_{\theta,e}[s]$ and $\Delta_e^{(\theta^\frown m_{\theta,e}{}^\frown 0^\omega \oplus \emptyset')\restriction d_{\theta,e}}[s]\downarrow = Q_e[s]\restriction(1+\max_\tau x_\tau)$.

For clarity of presentation, we assume that the enumeration of Q_e is fixed and independent of our actions. In particular we do not follow the customary practice of using a slowdown lemma in the enumeration of the jump. That is, when we define some $\Phi^{V_e}_{x_\sigma}(x_\sigma)$ to converge, we do not assume that $Q_e(x_\sigma)$ responds instantly. If this computation we defined is indeed correct then this will be reflected eventually in $Q_e(x_\sigma)$ and we can just wait for it; on the other hand if V_e changes before $Q_e(x_\sigma)$ responds, then we would have made some progress since the use on the axiom for x_σ was based on some other "real" computation reflected earlier by Q_e. Consequently we say that $Q_e(x)$ is *good at stage s*, if x is an index which we control by the recursion theorem, and $Q_e(x)[s] = 1$ iff there is a current axiom at s which applies for x.

For any set $X \subset 2^\omega$ and any number $n \in \omega$, we let $u(n, X)$ denote the (current) use of the computation $\Phi^X_n(n)$. At each future stage of the construction subsequent to the point at which we complete step 2, and for each $\sigma \in 2^{\gamma_e(z_{\theta,e})}$, x_σ will have a mode associated to it, which will be either IN or OUT, reflecting our desire to have x_σ in or out of V'_e. Initially begin with all x_σ in mode OUT. Unless a change in mode is explicitly stated in the ensuing construction, the mode will not change from one stage to the next.

At all successive stages, $m_{\theta,e}(s)$ outputs the previous value unless $z_{\theta,e}$ is toggled in which case we flip $m_{\theta,e}(s)$.

Stage s: Let $\sigma = Q_e\restriction\gamma_e(z_{\theta,e})[s]$.

If x_σ has mode IN, and there is no axiom that currently applies for x_σ, we enumerate an axiom for x_σ with use $V_{e,s}\restriction\max\{u(q, V_{e,s}) \mid \sigma(q) = 1\}$.

If s is a recovery stage and $Q_e(x_\sigma)$ is good, we call s a *good recovery stage* and proceed as follows.

Case 1: *no axiom for x_σ applies*. Declare x_σ to have mode IN for stage $s+1$.

Case 2: *an axiom for x_σ applies*. Declare x_σ to have mode OUT for stage $s+1$, and toggle $z_{\theta,e}$.

2.4. Verification. This completes the construction. If $\sigma \subset Q_e[s]$ then we will refer to s as a σ-stage. We first make the following observation.

Lemma 2.1. *At all stages s after Step* 2 *is completed, if an axiom applies for x_τ, then it has use* $\max\{u(q, V_{e,s}) \mid \tau(q) = 1\}$ *with all the uses defined. Moreover, if x_σ has mode IN at stage s and $\sigma = Q_e\restriction\gamma_e(z_{\theta,e})[s]$ then $u(q, V_{e,s})\downarrow$ for every q such that $\sigma(q) = 1$.*

Proof. The first statement follows directly from the second, while the second statement follows from the fact that if $q \in Q_e[s]$ then $\Phi^{V_e}_q(q)[s]\downarrow$. ⊣

Lemma 2.2. *If $\sigma = Q_e\restriction\gamma_e(z_{\theta,e})[s]$ is to the left of τ and an axiom for x_τ currently applies* (*with use u*), *then $V_e\restriction u$ cannot have been stable since the last τ-stage.*

Proof. Since τ is to the right of σ there is a least $q < \gamma_e(z_{\theta,e})$ such that $\tau(q) = 1$ and $\sigma(q) = 0$. Since $\sigma(q) = 0$, we have $q \notin Q[s]$. We know $V_{e,s}$ extends η for some x_τ-axiom η enumerated earlier (say at t), hence $\Phi_q^\eta(q)[t]\downarrow$. This means that $\Phi_q^{V_e}(q)[s]\downarrow$ which means that V_e cannot extend η at every stage between the last $\sigma\restriction q$-stage and s (otherwise $q \in Q_e[s]$). ⊣

We recall that we only enter Case 1 or 2 at good recovery stages.

