MINIMAL DEGREES AND DOWNWARDS DENSITY IN SOME STRONG POSITIVE REDUCIBILITIES AND QUASI-REDUCIBILITIES

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Abstract. We consider three strong reducibilities, s_1, s_2, Q_1 (where we identify a reducibility $\leqslant_r$ with its index r). The first two reducibilities can be viewed as injective versions of s-reducibility, whereas Q_1-reducibility can be viewed as an injective version of Q-reducibility. We have, with proper inclusions, $s_1 \subset s_2 \subset s$. It is well known that there is no minimal s-degree, and there is no minimal Q-degree. We show on the contrary that there exist minimal Δ^0_2 s_2-degrees, and minimal Δ^0_2 s_1-degrees. On the other hand both the Π^0_1 s_2-degrees, and the Π^0_1 s_1-degrees are downwards dense. By the isomorphism of the s_1-degrees with the Q_1-degrees induced by complementation of sets, it follows that there exist minimal Δ^0_2 Q_1-degrees, but the c.e. Q_1-degrees are downwards dense.

1. Introduction

Positive reducibilities formalize models of relative computability in which the computing agent accesses the information stored in an external database, called oracle, and therein coded as a set B of natural numbers, through questions of the form "Is some finite set $D \subseteq B$?", in such a way that an answer to any such question can be retrieved, and used by the computing agent to perform a corresponding action, only if the oracle, while enumerating the whole set B, does give positive evidence of the fact that D is a subset of B by eventually enumerating all the elements of D. Not surprisingly, therefore, positive reducibilities are often used to formalize in various ways the notion of relative enumerability of sets of natural numbers, in which a set A is informally reducible to a set B if there is an algorithm which enumerates A if given access to any enumeration of B. They have been an active field of research in computability theory in recent years, with special attention given to enumeration reducibility, see e.g. [1]. An important positive reducibility is s-reducibility: this paper is dedicated to two natural injective forms of s-reducibility, called s_1-reducibility and s_2-reducibility, for which we have, with proper inclusions, $s_1 \subset s_2 \subset s$. Whereas it is known that there is no minimal s-degree, we prove that there exist Δ^0_2 minimal s_1-degrees, and Δ^0_2 minimal s_2-degrees. On the other hand in both the degree structures corresponding to these reducibilities, the Π^0_1 degrees are downwards dense. The results about s_1-reducibility can be immediately translated into results about Q_1-reducibility, which can in turn be regarded as a natural injective version

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of Q-reducibility: via the isomorphism of s_1-reducibility with Q_1-reducibility provided by set-complementation, we derive that there is a Δ^0_2 minimal Q_1-degree, but the c.e. Q_1-degrees are downwards dense.

1.1. **The basic definitions.** The positive reducibility known as s-reducibility is obtained by restricting the class of enumeration operators, which provide enumeration reducibility, to a much smaller subclass of operators, called s-operators. By the proof of Gutteridge's theorem in [5] stating that there exists no minimal enumeration degree, it follows (see for instance [10]) that the degree structure corresponding to s-reducibility (and its isomorphic copy provided by Q-reducibility) does not possess minimal elements. Thus, non-minimality survives the drastic thinning of the class of operators which leads from enumeration reducibility to s-reducibility. In this paper we study minimality and downwards density questions concerning degree structures of some reducibilities arising from yet smaller, although natural, restrictions of the class of operators. We show that contrary to what happens for s-reducibility, the degree structures corresponding to these stronger reducibilities do have minimal elements. On the other hand, we also identify some natural downwards dense classes of degrees, and thus consisting of non-minimal degrees.

Tennenbaum (see [11, p. 159]) defined the notion of Q-reducibility on sets of natural numbers as follows: a set A is *Q-reducible* to a set B (in symbols: $A \leqslant_Q B$) if there exists a computable function f such that for every $x \in \omega$,

$$x \in A \Leftrightarrow W_{f(x)} \subseteq B,$$

where ω denotes the set of natural numbers and we refer to some fixed acceptable indexing $\{W_x : x \in \omega\}$ of the computably enumerable (or, simply, c.e.) sets, henceforth called the *standard* indexing of the c.e. sets (for notions and terminology relative to indexings, see for instance [11]). We say in this case that the function f *Q-reduces* A to B, or $A \leqslant_Q B$ *via* f. Esoteric as it might appear, Q-reducibility has been frequently and successfully applied to computability theory, abstract complexity theory, group theory and word problems: we refer the reader to Omanadze's paper [7] for an exhaustive survey of these applications of Q-reducibility. The degree structure corresponding to Q-reducibility has a least degree $\mathbf{0}_Q$ consisting of the Π^0_1 sets (it is convenient to exclude ω from the universe of the reducibility, as $A \leqslant_Q \omega$ if and only if $A = \omega$).

If $A \leqslant_Q B$ via a computable function f such that for all x, y,

$$x \neq y \Rightarrow W_{f(x)} \cap W_{f(y)} = \varnothing,$$

then we say that A is *Q_1-reducible* to B, denoted by $A \leqslant_{Q_1} B$, *via* f. One can view Q_1-reducibility as the "injective" version of Q-reducibility.

As already mentioned, the reducibility known as s-reducibility is a restricted version of enumeration reducibility. We recall that any c.e. set W *defines* an *enumeration operator* (for short: *e-operator*), i.e. a mapping Φ_W from the power set of ω to the power set of ω such that, for $A \subseteq \omega$,

$$\Phi_W(A) = \{x : (\exists u)\,[\langle x, u\rangle \in W \ \&\ D_u \subseteq A]\,\},$$

where D_u is the finite set with canonical index u: throughout the rest of the paper, we will often identify finite sets with their canonical indices, thus writing, for instance, $\langle x, D\rangle$ instead of $\langle x, u\rangle$ if $D = D_u$. If $A = \Phi(B)$ for some e-operator Φ then we say that A is *enumeration reducible* to B (or, more simply, A is *e-reducible* to B; in symbols: $A \leqslant_e B$) *via* Φ. An e-operator Φ is said to be an *s-operator*, if Φ is defined by a c.e. set W such that

$$(\forall \text{ finite } D)(\forall x)[\langle x, D\rangle \in W \Rightarrow \text{card}(D) \leqslant 1]$$

(where the symbol $\mathrm{card}(X)$ denotes the cardinality of a given set X). Following [4] we say that A is *s-reducible* to B (in symbols: $A \leqslant_s B$) if $A = \Phi(B)$, for some s-operator Φ. The degree structure corresponding to $\leqslant_s$ has a least degree $\mathbf{0}_s$ consisting of the c.e. sets.

If $B \neq \varnothing$ then it is just an exercise to see that $A \leqslant_s B$ if and only if there exists a computable function f such that

$$x \in \overline{A} \Leftrightarrow W_{f(x)} \subseteq \overline{B}$$

(where $\overline{X} = \omega \smallsetminus X$ for any subset $X \subseteq \omega$): in other words $A \leqslant_s B$ if and only if $\overline{A} \leqslant_Q \overline{B}$. (In fact, Friedberg and Rogers [4] define $\leqslant_s$ as $\leqslant_Q$ on complements: our characterization of s-reducibility, using s-operators, follows [6]). Hence the Q-degrees are order isomorphic with the s-degrees.

It is easy to isolate the subreducibility $\leqslant_{s_1}$ of $\leqslant_s$ which equals Q_1-reducibility on complements of sets. Precisely, $A \leqslant_{s_1} B$ if and only if $A = \Phi(B)$ via an *s_1-operator*, i.e. an s-operator such that:

(a) $\{z : \langle x, \{z\}\rangle \in \Phi\} \cap \{z : \langle y, \{z\}\rangle \in \Phi\} = \varnothing$ for all pair x, y of numbers such that $x \neq y$;
(b) there is no x with $\langle x, \varnothing\rangle \in \Phi$.

Again, an easy calculation shows that for any pair A, B of sets one has $A \leqslant_{s_1} B$ if and only if $\overline{A} \leqslant_{Q_1} \overline{B}$ (no need to assume now that $B \neq \varnothing$).

We will be interested in yet another strong form of s-reducibility, which we will call *s_2-reducibility*, denoted by the symbol $\leqslant_{s_2}$: we define $A \leqslant_{s_2} B$ if $A = \Phi(B)$ for some *s_2-operator* Φ, i.e. an s-operator such that for all distinct x, y, $\{z : \langle x, \{z\}\rangle \in \Phi\} \cap \{z : \langle y, \{z\}\rangle \in \Phi\} = \varnothing$, but Φ may contain axioms of the form $\langle x, \varnothing\rangle \in \Phi$. Then clearly every s_1-operator is an s_2-operator and thus $\leqslant_{s_1} \subseteq \leqslant_{s_2}$, the inclusion being proper, see for instance Corollary 1.4.

Let Φ_e be the enumeration operator defined by the c.e. set W_e. It is straightforward to see that for each $r \in \{s, s_1, s_2\}$ there is a computable functions f_r such that for every e, $\Phi_{f_r(e)}$ is an r-operator, and if Φ_e is already an r-operator then $\Phi_{f_r(e)} = \Phi_e$. Therefore we can refer to the indexing $\{\Psi_e : e \in \omega\}$ of the r-operators given by $\Psi_e = \Phi_{f_r(e)}$: this indexing will be called the *standard* indexing of the r-operators. We also refer to uniform computable approximations $\{\Psi_{e,s} : e, s\}$ to the r-operators, where each $\Psi_{e,s}$ is a finite set uniformly given by its canonical index, with $\Psi_{e,0} = \varnothing$, $\Psi_{e,s} \subseteq \Psi_{e,s+1}$ and $\Psi_e = \bigcup_s \Psi_{e,s}$: for instance take $\Psi_{e,s}$ to be the r-operator defined by $W_{r(e),s}$, where $\{W_{r(e),s} : s \in \omega\}$ is a uniform computable approximation to $W_{r(e)}$ via finite sets as defined for instance in [12, p.17].

We trust that there will be no confusion between the use of s to indicate an approximating stage, and the use of s to indicate s-reducibility.

1.2. **The degree structures and their local structures.** For any $r \in \{s, s_1, s_2, Q, Q_1\}$ the reducibility $\leqslant_r$ originates a degree structure which we denote by $\mathcal{D}_r$, the structure of the *r-degrees*; given a set A, the r-degree of A will be denoted by $\deg_r(A)$.

Definition 1.1. For $r \in \{s, s_1, s_2\}$ the *local structure* of the r-degrees consists of the r-degrees of the Σ^0_2 sets; for $r \in \{Q, Q_1\}$ the *local structure* of the r-degrees consists of the r-degrees of the Π^0_2 sets. (In any of the above cases, the local structure of the r-degrees will be denoted by the symbol $\mathcal{L}_r$.)