Lemma 2.3. *If Step 3 is started and V_e is Δ^0_2, then there are only finitely many good recovery stages. Consequently $z_{\theta,e}$ is toggled only finitely often.*

Proof. Assume for a contradiction that there are infinitely many good recovery stages. Let s be the stage by which $\emptyset'$ has settled on $d_{\theta,e}$. There are at most two possible configurations of $(\theta^\frown m_{\theta,e}{}^\frown 0^\omega \oplus \emptyset')\restriction d_{\theta,e}$ after s, which differ on the value of $m_{\theta,e}$. Every good recovery stage after s is either a σ_0-stage or a σ_1-stage, where σ_i corresponds to the configuration with $m_{\theta,e} = i$.

We first claim that there is a stage $s_1 > s$ such that Case 1 applies. Suppose not. Then at every good recovery stage after s, we must have Case 2 applies, whence $z_{\theta,e}$ is toggled. Thus as we visit the good recovery stages after stage s, we must be alternating between the two configurations σ_0 and σ_1, in order to recover the toggles. Also, after we have our first good recovery stage with configuration σ_i after stage s, we give x_{σ_i} mode OUT. Since we are assuming we never enter Case 1 after stage s, this means that x_{σ_i} will remain in mode OUT for the duration of the construction. In particular, it follows that only finitely many axioms are enumerated for x_{σ_1}. Let $t > s$ be a stage where $V_e\restriction\max\{\text{of the } x_{\sigma_1} \text{ axioms}\}$ is stable. Let $t_1 > t$ be a good recovery stage with configuration σ_1. Since we were in case 2, an axiom for x_{σ_1} applied at stage t_1. Let $t_2 > t_1$ be a good recovery stage with configuration σ_0. Since $t_1 > t$, the axiom for x_{σ_1} still applied at stage t_2. Now since $\sigma_0 = Q_e\restriction\gamma_e(z_{\theta,e})[t_2]$ is to the left of σ_1, Lemma 2.2 shows that V_e could not have been stable on the x_{σ_1} axiom since the previous σ_1-stage, giving the desired contradiction.

The above contradiction shows that s_1 exists. Suppose s_1 is a τ-stage. Let $s_2 > s_1$ be the next good recovery stage (we want to get a contradiction). Since $z_{\theta,e}$ is not toggled by the actions at s_1, it follows that the configuration of $(\theta^\frown m_{\theta,e}{}^\frown 0^\omega \oplus \emptyset')\restriction d_{\theta,e}$ at the beginning of s_2 is the same as at the beginning of s_1. Hence s_2 is also a τ-stage. Since x_τ receives mode IN at s_1, it follows that x_τ has mode IN at the beginning of s_2, where an axiom for x_τ will be enumerated (if there is not already one). At s_1, $Q_e(x_\tau)$ must be 0 because of its goodness, which means that at s_2, $Q_e(x_\tau)$ must be again 0 since s_2 is a recovery stage and a τ-stage. But an axiom for x_τ applies at stage s_2, and s_2 is a good stage, so $Q_e(x_\tau) = 1$, a contradiction. ⊣

Lemma 2.4. *$M_{\theta,e}$ satisfies* (P1) *and* (P2).

Proof. If V_e is Δ^0_2, then (P1) holds by Lemma 2.3. To show (P2) holds as well we assume that δ_e, γ_e are total (hence Step 3 is started). Let $\sigma = V'_e\restriction\gamma_e(z_{\theta,e})$, the true segment of V'_e. Also let $r = V'_e(x_\sigma)$. Hence there are infinitely many σ-stages s where $Q_e(x_\sigma)[s] = r$. We claim that $Q_e(x_\sigma)$ is good at almost every such stage.