The local structures are lower cones as can be easily seen from the following lemma.

Lemma 1.2. *Let A be any set. If $r \in \{s, s_1, s_2\}$ then $A \in \Sigma^0_2$ if and only if $A \leqslant_r \overline{K}$ (where $\overline{K}$ is the complement of the halting set). Taking complements, for $r \in \{Q, Q_1\}$ we have that $A \in \Pi^0_2$ if and only if $A \leqslant_r K$.*

Proof. We prove the claim for $r \in \{s, s_1, s_2\}$. Clearly if $A \leqslant_s B$ (even $A \leqslant_e B$) and $B \in \Sigma^0_2$ then $A \in \Sigma^0_2$. On the other hand let $A \in \Sigma^0_2$, and let R be a decidable relation such that $A = \{x : (\exists y)(\forall z)R(x, y, z)\}$. Thus $A = \{x : (\exists y)[\langle x, y\rangle \in B]\}$ where B is the Π^0_1 set $B = \{\langle x, y\rangle : (\forall z)R(x, y, z)\}$. Let now f be a 1-1 computable function reducing $B \leqslant_1 \overline{K}$. Then $\Gamma = \{\langle x, \{f(\langle x, y\rangle)\}\rangle : x, y \in \omega\}$ is an s_1-operator which reduces $A \leqslant_{s_1} \overline{K}$. □

If $r \in \{s, s_1, s_2\}$ then let $\mathbf{1}_r = \deg_r(\overline{K})$, and if $r \in \{Q, Q_1\}$ then let $\mathbf{1}_r = \deg_r(K)$. From Lemma 1.2 it follows that $\mathcal{L}_r = \mathcal{D}_r(\leqslant_r \mathbf{1}_r)$.

It is easy to check that the s- and s_2-degrees have a least element, denoted by $\mathbf{0}_r$, for each $r \in \{s, s_2\}$, consisting of exactly the c.e. sets. This follows from the fact that if A is c.e. then for every B, $A \leqslant_{s_2} B$ via the s_2-operator $\Phi = \{\langle x, \varnothing\rangle : x \in A\}$. Therefore, for $r \in \{s, s_2, Q\}$, the local structure $\mathcal{L}_r$ coincides with the closed interval $[\mathbf{0}_r, \mathbf{1}_r] = \{\mathbf{a} : \mathbf{0}_r \leqslant_r \mathbf{a} \leqslant_r \mathbf{1}_r\}$ of r-degrees (as already observed, the least Q-degree $\mathbf{0}_Q$ consists of the Π^0_1 sets, excluding ω).

Things are different for s_1- and Q_1-reducibility. We describe what happens for s_1, and leave to the reader the analysis for Q_1 using the isomorphism given by taking complements of sets. If A, B are c.e. sets of the same cardinality then A, B fall in the same s_1-degree. To see this let A, B be c.e. sets of the same cardinality, and φ a partial computable bijection of A onto B: then $\Gamma = \{\langle x, \{\varphi(x)\}\rangle : x \in A\}$ is an s_1-operator such that $A = \Gamma(B)$; similarly one gets an s_1-operator $\Delta = \{\langle x, \{\varphi^{-1}(x)\}\rangle : x \in B\}$ such that $B = \Delta(A)$. Moreover, if A, B are c.e. sets and $\text{card}(A) < \text{card}(B)$ then $A <_{s_1} B$. (This already shows also that s_1 is properly contained in s_2, as all c.e. sets fall in the same s_2-degree $\mathbf{0}_{s_2}$.) Therefore, similarly to what happens for 1-reducibility, it is easy to see that the finite sets have a special status for s_1: their s_1-degrees (called *finite* s_1-degrees) form an initial segment of order type ω, where the n-th place of the chain is the s_1-degree consisting of exactly the finite sets of cardinality n, and all other s_1-degrees are above these finite degrees. On the other hand the infinite c.e. sets constitute a single s_1-degree, which is a minimal upper bound of all finite s_1-degrees. Let us call $\mathbf{0}_{s_1}$ the s_1-degree containing all infinite c.e. sets: clearly, $\mathbf{0}_{s_1} = \deg_{s_1}(\omega)$ (the equivalent for the Q_1-degrees is $\mathbf{0}_{Q_1} = \deg_{Q_1}(\varnothing)$).

Lemma 1.3. *If $A \in \Delta^0_2$, then $\mathbf{0}_{s_1} \leqslant_{s_1} \deg_{s_1}(A)$ if and only if A is infinite and not hyperhyperimmune.*

Proof. If $\omega \leqslant_{s_1} A$ then going to complements we have that $\varnothing \leqslant_{Q_1} \overline{A}$, i.e. there is a computable function f such that for all x, $W_{f(x)} \cap A \neq \varnothing$. As $\overline{A} \in \Delta^0_2$, by [10, Corollary 5] we can assume that for every x, $W_{f(x)}$ is finite. Hence $\{W_{f(x)} : x \in \omega\}$ is disjoint weak array witnessing that A is not hyperhyperimmune. Conversely, suppose that A is infinite but not hyperhyperimmune, and let $\{W_{f(x)} : x \in \omega\}$ be a disjoint weak array of finite sets witnessing this fact: thus $\varnothing \leqslant_{Q_1} \overline{A}$ via f, giving by complements that $\omega \leqslant_{s_1} A$. □

Corollary 1.4. *There is no least non-finite s_1-degree. Moreover, reducibility s_1 is properly included in reducibility s_2, even if one restricts oneself only to infinite sets.*

Proof. If there were a least non-finite s_1-degree then this should be $\mathbf{0}_{s_1}$ as the c.e. sets are downwards closed under s_1. But by the previous lemma, $\mathbf{0}_{s_1}$ is not below the s_1-degree of any hyperhyperimmune Δ^0_2 set.

The previous lemma shows also that there exist infinite sets A such that $\omega \nleqslant_{s_1} A$, whereas, as already observed, $\omega \leqslant_{s_2} A$ for every A: this implies the second claim in the statement. □

1.3. **Minimality.** In a poset $\langle P, 0, \leqslant\rangle$ with least element 0, by a *minimal* element we usually mean an *atom*, i.e. an element $a \in P$ such that $a \nleqslant 0$, and $b < a$ implies $b \leqslant 0$: this is what is commonly meant when working in degree structures with a least element, and this is our definition of a minimal s_2-degree as well. Recall that neither $\mathcal{D}_s$ nor $\mathcal{D}_Q$ has minimal elements, as follows from the proof of Gutteridge's theorem in [5].

However, the above definition would be of very little interest in investigating the s_1-degrees and the Q_1-degrees, as there would be only one such minimal degree: in the s_1-degrees, this would be for instance the s_1-degree of any singleton. For $r \in \{s_1, Q_1\}$, although $\mathbf{0}_r$ is not the lest r-degree, we propose therefore a different definition, and say that an r-degree $\mathbf{a}$ is *minimal* if

$$\mathbf{a} \nleqslant_r \mathbf{0}_r \;\&\; [\mathbf{b} <_r \mathbf{a} \Rightarrow \mathbf{b} \leqslant_r \mathbf{0}_r],$$

for all r-degrees $\mathbf{b}$: this looks in any case like the familiar definition of a minimal element in a poset $\langle P, 0, \leqslant\rangle$ with least element 0, once we interpret 0 with $\mathbf{0}_r$.

Chitaia and Omanadze have extensively investigated the substructure of the Q_1-degrees (in fact of the local structure $\mathcal{L}_{Q_1}$) consisting of the c.e. Q_1-degrees [8, 9, 2]. When restricting the universe to the c.e. Q_1-degrees our definition of a minimal element is equivalent with what Chitaia and Omanadze call a minimal c.e. Q_1-degree, namely a degree of an undecidable c.e. set A so that if B is a c.e. set with $B <_{Q_1} A$ then B is decidable. In particular Chitaia [2] has shown the following result for the Q_1-degrees of c.e. sets.

Fact 1.5. [2] *The c.e. Q_1-degrees are not dense, and no c.e. Q_1-degree below a hyperhypersimple Q_1-degree can be minimal in the c.e. Q_1-degree, more precisely if A is hyperhypersimple and $B \leqslant_{Q_1} A$ is an undecidable c.e. set then there exists an undecidable c.e. set $Y <_{Q_1} B$.*

We show that the structures of Δ^0_2 s_2- and s_1-degrees both have minimal elements (Theorem 2.1 and Theorem 3.1, respectively). On the other hand, both the structures of Π^0_1 s_2- and s_1-degrees are downwards dense (Theorem 4.1 and Corollary 5.3 respectively). To show downwards density of the Π^0_1 s_1-degrees, we show that if $\mathbf{a}$ is a Π^0_1 s_1-degree which is not c.e. and not hyperhyperimmune, then $\mathbf{a}$ is not minimal in the Π^0_1 s_1-degrees: the full claim then follows by combining this result with Fact 1.5, and using the isomorphism between s_1- and Q_1-degrees induced by complementation of sets.

Notice also that each one of the local structures $\mathcal{L}_r$ considered in this paper is an initial segment of the corresponding degree structure $\mathcal{D}_r$, and consequently minimality results for $\mathcal{L}_r$ yield *ipso facto* minimality results for the full structure $\mathcal{D}_r$.

2. Minimal Δ^0_2 s_2-degrees

It is known that there is no minimal s-degree, as a consequence of the fact that Gutteridge's classical theorem showing that there is no minimal e-degree makes use of special e-operators (often known as *Gutteridge operators*) which are s-operators, in such a way that the proof works also for s-reducibility, and thus shows that there is no minimal s-degree. Given the isomorphism of the Q-degrees with the s-degrees, it follows that there is no minimal Q-degree either.

Contrary to downward density of the s-degrees, we show in this section that $\mathcal{D}_{s_2}$ does have minimal elements, in fact the local structure $\mathcal{L}_{s_2}$ does, as there exists a minimal Δ^0_2 s_2-degree.

Theorem 2.1. *There exists a Δ^0_2 set with minimal s_2-degree.*

Proof. The requirements for building A are (where W and Ψ are respectively a given c.e. set and a given s_2-operator):

P_W : $A \neq W$;

M_Ψ : $X = \Psi(A) \Rightarrow X$ c.e. $\vee A \leqslant_{s_2} X$.

We briefly sketch the strategies in isolation to meet the requirements, and their outcomes. The construction is organized in a tree of strategies, in which each node is responsible for passing along an infinite stream of numbers to its descendants.

Strategy for P_W. This is the usual Friedberg-Muchnik diagonalization strategy. We appoint a witness x and restrain it in A, waiting for it to show up in W: if and when this happens, we extract x from A, and restrain it out of A.

Outcomes. The outcomes of the strategy are clear.

Strategy for M_Ψ, in isolation. The strategy for M_Ψ first tries to figure out whether the set

$$Z = \{x : (\exists y)\langle x, \{y\}\rangle \in \Psi \;\&\; \langle x, \varnothing\rangle \notin \Psi\}$$

is finite. Notice that the set Z is 2-c.e. and so the finiteness of Z cannot be represented as a single Σ^0_2 or Π^0_2 outcome. In fact the strategy has three outcomes $d < \infty < w$: the outcomes d and ∞ are Π^0_2 outcomes, while w is a Σ^0_2 outcome. The strategy tries to build an infinite set Str_∞ (which starts anew as empty every time outcome ∞ is initialized, which happens when the strategy takes outcome d), or an infinite set Str_d.

In more detail:

(a) The strategy for M_Ψ waits for new elements to enter Z, in order to identify new ∞-setups to be used for building the set Str_∞. While waiting it plays outcome w.
Outcome w. If we get stuck waiting forever in outcome w, then either there are only finitely many x with an axiom $\langle x, D\rangle \in \Psi$, or almost every axiom of Ψ is of the form $\langle x, \varnothing\rangle \in \Psi$. In either case, $X = \Psi(F)$ for some finite set F, so X is c.e. and the requirement is satisfied.

(b) (*Appointing a new ∞-setup* $(x, \psi(x))$) Suppose that at a stage s we see a new element x entering Z via an axiom $\langle x, \{y\}\rangle \in \Psi$ with $x > m$ and $y > \psi(m)$, for all ∞-setups $(m, \psi(m))$ which have been appointed so far after the last initialization of outcome ∞. (Notice that if infinitely many new elements enter Z, then by injectivity of Ψ for every finite set F we are guaranteed that eventually we see an x entering Z via an axiom $\langle x, \{y\}\rangle \in \Psi$ with $x, y > \max F$.) If this happens then we play outcome ∞, we pick such a number x and a unique corresponding axiom $\langle x, \{y\}\rangle \in \Psi$, and we associate y with x: let us denote such a y by $\psi(x)$. We appoint the pair $(x, \psi(x))$ as a new ∞-*setup*, and add $\psi(x)$ to the set Str_∞. We then restrain out of A all $y < s$ such that $y \neq \psi(m)$ for all m (including x) such that $\psi(m) \in \mathrm{Str}_\infty$: this activity of restraining numbers out of A will be called *securing outcome* ∞. Doing this will ensure at the end of stage s that $m \in X$ if and only if $\psi(m) \in A$, for all m such that $(m, \psi(m))$ is a current ∞-setup, and as long as none of these numbers m leaves Z we have that $m \in X$ if and only if $\psi(m) \in A$ for each one of them.

Outcome ∞. Outcome ∞ indicates that eventually outcome ∞ stops being re-initialized, and after its last initialization, when Str_∞ started anew as empty, we appoint infinitely many ∞-setups $(x, \psi(x))$, of which none gets later lost because x leaves Z. In fact the set Str_∞, comprised of the numbers $\psi(x)$ corresponding to the setups which do not get lost after the last initialization of outcome ∞, is an infinite decidable set since it is computably enumerated in strictly ascending order. By securing outcome ∞, we make $A \subseteq \mathrm{Str}_\infty$. Finally, we define

$$\Phi = \{\langle \psi(x), \{x\}\rangle : \psi(x) \in \mathrm{Str}_\infty\}.$$

Then Φ is an s_2-operator with $A = \Phi(\Psi(A))$. The strategies below the strategy for M_Ψ will receive in this case an infinite stream of numbers, consisting of the numbers $y \in \mathrm{Str}_\infty$, which they can freely use, since extracting from A, or enumerating them in A, will not affect the equality $A = \Phi(\Psi(A))$. We need for this the fact that Ψ is injective: for every given m, whatever value is eventually assigned to any $A(\psi(x))$, with $x \neq m$, will not alter the fact that $(m, \psi(m))$ is an ∞-setup, and thus $A(\psi(m)) = \Phi(\Psi(A))(\psi(m))$, since there is no axiom $\langle m, \{\psi(x)\}\rangle \in \Psi$.

(c) (*Appointing a new d-setup* $(x, \psi(x))$) A pair $(x, \psi(x))$ which became an ∞-setup at some stage t can fail to be an ∞-setup at a stage $s > t$ if, at stage s, the number x is *dumped into* $\Psi(A)$, i.e. the pair $\langle x, \varnothing\rangle$ is enumerated into Ψ at stage s, causing x to leave Z. Notice that in this case x is in $\Psi(A)$ regardless of what we decide about the value $A(\psi(x))$. When at a stage s we discover such a pair $(x, \psi(x))$ we appoint the least one (by code) as a new *d-setup*, we add $\psi(x)$ to the set Str_d, we play outcome d, we permanently restrain out of A all elements $y < s$ with $y \neq \psi(m)$, for all current d-setups $(m, \psi(m))$ (including the new d-setup $(x, \psi(x))$): this activity of restraining numbers out of A will be called *securing outcome d*. Finally, we initialize again outcome ∞, letting Str_∞ being empty again.

Outcome d. Outcome d indicates that we play outcome d infinitely many times, and thus there are infinitely many d-setups. We come up in this case with an infinite c.e. set Str_d consisting of all $\psi(x)$ such that $(x, \psi(x))$ is appointed as a d-setup. The strategies below outcome d for M_Ψ will receive the infinite stream consisting of the numbers $y \in \mathrm{Str}_d$. This means that $X = \Psi(A)$ is c.e., since by the d-securing activity, X consists of the numbers m such that $\langle m, \varnothing\rangle \in \Psi$. The lower priority strategies can freely use the numbers $\psi(x)$ lying in the infinite stream Str_d handed down to them, and define the values $A(\psi(x))$ as it is most conveniente for them.

Before looking at how the various strategies interact with each other let us introduce the tree of outcomes for our construction.

2.1. **The tree of outcomes.** We briefly introduce some basic notions relative to the priority argument used in this proof: for more on priority arguments, see Soare's textbook [12]. The *tree of outcomes* (or *tree of strategies*) is the smallest set of strings $T \subseteq \{d, \infty, w\}^*$ (where, for any set X, we denote by X^* the set consisting of all finite strings of elements of X) such that: T contains the empty string λ; if $\sigma \in T$ and its length $|\sigma|$ is even, then $\{\sigma^\frown\langle d\rangle, \sigma^\frown\langle w\rangle\} \subseteq T$ (the symbol $^\frown$ denotes concatenation of strings); if $\sigma \in T$ and $|\sigma|$ is odd, then $\{\sigma^\frown\langle d\rangle, \sigma^\frown\langle \infty\rangle, \sigma^\frown\langle w\rangle\} \subseteq T$. If σ is an initial segment of τ we write $\sigma \subseteq \tau$; we write $\sigma <_L \tau$ to denote that there is a string ρ and $o, o' \in \{d < \infty < w\}$ with $o < o'$ such that $\rho^\frown\langle o\rangle \subseteq \sigma$ and $\rho^\frown\langle o'\rangle \subseteq \tau$; finally we write $\sigma \leqslant \tau$ if $\sigma <_L \tau$ or $\sigma \subseteq \tau$, and $\sigma < \tau$ if $\sigma \leqslant \tau$ but $\sigma \neq \tau$. We say that τ *has higher priority than* σ (or σ *has lower priority than* τ) if $\tau < \sigma$.

Each string $\sigma \in T$ will be viewed as a strategy aiming at meeting a requirement $R(\sigma)$ as follows. Let us index the requirements by letting $P_e = P_{W_e}$ and $M_e = M_{\Psi_e}$ (where we refer to the standard indexings $\{W_e\}_{e\in\omega}$ and $\{\Psi_e\}_{e\in\omega}$ of the c.e. sets and of the s_2-operators, respectively), and linearly order them as

$$P_0 < M_0 < P_1 < M_1 < \cdots .$$

Finally let

$$R(\sigma) = \begin{cases} P_e, & \text{if } |\sigma| = 2e, \\ M_e, & \text{if } |\sigma| = 2e+1. \end{cases}$$

The way the strategy σ will try to meet $R(\sigma)$ if $R(\sigma) = M_e$ is by building a c.e. set X and an s_2-operator Φ such that $X = \Psi_e(A)$ or $A = \Phi(\Psi_e(A))$.

A strategy σ is a *P-node* or a *P-strategy* if $R(\sigma)$ is a P requirement, otherwise σ is an *M-node* or an *M-strategy*. If σ is an M-strategy, and $R(\sigma) = M_e$, then we often denote $\Psi_e = \Psi_\sigma$.

The environment of a strategy σ. In the construction, at each stage s we will define a string $\delta_s \in T$. For every $\sigma \in T$, a stage s is a *σ-stage* if $\sigma \subseteq \delta_s$. The construction will make use of additional parameters, described as follows. If σ is a P-node then $x_{\sigma,s}$ is the approximation at stage s of the witness chosen by strategy σ for its diagonalization; if σ is an M-node, and (x, y) is appointed as a new ∞-setup at s, then we denote $y = \psi_\sigma(x)$: the value $\psi_\sigma(x)$ will never be redefined, as follows from the remarks accompanying the formal definition of a (σ, ∞)-setup which we give later; every node σ at stage s is equipped with a set $\mathrm{Str}_{\sigma,s}$, called the *stream of σ at s*, which collects the numbers among which the lower priority strategies $\tau \supseteq \sigma$ may choose their witnesses or the elements of their streams at stage s. When a strategy σ acts, it specifies which outcome o it takes, and the values of the various parameters for $\sigma^\frown\langle o\rangle$.

To *initialize* a strategy σ at stage s means: to set $\mathrm{Str}_{\sigma,s} = \varnothing$, and to set as undefined the parameter $x_{\sigma,s}$. If σ is a strategy and s is a stage, denote by s_σ the maximum between $|\sigma|$ and the last stage $t < s$ at which σ was initialized. Since by construction all strategies are initialized at stage 0, we have that for every $s > 0$ the parameter s_σ is defined. Clearly s_σ is a function of s, σ. When σ is a strategy that eventually stops being re-initialized, then we will use s_σ^{fin} to denote the limit value of this function, i.e. the maximum between $|\sigma|$ and the last stage at which σ has been re-initialized.

Finally we assume, without loss of generality, that all numbers which appear in the construction by the end of stage s are $< s$.

2.2. **Interactions between strategies.**

Most of the concerns about the interactions between strategies are resolved through the use of the initialization mechanism. In deciding which outcome o to choose at s, strategy σ will in general look only at numbers $y \geqslant s_{\sigma^\frown\langle o\rangle}$, as it will regard numbers $z < s_{\sigma^\frown\langle o\rangle}$ as not usable because restrained in, or out of, A by the actions previously taken by higher priority strategies. Thus each strategy τ will use, at any stage s, only numbers $x \geqslant s_\tau$ so that x is bigger than the last stage $t < s$ at which τ was initialized, and thus x is bigger than all numbers used by any higher priority strategy whose action has initialized τ. Notice that by definition $s_\tau \geqslant |\tau|$ for every τ: this feature will be used to argue that the set A is Δ^0_2.