For every p such that $\sigma(p) = 1$, p must be in the real V'_e and so it is easy to see that once V_e is stable below these uses, any x_σ-axiom we enumerate applies forever. Hence we only enumerate finitely many axioms for x_σ. If V_e extends one of these axioms then at almost every stage $V_e[s]$ extends the axiom and also $Q_e(x_\sigma) = 1$. If V_e extends none of these axioms then at almost every σ-stage where $Q_e(x_\sigma)[s] = 0$, we have $V_e[s]$ extending none of these axioms. Hence $Q_e(x_\sigma)$ is good at almost every σ-stage s where $Q_e(x_\sigma)[s] = r$.

Assume for a contradiction that the last condition in (P2) fails. By Lemma 2.3 let s_0 be a stage by which $(\theta^\frown m_{\theta,e}{}^\frown 0^\omega \oplus \emptyset')\upharpoonright d_{\theta,e}$ has settled. There are infinitely many stages after s_0 where $Q_e[t]\upharpoonright 1 + \max\{x_\tau\}$ is correct, and each of these is a recovery stage. By the above paragraph we will have infinitely many good recovery stages, contradicting Lemma 2.3. ⊣

We make a further comment. If we further assumed that $\{V_e\}$ is a c.e. approximation for every e, then the function $(\theta, e) \mapsto \lim_s m_{\theta,e}[s]$ is $\omega+1$-c.e.. To see this, suppose s is a stage where we toggled $z_{\theta,e}$. Follow the proof of Lemma 2.3 and see that after s, as long as there is no change to the $\emptyset'$ portion of $(\theta^\frown m_{\theta,e}{}^\frown 0^\omega \oplus \emptyset')\upharpoonright d_{\theta,e}$, we only toggle $z_{\theta,e}$ at most 4 times under Case 2 before Case 1 must apply at a good recovery stage. The second paragraph in the proof of Lemma 2.3 shows that $z_{\theta,e}$ is never toggled again, unless there is a change to the $\emptyset'$ portion of $(\theta^\frown m_{\theta,e}{}^\frown 0^\omega \oplus \emptyset')\upharpoonright d_{\theta,e}$. Hence, if V_e is c.e., then $z_{\theta,e}$ will be toggled no more than $2d_{\theta,e}$ many times.

§3. The failure of the analogs of jump inversion. Towards proving our main theorems, the module $M_{\theta,e}$ will meet requirement R_e, provided that θ is indeed the initial segment of the characteristic function of S. We now show how to combine the modules in such a way that for each e, there is a successful $M_{\theta,e}$ module.

Given a sequence $\{V_e\}$, we apply the previous section to get $m_{\theta,e}, d_{\theta,e}$. Hence $m_{\theta,e}(t)$ is the result after running module $M_{\theta,e}$ for t many steps. We now use the result from the different modules to specify an approximation $S[s]$ in the following way. First we order the finite binary strings: $\lambda \prec 1 \prec 0 \prec 11 \prec 10 \prec 01 \prec 00 \prec 111 \prec \cdots$. Hence $\prec$ refers to the ordering obtained by first considering increasing length, and then reverse lexicographic ordering. For each η we will have an associated binary string θ_η, and the corresponding $z_{\theta_\eta,|\eta|}$ as defined in module $M_{\theta_\eta,|\eta|}$. That is, $z_{\theta_\eta,|\eta|} = |\theta_\eta|$. For convenience, we will let z_η denote $z_{\theta_\eta,|\eta|}$. We will arrange it so that for $\eta' \prec \eta$, we have $z_{\eta'} < z_\eta$. Basically z_η serves as a pointer, and points to a location of S where $S(z_\eta)[s]$ will be approximated by the digits of $m_{\theta_\eta,|\eta|}$. Each η codes a guess as to the membership of $z_{\eta'}$ in S for $\eta' \prec \eta$. We will have θ_η represent the η-guess as to the correct initial segment $S\upharpoonright|\theta_\eta|$. As we give the stage by stage construction of S, we will move the pointers z_η, but each z_η will only be moved finitely often. Although z_η is defined to be the length of θ_η, in practice we will define z_η first, and later define θ_η. At every stage s, if y is not being pointed at (i.e., $y \neq z_\eta$ for any η), then we will have $S(y)[s] = 0$.