Definition 2.2. If σ is an M-strategy and $o \in \{d, \infty\}$ then we say that *x is restrained in $\Psi_\sigma(A)$ for $\sigma^\frown\langle o\rangle$ at stage $s+1$ by higher priority strategies*, if $x \in \Psi_{\sigma,s+1}(A^s \cap [0,(s+1)_{\sigma^\frown\langle o\rangle}))$ (in other words, if x has already been dumped into $\Psi_\sigma(A)$, i.e. by stage $s+1$ the axiom $\langle x, \varnothing\rangle \in \Psi_\sigma$ has appeared, or by stage $s+1$ an axiom $\langle x, \{z\}\rangle \in \Psi_\sigma$ has appeared such that $z \in A^s \cap [0,(s+1)_{\sigma^\frown\langle o\rangle}))$.

Notice that if $\sigma^\frown\langle o\rangle$ eventually stops being re-initialized, then for any $x \in [0, s^{fin}_{\sigma^\frown\langle o\rangle})$ the value $A(x)$ will never change after $s^{fin}_{\sigma^\frown\langle o\rangle}$, because this value may change only if x is involved in the action of a strategy of higher priority than $\sigma^\frown\langle o\rangle$, which would consequently re-initialize $\sigma^\frown\langle o\rangle$.

We now refine and make more precise the notion of a setup, taking this time into account interactions between strategies.

Definition 2.3. We say that a pair (x, y) is a *potential new* (σ, ∞)*-setup at stage* $s+1$ if the following happen: $y \in \mathrm{Str}_{\sigma,s+1}$; $\langle x, \{y\}\rangle \in \Psi_{\sigma,s+1}$, and $x > m$ and $y > \psi_\sigma(m)$ for each (σ, ∞)-setup $(m, \psi_\sigma(m))$ previously appointed after $(s+1)_{\sigma^\frown\langle\infty\rangle}$; x is not restrained in $\Psi_\sigma(A)$ for $\sigma^\frown\langle\infty\rangle$ at $s+1$ by higher priority strategies; finally $x, y \geqslant (s+1)_{\sigma^\frown\langle\infty\rangle}$ (thus if $\sigma^\frown\langle\infty\rangle$ is re-initialized at some later stage which will be $> x, y$, then after this re-initialization the number x will never be used to build a new (σ, ∞)-setup again, and therefore we never need to re-define $\psi_\sigma(x)$ again).

If there is a potential new (σ, ∞)-setup at $s+1$, we *select* the one, say (x, y), with least code, and define $y = \psi_\sigma(x)$, so that we have appointed a *new* (σ, ∞)*-setup at* $s+1$, and we enumerate $y \in \mathrm{Str}_{\sigma^\frown\langle\infty\rangle,s+1}$.

Definition 2.4. We say that a pair $(x, \psi_\sigma(x))$ with $\psi_\sigma(x) \in \mathrm{Str}_{\sigma^\frown\langle\infty\rangle,s}$ is a *potential new* (σ, d)*-setup at stage* $s+1$ if x is restrained in $\Psi_\sigma(A)$ for $\sigma^\frown\langle d\rangle$ at $s+1$ (thus including also when at $s+1$ we have the axiom $\langle x, \varnothing\rangle \in \Psi_\sigma$) by higher priority strategies (notice that since $(s+1)_{\sigma^\frown\langle d\rangle} \leqslant (s+1)_{\sigma^\frown\langle\infty\rangle}$ it automatically follows that $x, \psi_\sigma(x) \geqslant (s+1)_{\sigma^\frown\langle d\rangle}$). (With respect to our description of the strategy in isolation, we now regard as dumped into $\Psi_\sigma(A)$ not only those numbers x for which we have $\langle x, \varnothing\rangle \in \Psi_\sigma$, but also those x which are currently restrained in $\Psi_\sigma(A)$ by higher priority strategies via axioms of the form $\langle x, \{z\}\rangle \in \Psi_\sigma$.)

If there is a potential new (σ, d)-setup at stage $s+1$, then we *select* the least one, say (x, y), which becomes a *new* (σ, d)*-setup at stage* $s+1$, and we enumerate $y \in \mathrm{Str}_{\sigma^\frown\langle d\rangle,s+1}$.

In addition to the restraints demanded by higher priority strategies as a consequence of their actions at previous stages, a strategy σ must deal at stage $s+1$ with the new restraints demanded at $s+1$ by M-strategies $\tau \subset \sigma$, when $\tau^\frown\langle o\rangle \subseteq \sigma$ for $o \in \{d, \infty\}$. These new restraints (described earlier as aiming at "securing outcome o") want to restrain out of A certain numbers y such that $y \neq \psi_\tau(m)$ where $\psi_\tau(m)$ belongs to the current $\mathrm{Str}_{\sigma^\frown\langle o\rangle}$, so that $y \notin \mathrm{Str}_{\sigma^\frown\langle o\rangle}$. These extractions will not cause any problem to σ, because σ uses only numbers in $\mathrm{Str}_{\tau^\frown\langle o\rangle}$, none of which will be restrained out of A by strategy $\tau^\frown\langle o\rangle$.

We say that a σ-stage $s+1$ is (σ, ∞)*-expansionary* if a potential new (σ, ∞)-setup at $s+1$ has appeared; and $s+1$ is (σ, d)*-expansionary* if a potential (σ, d)-setup at $s+1$ has appeared.

2.3. **The construction.** The construction is in stages. At stage s we define a finite approximation A^s to A. The set A will be eventually defined as

$$A = \{x : (\exists t)(\forall s \geqslant t)[x \in A^s]\},$$

and will be shown to be a Δ^0_2 set. As already anticipated, at stage s we also define a string $\delta_s \in T$.

Stage 0. Let $A^0 = \varnothing$. Initialize all strategies, and let $\delta_0 = \lambda$.

Stage $s+1$. We define δ_{s+1} in substages: if $t \leqslant s+1$ then unless we have previously stopped the stage at some substage $u < t$, at substage t we define σ_t with $|\sigma_t| = t$, so that $\sigma_u \subset \sigma_t$ for all $u < t$. Eventually we take δ_{s+1} to be the greatest σ_t which has been defined at stage $s+1$.

When, in the description of an action taken at $s+1$, we need to refer to the current approximation of a parameter, we will often omit to mention the stage at which the parameter is evaluated, i.e. we omit to mention s if the parameter has not as yet been redefined at $s+1$, or we omit to mention $s+1$ if the parameter has been already redefined at stage $s+1$.

Substage $t=0$. Let $\sigma_0=\lambda$ and $\mathrm{Str}_\lambda=[0,s+1)$, where, using the usual interval notation, we understand that $[0,s+1)=\{y\in\omega : y<s+1\}$.

Substage $t+1$. Assume that $t<s+1$, and for simplicity denote by σ the string σ_t obtained at the previous substage: if $t=s+1$ then stop the stage by letting $\delta_{s+1}=\sigma$, and go to next stage. If we omit to explicitly define $\mathrm{Str}_{\sigma_{t+1}}$ then it is understood that we let $\mathrm{Str}_{\sigma_{t+1}}=\mathrm{Str}_\sigma\cap\big[(s+1)_{\sigma_{t+1}},\infty\big)$ (here and below we use again the usual interval notation for subsets of ω).

We distinguish the following two cases depending on whether σ is a P-strategy, or an M-strategy.

Case $R(\sigma)=P_W$. If s^- is the previous σ-stage at which we had $\sigma^\frown\langle o\rangle\subseteq\delta_{s^-}$, the strategy $\sigma^\frown\langle o\rangle$ has not been initialized after s^-, and σ need not act (i.e. x_σ was defined at s^- and $W[s^-](x_\sigma)=W[s](x_\sigma)$, where for any stage u, the symbol $W[u]$ denotes the approximation of the c.e. set W at stage u), then let $\sigma_{t+1}=\sigma^\frown\langle o\rangle$.

Otherwise, we *act* according to which of the following cases happens first:

1. if x_σ is still undefined and $\mathrm{Str}_\sigma=\varnothing$ then let $\sigma_{t+1}=\sigma^\frown\langle w\rangle$;
2. if x_σ is still undefined and $\mathrm{Str}_\sigma\neq\varnothing$ then
   (a) *appoint* as x_σ the least $x\in\mathrm{Str}_\sigma$;
   (b) let $\sigma_{t+1}=\sigma^\frown\langle w\rangle$;
   (c) *enumerate* x_σ in A;
3. if x_σ is defined and $x_\sigma\in W[s]$ (i.e. $x\in W[s]\cap A^s$) then
   (a) let $\sigma_{t+1}=\sigma^\frown\langle d\rangle$;
   (b) *extract* x_σ from A.

After acting, stop the stage by letting $\delta_{s+1}=\sigma_{t+1}$ and go to stage $s+2$. Notice that if this happens because of (2) or (3) then, by initialization (as requested at the end of the stage), $\mathrm{Str}_{\sigma_{t+1}}$ starts anew as empty, and x_σ will never be used again by any lower priority strategy τ because $x_\sigma\leqslant s+1\leqslant u_\tau$, for every $u>s+1$.

Case $R(\sigma)=M_\Psi$.

1. If $s+1$ is neither (σ,∞)-expansionary nor (σ,d)-expansionary, then:
   (a) let $\sigma_{t+1}=\sigma^\frown\langle w\rangle$;
2. If $s+1$ is (σ,∞)-expansionary but not (σ,d)-expansionary then
   (a) let $\sigma_{t+1}=\sigma^\frown\langle\infty\rangle$;
   (b) *select* a new (σ,∞)-setup $(x,\psi_\sigma(x))$ at $s+1$, add $\psi_\sigma(x)$ to $\mathrm{Str}_{\sigma^\frown\langle\infty\rangle}$;
   (c) (*securing outcome* ∞) *extract* from A the numbers lying in $\big[(s+1)_{\sigma^\frown\langle\infty\rangle},s+1\big)\smallsetminus\mathrm{Str}_{\sigma^\frown\langle\infty\rangle}$;
3. if $s+1$ is (σ,d)-expansionary then
   (a) let $\sigma_{t+1}=\sigma^\frown\langle d\rangle$;
   (b) *select* a new (σ,d)-setup $(x,\psi_\sigma(x))$ at $s+1$ and *add* $\psi_\sigma(x)$ to $\mathrm{Str}_{\sigma^\frown\langle d\rangle}$;
   (c) (*securing outcome* d) *extract* from A the numbers in $\big[(s+1)_{\sigma^\frown\langle d\rangle},s+1\big)\smallsetminus\mathrm{Str}_{\sigma^\frown\langle d\rangle}$.

Initialize all strategies $\tau\geqslant\delta_{s+1}$. Define A^{s+1} to equal A^s *plus* the elements that have been enumerated in A at $s+1$, *minus* the elements that have been extracted from A at $s+1$.

2.4. **The verification.** The verification depends upon the following lemmata.

Lemma 2.5. *For every n there exist a string σ_n, with $|\sigma_n| = n$, and a stage s_n such that*

(1) $\sigma_i \subseteq \sigma_n$ for every $i \leqslant n$; s_n is the last σ_n-stage at which σ_n is re-initialized, thus $s^{fin}_{\sigma_n}$ exists; there are infinitely many σ_n-stages $s+1$, and at cofinitely many of them σ_n does not end the stage, i.e. δ_{s+1} properly extends σ_n;
(2) for every $s+1 > s_n$, for every $m \leqslant n$, $\mathrm{Str}_{\sigma_n,s+1} \subseteq \mathrm{Str}_{\sigma_m,s+1}$;
(3) the stream Str_{σ_n} increases with respect to inclusion at all σ_n-stages after s_n, and strictly increases at cofinitely many σ_n-stages: in fact if $n > 0$ and $\sigma_n = \sigma_{n-1}{}^\frown\langle o\rangle$ with $o \in \{d, \infty\}$, then Str_{σ_n} strictly increases at all σ_n-stages after s_n.

Proof. We prove the lemma by induction on n. The case $n = 0$ follows immediately by the definitions, with $\sigma_0 = \lambda$, and $s_0 = 0$. Suppose that the three claims are true of n, with σ_n and s_n as in the statement of the lemma. We are now going to show the claims for $n + 1$.

Since the tree is finitely branching, by the inductive assumption in (1) about the existence of infinitely (in fact, cofinitely) many σ_n-stages at which σ_n does not end the stage, clearly there is a $\leqslant$-least string $\tau \supset \sigma_n$ with $|\tau| = n+1$, for which there exist infinitely many τ-stages. Notice that at a stage $s+1 > s_n$, we have that σ_{n+1} may be initialized only by strategies $\tau \supseteq \sigma_n$ with $\tau <_L \sigma_{n+1}$, but this may happen only finitely many times; or it may be initialized if $s+1$ is a σ_{n+1}-stage and $s+1 = n+1$, but this may happen only once; or it can be initialized if $s+1$ is a σ_{n+1}-stage and σ_n is a P-strategy: namely, if $\sigma_{n+1} = \sigma_n{}^\frown\langle w\rangle$ and we act through (1) or (2) of the construction, but this may happen only finitely many times, until the final value of x_{σ_n} has been appointed; or $\sigma_{n+1} = \sigma_n{}^\frown\langle d\rangle$ and we act through (3) of the construction, but this may happen only once. This shows that s_{n+1} exists, and at no σ_{n+1}-stage $s+1 > s_{n+1}$ and $s+1 > n+1$ does σ_{n+1} end the stage.

Item (2) for $n+1$ comes straight from the definitions: notice that if $\sigma_{n+1} = \sigma_n{}^\frown\langle\infty\rangle$ then the elements of the stream $\mathrm{Str}_{\sigma_{n+1}}$ comes from Str_{σ_n}, and if $\sigma_{n+1} = \sigma_n{}^\frown\langle d\rangle$ then the elements of the stream $\mathrm{Str}_{\sigma_{n+1}}$ comes from $\mathrm{Str}_{\sigma_n{}^\frown\langle\infty\rangle}$ and thus from Str_{σ_n}.

To show (3) it is sufficient to show that no number which enters $\mathrm{Str}_{\sigma_{n+1}}$ at stages $s+1 > s_{n+1}$ ever leaves the stream later. Let us consider any $s+1 > s_{n+1}$, and $y \in \mathrm{Str}_{\sigma_{n+1},s+1}$. Suppose that y leaves $\mathrm{Str}_{\sigma_{n+1}}$ at some $s'+1 > s+1$. By definition of s_{n+1} this can not be due to the fact that y is used by some P-strategy σ of higher priority, or some M-strategy $\sigma <_L \sigma_{n+1}$, because this would entail initialization of σ_{n+1}. So the extraction of y from $\mathrm{Str}_{\sigma_{n+1}}$ at $s'+1$ must be due to some M-strategy $\sigma_m \subseteq \sigma_n$ (we may assume m to be least with this property) so that y is extracted at $s'+1$ by $\sigma_m{}^\frown\langle o\rangle$, with $o \in \{\infty, d\}$, for the sake of the activity of securing outcome o. But this cannot happen because $\sigma_m{}^\frown\langle o\rangle$ never extracts number which are already in $\mathrm{Str}_{\sigma_m{}^\frown\langle o\rangle}$ after s_{m+1}.

Finally notice that for every M-strategy σ, a new number *always* enters $\mathrm{Str}_{\sigma^\frown\langle o\rangle}$ at any $\sigma^\frown\langle o\rangle$-stage if $o \in \{d, \infty\}$. Therefore if σ_n is an M-strategy and $\sigma_{n+1} = \sigma_n{}^\frown\langle o\rangle$ with $o \in \{d, \infty\}$, then $\mathrm{Str}_{\sigma_{n+1}}$ strictly increases at all σ_{n+1}-stages after s_{n+1}. □

An immediate consequence of the previous lemma is that if $R(\sigma_n)$ is a P-strategy then $x_{\sigma_n} = \lim_s x_{\sigma_n,s}$ exists.

For every number n, let σ_n and s_n be as given by Lemma 2.5.

Definition 2.6. The infinite path f in T, defined by $f{\restriction}n = \sigma_n$, is called the *true path* of the construction. For every n let also $\mathrm{Str}_{\sigma_n} = \bigcup_{s+1>s_n} \mathrm{Str}_{\sigma_n,s+1}$.

Assume $n > 0$: if $\sigma_n = \sigma_{n-1}{}^\frown\langle\infty\rangle$ then Str_{σ_n} is an infinite decidable set; if $\sigma_n = \sigma_{n-1}{}^\frown\langle d\rangle$ then Str_{σ_n} is an infinite c.e. set.

We will now show that for every $\sigma \subset f$ the actions taken by strategy σ are sufficient to meet the corresponding requirement.

Lemma 2.7. *If $R(\sigma_n)$ is an M-requirement and $\sigma_{n+1} = \sigma_n{}^\frown\langle o\rangle$ with $o \in \{d, \infty\}$ and $y \geqslant s^{fin}_{\sigma_{n+1}}$ then eventually $y \in \mathrm{Str}_{\sigma_{n+1}}$ or $y \notin A$.*

Proof. Immediate by definitions and construction, in particular by the extracting activity of σ_{n+1} towards securing outcome o. □

Lemma 2.8. *For every n, $R(\sigma_n)$ is satisfied.*

Proof. If $R(\sigma_n) = P_W$ then using that $x_{\sigma_n} = \lim_s x_{\sigma_n,s}$ exists, it is easy to see that $A(x_{\sigma_n}) \neq W(x_{\sigma_n})$ since this is how we define $A(x_{\sigma_n})$ at s_{n+1}, and at no stage $s+1 > s_{n+1}$ can any τ modify this value. Indeed, consider first the strategies $\tau \leqslant \sigma_{n+1}$: if τ is a P-strategy, or τ is an M-strategy with $\tau <_L \sigma_{n+1}$, this is due to the fact that σ_{n+1} does never get re-initialized after s_{n+1}; if τ is an M-strategy with $\tau \subset \sigma_{n+1}$ then this is due to the fact that $x_{\sigma_n} \in \mathrm{Str}_{\tau^\frown\langle o\rangle}$ by Lemma 2.5(2), and the fact that $\tau^\frown\langle o\rangle$ does not extract elements of its stream; and clearly the value $A(x_{\sigma_n})$ will not be modified by σ_{n+1} itself after s_{n+1}. At the same time by initialization due to priority, no $\tau > \sigma_{n+1}$ can modify $A(x_{\sigma_n})$.

Assume now that $R(\sigma_n) = M_\Psi$. Suppose first that $\sigma_{n+1} = \sigma_n{}^\frown\langle w\rangle$. Then there are only finitely many x with an axiom $\langle x, D\rangle \in \Psi$ such that $D \neq \varnothing$. Then

$$\Psi(A) = \{x : \langle x, \varnothing\rangle \in \Psi\} \cup \{n : (\exists y)[\langle x, \{y\}\rangle \in \Psi \,\&\, y \in A]\}.$$

This shows that $\Psi(A)$ is c.e.: the first summand of the previous union is clearly c.e., and the second summand is finite.

Now suppose that $\sigma_{n+1} = \sigma_n{}^\frown\langle\infty\rangle$. Let

$$\Phi = \{\langle \psi_{\sigma_n}(m), \{m\}\rangle : \psi_{\sigma_n}(m) \in \mathrm{Str}_{\sigma_{n+1}}\} \cup \{\langle y, \varnothing\rangle : y \in A \cap [0, s^{fin}_{\sigma_{n+1}})\}.$$

Then Φ is an s_2-operator and $A = \Phi(\Psi(A))$. To see this, let us consider any y. If $y \in \mathrm{Str}_{\sigma_{n+1}}$ then the claim follows from the fact that each $(m, \psi_{\sigma_n}(m))$ with $\psi_{\sigma_n}(m) \in \mathrm{Str}_{\sigma_{n+1}}$ is a (σ_n, ∞)-setup. If $y \notin \mathrm{Str}_{\sigma_{n+1}}$ then by Lemma 2.7, either $y \in A \cap [0, s^{fin}_{\sigma_{n+1}})$ and in this case $y \in \Phi(\Psi(A))$; or $y \notin A$, but then also $y \notin \Phi(\Psi(A))$ since there is no axiom in Φ with first component y.