We give a few notations to be used. Define T_s to be a string of finite length, which can be thought of as the current approximation to the "true strategies". Loosely speaking, only those z_η where strategy $\eta \subset T$ will be the important ones; the other z_η with η not on T are just red herrings; they are the artifacts produced by our wrong guesses. T_s is defined inductively by: $T_s(n) = S(z_{T_s\upharpoonright n})[s]$. Proceed this way until we hit the first undefined z_η.

At stage s to *read the next digit of* $m_{\theta,e}$ means to do the obvious thing: if this is the first time we encounter this instruction then we output $m_{\theta,e}(0)$. Otherwise output $m_{\theta,e}(k+1)$ where $m_{\theta,e}(k)$ was the previous digit read by the construction.

Construction of S: at $s = 0$ make every z_η, θ_η undefined. At stage $s > 0$, only finitely many z_η, θ_η have been defined at the end of stage $s-1$. Go through all such η in increasing order, and for each we (inductively) update θ_η and specify $S(z_\eta)[s]$. For z_λ we let $\theta_\lambda = \lambda$ and set $S(z_\lambda)[s] =$ the next digit of $m_{\theta_\lambda,0}$.

Now assume that $S(z_{\eta'})[s]$ has been defined for all $\eta' \prec \eta$. We define θ_η as follows. For $y < z_\eta$ such that $y \neq z_{\eta'}$ for any η', set $\theta_\eta(y) = 0$. If $y < z_\eta$ is such that $y = z_{\eta'}$, then necessarily $\eta' \prec \eta$. If η' is lexicographically to the right of $\eta\restriction|\eta'|$ then set $\theta_\eta(z_{\eta'}) = 0$. If η' is left of $\eta\restriction|\eta'|$ then set $\theta_\eta(z_{\eta'}) = S(z_{\eta'})[s]$. Otherwise $\eta' = \eta\restriction|\eta'|$ and we let $\theta_\eta(z_{\eta'}) = \eta(|\eta'|)$. Next we define $S(z_\eta)[s]$ in the following way. Note first of all that $T_s\restriction|\eta|$ can be evaluated at this point. If $T_s\restriction|\eta| = \eta$ then we let $S(z_\eta)[s]$ be the next digit of $m_{\theta_\eta,|\eta|}$. If $T_s\restriction|\eta|$ is left of η then let $S(z_\eta)[s] = 0$. Otherwise if $T_s\restriction|\eta|$ is right of η we let $S(z_\eta)[s] = S(z_\eta)[s-1]$.

If some $d_{\theta_\eta,|\eta|}$ has converged at stage s, we make all $z_{\eta'}, \theta_{\eta'}$ undefined for all $\eta' \succ \eta$ and go to the next stage. Otherwise the above stops naturally when we find some least η with z_η not defined at stage $s-1$. We then pick a fresh value for z_η and set $S(z_\eta)[s] = 0$.

Finally let $S(x) = \liminf_s S(x)[s]$. It is clear that z_η eventually settles on a final value for each η, and also that $|T_s| \to \infty$. Let T be the leftmost path specified infinitely often by T_s. We first show that T actually reflects the correct η's:

LEMMA 3.1. *For every $\eta \subset T$, we have θ_η eventually settles, $\theta_\eta \subset S$ and $S(z_\eta) = T(|\eta|) = \liminf m_{\theta_\eta,|\eta|}$.*

PROOF. We proceed inductively on $|\eta|$. The statement clearly holds if $|\eta| = 0$ so take $|\eta| > 0$. After z_η settles, the value of θ_η and also $S\restriction z_\eta$ will be decided on the places $\{z_{\eta'} \mid \eta' \prec \eta\}$. There are three cases. If η' is right of $\eta\restriction|\eta'|$ then $\theta_\eta(z_{\eta'})$ is always 0, while at infinitely many stages s, $T_s \supset \eta$ which makes $S(z_{\eta'})[s] = 0$ infinitely often. If η' is left of $\eta\restriction|\eta'|$ then T_s is right of η' at every stage after some s_0. Hence $S(z_{\eta'})[s] = S(z_{\eta'})[s_0]$ for all $s > s_0$ and also $\theta_\eta(z_{\eta'})$ will agree with $S(z_{\eta'})[s_0]$. Finally if $\eta' \subset \eta$ then inductively let $\theta_{\eta'}$ be the limit value. It is easy to see that the value of $\theta_\eta(z_{\eta'}) = \eta(|\eta'|) = T(|\eta'|) = S(z_{\eta'})$. Hence θ_η eventually settles and $\theta_\eta \subset S$.