Finally it remains to consider the case when $\sigma_{n+1} = \sigma_n{}^\frown\langle d\rangle$. In this case we claim that

$$\Psi(A) = \Psi(A \cap [0, s^{fin}_{\sigma_{n+1}})),$$

which is a c.e. set since $A \cap [0, s^{fin}_{\sigma_{n+1}})$ is finite. The above equality follows from the fact that if $x \in \Psi(A)$ via an axiom $\langle x, \{y\}\rangle \in \Psi$ with $y \in A$ and $y \geqslant s^{fin}_{\sigma_{n+1}}$, then by Lemma 2.7 $y \in \mathrm{Str}_{\sigma_{n+1}}$ and thus there is an axiom $\langle x, \varnothing\rangle \in \Psi$, or otherwise an axiom $\langle x, \{z\}\rangle \in \Psi$ with $z \in A \cap [0, s^{fin}_{\sigma_{n+1}})$. □

Lemma 2.9. *The set $A = \{x : (\exists t)(\forall s \geqslant t)[x \in A^s]\}$ is Δ^0_2.*

Proof. We prove that the approximation $\{A^s : s \in \omega\}$ is a Δ^0_2 approximation. Consider any number y. By the way we define s_τ for every stage s and strategy τ, a number y can be used only by strategies τ with $y \geqslant |\tau|$, so by finitely many strategies. On the other hand if a strategy τ is using y and later the value $A(y)$ changes, then this is either due to an action performed by τ itself, or by the action of some higher priority strategy which thus initializes τ. In either case τ will not use y again. Consequently, the value $A(y)$ may change at most finitely often. □

The proof is now complete. □

Corollary 2.10. *The local structures $\mathcal{L}_s$ and $\mathcal{L}_{s_2}$ are not elementarily equivalent; the structures $\mathcal{D}_s$ and $\mathcal{D}_{s_2}$ are not elementarily equivalent.*

Proof. By Theorem 2.1 and downwards density of the s-degrees. □

3. Minimal Δ^0_2 s_1-degrees and minimal Δ^0_2 Q_1-degrees

We now consider the problem of existence of minimal s_1- and Q_1-degrees. Given the isomorphism between s_1- and Q_1-reducibility, it does not matter whether we work with s_1- or Q_1-reducibility: as a matter of fact, our proof refers to s_1 instead of Q_1. Contrary to the fact that there are no minimal s- or Q-degrees, but in analogy with the existence of minimal Δ^0_2 s_2-degrees given by Theorem 2.1, we are now going to show that there exists a minimal Δ^0_2 s_1-degree, and consequently, by the isomorphism given by complements, a minimal Δ^0_2 Q_1-degree.

Theorem 3.1. *There exists a Δ^0_2 set with minimal s_1-degree.*

Proof. The proof is similar to the proof of Theorem 2.1 of which we borrow notations and terminology. The similarities are not of course surprising since s_1-operators are also s_2-operators.

As in the previous theorem, the P-strategies employ the familiar Friedberg-Muchnik strategy. We only discuss here in more detail the strategy to meet requirement M_Ψ, where Ψ is a given s_1-operator. An obvious difference with the case of s_2-reducibility arises from the fact that no number leaves Z after entering it: in fact, now Z is a c.e. set, being $Z = \Psi$. Thus let σ be a strategy trying to meet M_Ψ, i.e. $R(\sigma) = M_\Psi$. As in the proof of Theorem 2.1, the strategy looks for new (σ, ∞)-setups, defined as in Definition 2.3, and in isolation we recognize two possible outcomes: outcome w if Ψ is finite (giving $\Psi(A)$ finite), or otherwise outcome ∞ which builds infinitely many (σ, ∞)-setups $\{(m_i, \psi(m_i)) : i \in \omega\}$ (with $m_i < m_j$ and $\psi(m_i) < \psi(m_j)$ if $i < j$) giving a decidable stream $\mathrm{Str}_{\sigma^\frown\langle\infty\rangle} = \{\psi(m_i) : i \in \omega\}$. Accompanied by the usual extracting activity aiming at securing outcome ∞, this gives $A \subseteq \mathrm{Str}_{\sigma^\frown\langle\infty\rangle}$, and $A = \Phi(\Psi(A))$ where $\Phi = \{\langle\psi(m_i), \{m_i\}\rangle : i \in \omega\}$. There is however to solve the potential conflicts between this strategy and the higher priority strategies. In the proof of Theorem 2.1, the analogous strategy, with σ on the true path, had to deal with the finite set $F = A \cap [0, s^{fin}_\sigma]$ restrained in A by the higher priority strategies. In that case the difficulties caused by F to the definition of a correct s_2-operator Φ guaranteeing that $A = \Phi(\Psi(A))$ were solved by $\sigma^\frown\langle\infty\rangle$ by separately adding to Φ, axioms of the form $\langle y, \emptyset\rangle \in \Phi$ when $y \in F$. Clearly, this is something which we cannot afford here, because no axiom of the form $\langle x, \emptyset\rangle \in \Phi$ is consistent with Φ being an s_1-operator. So what should we do when we move from s_2-reducibility to s_1-reducibility? Suppose that σ takes outcome ∞, and wants to construct an s_1-operator Φ such that $A = \Phi(\Psi(A))$ (with $R(\sigma) = M_\Psi$ where Ψ is an s_1-operator): an obvious way of adding axioms to Φ which are allowed in s_1-reducibility, and still guaranteeing that $z \in \Phi(\Psi(A))$ for a number z is by reserving for z a number x_1 for which there is an axiom $\langle x_1, \{y_1\}\rangle \in \Psi$, so that we enumerate the axiom $\langle z, \{x_1\}\rangle \in \Phi$, and guarantee $x_1 \in \Psi(A)$ by permanently enumerating $y_1 \in A$; on the other hand, to permanently achieve $y_1 \in \Phi(\Psi(A))$ we need in turn to resort to another axiom $\langle x_2, \{y_2\}\rangle \in \Psi$, to enumerate the axiom $\langle y_1, \{x_2\}\rangle \in \Phi$, to guarantee x_2 in $\Psi(A)$ by permanently enumerating y_2 in A; and so on. This yields a cascade effect, which makes it necessary to resort to this mechanism infinitely many times. To do this, in case of outcome ∞ at σ we split what in the proof of Theorem 2.1 is $\mathrm{Str}_{\sigma^\frown\langle\infty\rangle} = \{y_0, y_1, \ldots\}$ (computably enumerated in strictly ascending order $y_0 < y_1 < \ldots$) into two disjoint infinite decidable subsets $\mathrm{Str}^0_{\sigma^\frown\langle\infty\rangle} = \{y^0_i : i \in \omega\}$ where $y^0_i = y_{2i}$, and $\mathrm{Str}^1_{\sigma^\frown\langle\infty\rangle} = \{y^1_i : i \in \omega\}$ where $y^1_i = y_{2i+1}$. We permanently enumerate in A at $\sigma^\frown\langle\infty\rangle$ the set $\mathrm{Str}^0_{\sigma^\frown\langle\infty\rangle}$. For each $y^0_i \in \mathrm{Str}^0_{\sigma^\frown\langle\infty\rangle}$ take x^0_i so that $y^0_i = \psi(x^0_i)$, i.e. the pair (x^0_i, y^0_i) is the (σ,∞)-setup responsible for enumerating $y_{2i} \in \mathrm{Str}_{\sigma^\frown\langle\infty\rangle}$: since $y^0_i \in A$ and, by the definition of a (σ,∞)-setup, $y^0_i \in A$ if and only if $x^0_i \in \Psi(A)$, it follows that $x^0_i \in \Psi(A)$, for every i. Now, given any c.e. set V (we assume that V is infinite since in our later applications it will be $\mathrm{Str}^0_{\sigma^\frown\langle\infty\rangle} \subseteq V$) we can use $\mathrm{Str}^0_{\sigma^\frown\langle\infty\rangle} \subseteq A$ to guarantee that $V \subseteq \Phi(\Psi(A))$ as is described by the following definition:

Definition 3.2. Suppose that V is an infinite c.e. set, and let $i : V \to \omega$ be a partial computable bijection. Let

$$\mathrm{Ax}_V = \{\langle z, \{x^0_{i(z)}\}\rangle : z \in V\}.$$

It now easy to see that Ax_V is an s_1-operator, and $V \subseteq \mathrm{Ax}_V(\Psi(A))$.

The other half $\mathrm{Str}^1_{\sigma^\frown\langle\infty\rangle}$ will play the role of what in the proof of Theorem 2.1 is the stream $\mathrm{Str}_{\sigma^\frown\langle\infty\rangle}$. Hence the lower priority strategies $\tau \supseteq \sigma^\frown\langle\infty\rangle$ will use only numbers coming from $\mathrm{Str}^1_{\sigma_{n+1}}$. Notice that for a given strategy τ, the partition of Str_τ into two halves is only relevant when τ is of the form $\tau = \rho^\frown\langle\infty\rangle$ and ρ is an M-strategy with outcome ∞, so that it needs to define a suitable s_1-operator. In all other cases we will take $\mathrm{Str}^1_\tau = \mathrm{Str}_\tau$ and $\mathrm{Str}^0_\tau = \varnothing$.

Of course, this adds the extra complication that now every $\tau^\frown\langle\infty\rangle \subseteq \sigma$, where τ is an M-strategy, restrains in A stage by stage what eventually grows up to an infinite decidable set, namely $\mathrm{Str}^0_{\tau^\frown\langle\infty\rangle}$. Luckily, our M-strategy σ has really no problem to account for this set restrained in A when defines its s_1-operator Φ, i.e. when it takes outcome ∞ (in other words, when it succeeds in appointing infinitely many (σ,∞)-setups $(x, \psi(x))$ for which $x \in A$ if and only if $\psi(x) \in \Psi(A)$): it simply makes $\mathrm{Ax}_V \subseteq \Phi$ where $V \supseteq \mathrm{Str}^0_{\tau^\frown\langle\infty\rangle}$ is a suitable c.e. set (it will be clear in the verification that the axioms in Ax_V, plus the other axioms added to Φ to get $A = \Phi(\Psi(A))$, are all consistent with Φ being an s_1-operator).

Another problem caused by $\tau^\frown\langle\infty\rangle$ to σ is that $\mathrm{Str}^0_{\tau^\frown\langle\infty\rangle}$ may contribute to destroy (σ,∞)-setups already created by σ. Indeed, an already appointed (σ,∞)-setup $(x, \psi(x))$ for σ (it does not matter whether $\psi(x)$ lies in $\mathrm{Str}^0_{\sigma^\frown\langle\infty\rangle}$ or in $\mathrm{Str}^1_{\sigma^\frown\langle\infty\rangle}$) may cease to be a (σ,∞)-setup if a new axiom $\langle x, \{y\}\rangle \in \Psi$ shows up after the setup has been appointed, such that y lies already in $\mathrm{Str}^0_{\tau^\frown\langle\infty\rangle}$: we say in this case that $\mathrm{Str}^0_{\tau^\frown\langle\infty\rangle}$ *destroys* the (σ,∞)-setup $(x, \psi(x))$. The possibility of (σ,∞)-setups being destroyed introduces again outcome d for the M-strategies: this time a (σ,∞)-setup $(x, \psi(x))$ may cease to be so not because the axiom $\langle x, \varnothing\rangle \in \Psi$ shows up or an axiom $\langle x, \{z\}\rangle \in \Psi$ appears such that $z \in A \cap [0, (s+1)_{\sigma^\frown\langle o\rangle})$, but because some set $\mathrm{Str}^0_{\tau^\frown\langle\infty\rangle}$ with $\tau^\frown\langle\infty\rangle \subset \sigma$ destroys the (σ,∞)-setup.