Since η is on T, hence for almost all s we have T_s is right of η (where $S(z_\eta)[s]$ is unchanged from the previous stage) or $T_s \supset \eta$ (in which the next digit of $m_{\theta_\eta,|\eta|}$ is read). Hence $S(z_\eta) = \liminf m_{\theta_\eta,|\eta|}$. To see that this value is the same as $T(|\eta|)$, observe that $T(|\eta|) = \liminf\{T_s(|\eta|) \mid T_s \supset \eta\} = \liminf\{S(z_\eta)[s] \mid T_s \supset \eta\} = \liminf m_{\theta_\eta,|\eta|}$. ⊣

LEMMA 3.2. *For every e, if V_e is Δ^0_2, then $V'_e \not\equiv_{wtt} S \oplus \emptyset'$.*

PROOF. We assume that $V'_e = \Delta_e^{S\oplus\emptyset'}$ and $S = \Gamma_e^{V'_e}$ with use bounded by δ_e, γ_e (which are total). Let $\eta = T\restriction e$. By Lemmas 2.4 and 3.1 $S(z_\eta) = \lim m_{\theta_\eta,e}$ where $\theta_\eta \subset S$. Since $d_{\theta_\eta,e}\downarrow$ then the initialization in the construction of S ensures that in fact $(S \oplus \emptyset')\restriction d_{\theta_\eta,e} = (\theta_\eta{}^\frown \lim m_{\theta_\eta,e}{}^\frown 0^\omega \oplus \emptyset')\restriction d_{\theta_\eta,e}$. A contradiction to the last condition of (P2) follows. ⊣

We now obtain as corollaries, the following three statements.

THEOREM 3.3. *For any computable sequence of Δ^0_2 sets $\{V_e\}_{e\in\mathbb{N}}$ (given by their Δ^0_2 indices), there exists a Δ^0_2 set $S \geq_{tt} \emptyset'$ such that for every e, $V'_e \not\equiv_{wtt} S$.*

PROOF. Apply the results of the past two sections, and $S \oplus \emptyset'$ is the desired set. Note that S is Δ^0_2 because of (P2) and the fact that it is easy to prove that $\{T_s\}$ itself is a Δ^0_2 approximation. ⊣

THEOREM 3.4. *There exists an $\omega + 1$-c.e. set $S >_{tt} \emptyset'$ such that there is no c.e. set A with $A' \equiv_{wtt} S$.*

PROOF. Theorem 3.3 gives a us a Δ^0_2 set S. To see that S can be made $\omega + 1$-c.e., use the fact that every module will reach a limit and modify the construction of S slightly to ensure that each time $S \upharpoonright z_\eta[s]$ changes we also reset z_η. It is not hard to see that the ensuing approximation for S will be $\omega + 1$-c.e.. We sketch the reason why, and leave the details to the reader. The value $S(z_\eta)[s]$ depends directly on the value of $T_s \upharpoonright |\eta|$, which in turn depend on $S(z_\nu)[s]$ where $\nu \prec \eta$. As long as $S \upharpoonright z_\eta[s]$ remains unchanged, we will either output 0 for $S(z_\eta)[s]$, or the digits of $m_{\theta_\eta,|\eta|}$. θ_η will also not change as long as $S \upharpoonright z_\eta[s]$ remains fixed. Hence the number of changes in $S(z_\eta)$ is at most the number of flips in $m_{\theta_\eta,|\eta|}$ (until z_η is cancelled). This number can be computed by the comments after Lemma 2.4. ⊣

THEOREM 3.5. *There exists a Σ^0_2 set $S >_{tt} \emptyset'$ such that there is no Δ^0_2 set A with $A' \equiv_{wtt} S$.*

PROOF. Use a list of all possible Δ^0_2 indices. ⊣

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