This introduces again the distinction between (σ,∞)-setups, and (σ, d)-setups. We first revise Definition 2.2.

Definition 3.3. If σ is an M-strategy and $o \in \{d, \infty\}$ then we say that *x is restrained in $\Psi_\sigma(A)$ for $\sigma^\frown\langle o\rangle$ at stage $s+1$ by higher priority strategies*, if the axiom $\langle x, \varnothing\rangle \in \Psi_\sigma$ has appeared, or at stage $s+1$ there exists an axiom $\langle x, \{z\}\rangle \in \Psi_\sigma$ such that

$$z \in \big(A^s \cap [0,(s+1)_{\sigma^\frown\langle o\rangle})\big) \cup \bigcup_{\tau\subset\sigma} \mathrm{Str}^0_{\tau,s+1}\,.$$

(Recall that $\mathrm{Str}^0_{\tau,s+1} = \varnothing$ if τ is not of the form $\tau = \rho^\frown\langle\infty\rangle$.)

The definitions of a *potential* (σ,∞)*-setup*, and of a *potential* (σ,d)*-setup* are now exactly as in Definitions 2.3 and 2.4, except for the fact that now these definitions use Definition 3.3 as the new meaning for the expression "being restrained in $\Psi_\sigma(A)$ for $\sigma^\frown\langle o\rangle$ at stage $s+1$ by higher priority strategies".

With this in mind, we are now going to sketch the construction and the proof of the theorem.

If there is a potential new (σ,∞)-setup at $s+1$, we *select* the one, say (x,y), with least code, and we define $y = \psi_\sigma(x)$, so that we have appointed a *new* (σ,d)*-setup at* $s+1$, and $\psi_\sigma(x)$ is enumerated either into $\mathrm{Str}^0_{\sigma^\frown\langle\infty\rangle}$, or $\mathrm{Str}^1_{\sigma^\frown\langle\infty\rangle}$ as described later in the construction. This action is accompanied by the usual extracting activity of securing outcome ∞, as described in the construction.

If there is a potential new (σ,d)-setup at $s+1$, we *select* the one, say $(x,\psi(x))$, with least code, we appoint the *new* σ*-setup* $(x,\psi(x))$ *at* $s+1$, and enumerate $\psi_\sigma(x)$ into $\mathrm{Str}^1_{\sigma^\frown\langle d\rangle}$. This action is accompanied by the usual extracting activity of securing outcome ∞.

We say that a σ-stage $s+1$ is (σ,∞)*-expansionary* if a potential new (σ,∞)-setup at $s+1$ has appeared; we say that $s+1$ is (σ,d)*-expansionary* if a potential new (σ,d)-setup at $s+1$ has appeared.

3.1. **The construction.** The construction follows along the lines of the construction in the proof of Theorem 2.1. We sketch as follows.

To *initialize* a strategy σ at stage s now means: to set $\mathrm{Str}^0_{\sigma,s} = \mathrm{Str}^1_{\sigma,s} = \varnothing$, and to set as undefined $x_{\sigma,s}$.

Stage 0 is as in the proof of Theorem 2.1. In particular all strategies are initialized.

We now consider the various substages of Stage $s+1$. As in Theorem 2.1, when in the description of an action taken by a strategy we omit to mention the stage at which a given parameter was evaluated, we always mean to refer to the last stage at which the parameter was defined: this can also be $s+1$ if the parameter has been defined by previous actions taken at stage $s+1$.

Substage $t=0$. Let $\sigma_0 = \lambda$, and $\mathrm{Str}^1_\lambda = [0,s+1)$.

Substage $t+1$. Assume that $t < s+1$, and denote by σ the string σ_t obtained at the previous substage: if $t = s+1$ then stop the stage by letting $\delta_{s+1} = \sigma$, and go to next stage. Otherwise we proceed as follows. If we omit to explicitly define $\mathrm{Str}_{\sigma_{t+1}}$ (and consequently $\mathrm{Str}^0_{\sigma_{t+1}}$ and $\mathrm{Str}^1_{\sigma_{t+1}}$) then it is understood that we let $\mathrm{Str}_{\sigma_{t+1}} = \mathrm{Str}_\sigma \cap \big[(s+1)_{\sigma_{t+1}},\infty\big)$.

Case $R(\sigma) = P_W$. *Case* $R(\sigma) = P_W$. If s^- is the previous σ-stage, with $\sigma^\frown\langle o\rangle \subseteq \delta_{s^-}$, the strategy $\sigma^\frown\langle o\rangle$ has not been initialized after s^-, and σ need not act, i.e. x_σ was defined at s^- and $W[s^-](x_\sigma) = W[s](x_\sigma)$, then let $\sigma_{t+1} = \sigma^\frown\langle o\rangle$.

Otherwise, we *act* according to which of the following cases happens first:

(1) if x_σ is still undefined and $\mathrm{Str}^1_\sigma = \varnothing$ then let $\sigma_{t+1} = \sigma^\frown\langle w\rangle$;
(2) if x_σ is still undefined and $\mathrm{Str}^1_\sigma \neq \varnothing$ then

   (a) *appoint* as x_σ the least $x \in \mathrm{Str}^1_\sigma$;
   (b) *enumerate* x_σ in A;
(3) if x_σ is defined and $x_\sigma \in W[s]$ (i.e. $x \in W[s] \cap A^s$) then
   (a) let $\sigma_{t+1} = \sigma^\frown\langle d\rangle$;
   (b) *extract* x_σ from A.

After acting, stop the stage by letting $\delta_{s+1} = \sigma_{t+1}$ and go to stage $s + 2$.

Case $R(\sigma) = M_\Psi$.

(1) If $s + 1$ is neither (σ, ∞)-expansionary nor (σ, d)-expansionary then:
   (a) let $\sigma_{t+1} = \sigma^\frown\langle w\rangle$;
(2) If $s + 1$ is (σ, ∞)-expansionary then
   (a) let $\sigma_{t+1} = \sigma^\frown\langle\infty\rangle$;
   (b) *select* a new σ-setup $(x, \psi_\sigma(x))$ at $s + 1$: if the cardinality of the previous value of $\mathrm{Str}^0_{\sigma^\frown\langle\infty\rangle}$ is even then add $\psi_\sigma(x)$ to $\mathrm{Str}^0_{\sigma^\frown\langle\infty\rangle}$, otherwise add it to $\mathrm{Str}^1_{\sigma^\frown\langle\infty\rangle}$;
   (c) (*securing outcome* ∞) *extract* from A the numbers lying in
$$[(s+1)_{\sigma^\frown\langle\infty\rangle}, s+1) \smallsetminus \left(\mathrm{Str}^1_{\sigma^\frown\langle\infty\rangle} \cup \bigcup_{\tau\subseteq\sigma} \mathrm{Str}^0_\tau\right);$$
(3) if $s + 1$ is (σ, d)-expansionary then
   (a) let $\sigma_{t+1} = \sigma^\frown\langle d\rangle$;
   (b) *select* a new (σ, d)-setup $(x, \psi_\sigma(x))$ at $s + 1$ and *add* $\psi_\sigma(x)$ to $\mathrm{Str}^1_{\sigma^\frown\langle d\rangle}$;
   (c) (*securing outcome* d)*extract* from A the numbers in
$$[(s+1)_{\sigma^\frown\langle d\rangle}, s+1) \smallsetminus \left(\mathrm{Str}^1_{\sigma^\frown\langle d\rangle} \cup \bigcup_{\tau\subseteq\sigma} \mathrm{Str}^0_\tau\right).$$

Initialize all strategies $\tau \geqslant \delta_{s+1}$. The definition of A^{s+1} is as in the proof of Theorem 2.1, with the difference that now we must take into account also the additional activity of *enumerating* elements into A^{s+1} pursued by the M-strategies when they *restrain* in A sets of the form $\mathrm{Str}^0_{\tau^\frown\langle\infty\rangle}$, in addition to their usual extracting activity to secure their outcomes. Thus, A^{s+1} consists of A^s *plus* the elements which have been enumerated into A at $s + 1$, *minus* the elements which have been extracted from A at $s + 1$.

3.2. **The verification.**

Lemma 3.4. *For every n there exist a string σ_n, with $|\sigma_n| = n$, and a stage s_n such that:*

(1) $\sigma_i \subseteq \sigma_n$ for every $i \leqslant n$; s_n is the last σ_n-stage at which σ_n has been re-initialized (hence $s^{fin}_{\sigma_n}$ exists); there are infinitely many σ_n-stages $s+1$, and at cofinitely many of them σ_n does not end the stage, i.e. δ_{s+1} properly extends σ_n;
(2) for every $s + 1 > s_n$, for every $m \leqslant n$, $\mathrm{Str}_{\sigma_n,s+1} \subseteq \mathrm{Str}_{\sigma_m,s+1}$;
(3) the stream halves $\mathrm{Str}^0_{\sigma_n}$ and $\mathrm{Str}^1_{\sigma_n}$ increase with respect to inclusion at all σ_n-stages $s+1 > s_n$, and $\mathrm{Str}^1_{\sigma_n}$ strictly increases at cofinitely many σ_n-stages after s_n; moreover if $n = 0$ or $\sigma_n \neq \sigma_{n-1}{}^\frown\langle\infty\rangle$ then $\mathrm{Str}^0_{\sigma_n,s+1} = \emptyset$; if $n > 0$ and $\sigma_n = \sigma_{n-1}{}^\frown\langle\infty\rangle$ then $\mathrm{Str}^0_{\sigma_n}$ and $\mathrm{Str}^1_{\sigma_n}$ alternately strictly increase at every other σ_n-stage $s + 1 > s_n$: in more detail, if $\{u_i : i \in \omega\}$ is the list in increasing order of all σ_n-stage $s + 1 > s_n$, then $|\,\mathrm{Str}^0_{\sigma_n,u_{2i}}| =$

$|\operatorname{Str}^0_{\sigma_n,u_{2i}-1}|+1$ *and* $\operatorname{Str}^1_{\sigma_n,u_{2i}} = \operatorname{Str}^1_{\sigma_n,u_{2i}-1}$*, whereas* $|\operatorname{Str}^1_{\sigma_n,u_{2i+1}}| = |\operatorname{Str}^1_{\sigma_n,u_{2i}}|+1$ *and* $\operatorname{Str}^0_{\sigma_n,u_{2i}+1} = \operatorname{Str}^0_{\sigma_n,u_{2i}}$.

Proof. The proof follows along the lines of the proof of Lemma 2.5 with obvious modifications.

First of all, if $R(\sigma_n) = P_W$, then σ_n picks the final value of its witness x_{σ_n} from $\operatorname{Str}^1_{\sigma_n,s+1}$, where $s+1$ is the least σ_n-stage after s_n, with $s+1 \geqslant n+1$, at which $\operatorname{Str}^1_{\sigma_n} \neq \varnothing$: such a stage exists by the inductive assumption (3) for σ_n. The rest of the verification is as in the corresponding case of Lemma 2.5.

To prove (3) the tedious part is again to show that no number which enters $\operatorname{Str}^0_{\sigma_{n+1}}$ or $\operatorname{Str}^1_{\sigma_{n+1}}$ at stages $s+1 > s_{n+1}$ ever leaves the set at later stages. To see this, by initialization we may assume that the potential extraction of a number y from $\operatorname{Str}^0_{\sigma_{n+1}}$ or $\operatorname{Str}^1_{\sigma_{n+1}}$ at a later stage $s'+1$ must be due to some M-strategy σ_m with $\sigma_m{}^\frown\langle o\rangle \subseteq \sigma_n$ and $o \in \{d,\infty\}$, so that y is extracted at $s'+1$ by $\sigma_m{}^\frown\langle\infty\rangle$, for the sake of the activity of securing outcome ∞, or y is placed in $\operatorname{Str}^0_{\sigma_{n+1}}$. But this cannot happen because $\sigma_m{}^\frown\langle o\rangle$ never extracts numbers which are already in $\operatorname{Str}^1_{\sigma_m{}^\frown\langle o\rangle}$, nor does it move to $\operatorname{Str}^0_{\sigma_m{}^\frown\langle o\rangle}$ numbers which are already in $\operatorname{Str}^1_{\sigma_m{}^\frown\langle o\rangle}$. □

For the *true path* f of this construction, we let again $f{\upharpoonright}n = \sigma_n$. Let also $\operatorname{Str}^0_{\sigma_n} = \bigcup_{s+1>s_n}\operatorname{Str}^0_{\sigma_n,s+1}$ and $\operatorname{Str}^1_{\sigma_n} = \bigcup_{s+1>s_n}\operatorname{Str}^1_{\sigma_n,s+1}$.

Assume $n>0$: if $\sigma_n = \sigma_{n-1}{}^\frown\langle\infty\rangle$ then $\operatorname{Str}^0_{\sigma_n}$ and $\operatorname{Str}^1_{\sigma_n}$ are infinite decidable set; if $\sigma_n = \sigma_{n-1}{}^\frown\langle d\rangle$ then $\operatorname{Str}^1_{\sigma_n}$ is an infinite c.e. set.

With the same proof as for Lemma 2.7, we can prove:

Lemma 3.5. *If $R(\sigma_n)$ is an M-requirement and $\sigma_{n+1} = \sigma_n{}^\frown\langle o\rangle$ with $o \in \{d,\infty\}$ and $y \geqslant s^{fin}_{\sigma_{n+1}}$ then eventually $y \in \operatorname{Str}_{\sigma_{n+1}}$ or $y \notin A$.*

Proof. This follows by the extracting activity described in the construction as "securing outcome o" for σ_n. □

Lemma 3.6. *For every n, $R(\sigma_n)$ is satisfied.*

Proof. If $R(\sigma_n) = P_W$ then the verification is as in Lemma 2.8, observing that no M-strategy τ with $\tau \subset \sigma_{n+1}$ can modify the value $A(x_{\sigma_n})$ which has been defined at s_{n+1}, as x_{σ_n} has been chosen from $\operatorname{Str}^1_{\tau^\frown\langle o\rangle}$ if $\tau^\frown\langle o\rangle \subseteq \sigma_n$, and $\tau^\frown\langle o\rangle$ neither moves numbers from $\operatorname{Str}^1_{\tau^\frown\langle o\rangle}$ to $\operatorname{Str}^0_{\tau^\frown\langle o\rangle}$ nor does it extract from A elements which lie in $\operatorname{Str}^1_{\tau^\frown\langle o\rangle}$. By initialization no other strategy can change the value $A(x_{\sigma_n})$, nor can this valued be further changed by σ_{n+1} itself.

If $R(\sigma_n) = M_\Psi$ and $\sigma_{n+1} = \sigma_n{}^\frown\langle w\rangle$ then there are only finitely many numbers x with an axiom $\langle x, D\rangle \in \Psi$. Then $\Psi(A)$ is a finite set, and the requirement is satisfied.

If $R(\sigma_n) = M_\Psi$ and $\sigma_{n+1} = \sigma_n{}^\frown\langle\infty\rangle$, then let

$$V = \left(A \cap [0, s^{fin}_{\sigma_{n+1}})\right) \cup \bigcup_{\tau \subseteq \sigma_{n+1}} \operatorname{Str}^0_\tau,$$

and define

$$\Phi = \operatorname{Ax}_V \cup \{\langle y^1_i, \{x^1_i\}\rangle : i \in \omega\},$$

where we recall $\mathrm{Str}^1_{\sigma^\frown\langle\infty\rangle} = \{y^1_i : i \in \omega\}$ and (x^1_i, y^1_i) is the (σ_n, ∞)-setup corresponding to y^1_i. The operator Φ is an s_1-operator since the elements of the set $\{x^0_i : i \in \omega\} \cup \{x^1_i : i \in \omega\}$ are pairwise distinct. Finally by Lemma 3.5 and the discussion on Ax_V accompanying Definition 3.2, we have that $A = \Phi(\Psi(A))$.

If $R(\sigma_n) = M_\Psi$ and $\sigma_{n+1} = \sigma_n{}^\frown\langle d\rangle$, then (as in the proof of Lemma 2.8, with the only difference that in that proof we had simply $V = A \cap [0, s^{fin}_{\sigma_{n+1}}))$, whereas we have now $V = \left(A \cap [0, s^{fin}_{\sigma_{n+1}})\right) \cup \bigcup_{\tau\subseteq\sigma_{n+1}} \mathrm{Str}^0_\tau$) it follows that $\Psi(A) = \Psi(V)$ is c.e., as V is c.e.. □

Lemma 3.7. *The set $A = \{x : (\exists t)(\forall s \geqslant t)[x \in A^s]\}$ is Δ^0_2.*

Proof. The proof is the same as for the analogous lemma of Theorem 2.1. □

The proof of the theorem is now complete. □

The above result has obvious consequences for the Q_1-degrees. We only mention the following two corollaries.

Corollary 3.8. *There is a Δ^0_2 set with minimal Q_1-degree.*

Proof. By the isomorphism between the Δ^0_2 s_1-degrees and the Δ^0_2 Q_1-degrees. □

Corollary 3.9. *The local structures $\mathcal{L}_s$ and $\mathcal{L}_{s_1}$ are not elementarily equivalent; the structures $\mathcal{D}_s$ and $\mathcal{D}_{s_1}$ are not elementarily equivalent. Similarly, the local structures $\mathcal{L}_Q$ and $\mathcal{L}_{Q_1}$ are not elementarily equivalent; the structures $\mathcal{D}_Q$ and $\mathcal{D}_{Q_1}$ are not elementarily equivalent.*

Proof. As in Corollary 2.10. □

4. Downwards density in the Π^0_1 s_2-degrees

We are now going to restrict attention to the Π^0_1 s_2-degrees. We first show that Theorem 2.1 is sharp in terms of the arithmetical hierarchy, as the Π^0_1 s_2-degrees are downwards dense.

Theorem 4.1. *For every non-c.e. Π^0_1 set A there exists a non-c.e. Π^0_1 set Y such that $Y <_{s_2} A$.*

Proof. Let $A \in \Pi^0_1$, A not c.e., and let $\{A^s : s \in \omega\}$ be a fixed Π^0_1 approximation to A, for which we assume that the predicate "$x \in A^s$" is decidable (in fact, the complements $\overline{A^s}$ of the sets A^s form a uniform sequence of finite sets given by their canonical indices), $A^0 = \omega$, $A^s \supseteq A^{s+1}$ for every s, and $A = \bigcap_s A^s$. We construct an s_2-operator Γ so that the set $Y = \Gamma(A)$ satisfies our desiderata. The construction of Γ is in stages. At stage 0, let Γ be the s_2-operator $\Gamma = \{\langle n, \{n\}\rangle : n \in \omega\}$. At the beginning of any stage s a number n is said to be *dumped*, or *non-dumped*, *in* Γ, depending on whether we have, or not, previously added to Γ the axiom $\langle n, \varnothing\rangle \in \Gamma$; during stage s we say that we *dump* n if at this stage we add the axiom $\langle n, \varnothing\rangle \in \Gamma$. If Ψ_e is the e-th s_2-operator then at the beginning of stage s we say that n *has already been dumped into* $\Psi_e(\Gamma(A))$ if $n \in \Psi_{e,s}(\Gamma_{s-1}(\varnothing))$, i.e. if there is an axiom $\langle n, \varnothing\rangle \in \Psi_{e,s}$, or there is an axiom $\langle n, \{y\}\rangle \in \Psi_{e,s}$, and y is dumped; during stage s we say that we *dump n into* $\Psi_e(\Gamma(A))$ if at this stage we dump y, for some y such that there is an axiom $\langle n, \{y\}\rangle \in \Psi_{e,s}$. It will follow by the construction that $\Gamma(A)$ is Π^0_1 as well.

The proof will employ a simple finite priority argument. We shall need to satisfy the requirements

$$P_e : \Gamma(A) = W_e \Rightarrow A = Z_e,$$
$$N_e : A = \Psi_e(\Gamma(A)) \Rightarrow A = V_e,$$

where Ψ_e is the e-th s_2-operator, and Z_e and V_e are c.e. sets built by us. Throughout the proof, we define the *use of* Ψ_e *for* m to be the partial computable function ψ_e of two variables such that, for $s > 0$, $\psi_e(n, s)$ is defined (notation: $\psi_e(n, s) \downarrow$) if $n \notin \Psi_{e,s}(\Gamma_{s-1}(\emptyset))$ and there exists an axiom $\langle n, \{y\}\rangle \in \Psi_{e,s}$ with $y \in \Gamma_{s-1}(A^s)$ (hence $y \in A^s$), and in this case $\psi_e(n, s)$ equals the least such y. Notice that $\psi_e(n, s)$ may be undefined even if it was previously defined but now $n \in \Psi_{e,s}(\Gamma_{s-1}(\emptyset))$.

Let $R_{2e} = P_e$ and $R_{2e+1} = N_e$. The priority ordering of the requirements is given by $R_i < R_j$ if $i < j$: if $R_i < R_j$ then we say that R_i has *higher priority than* R_j (or, equivalently, R_j has *lower priority than* R_i).

We now describe the strategies to meet the requirements, which will interact with each other through initialization, as is typical in most priority arguments: when a requirement acts, it initializes all lower priority requirements.

Strategy for P_e. The strategy for P_e is simple. We enumerate in stages in Z_e all non-dumped numbers $y \in \Gamma(A) \cap W_e$ until a stage at which some non-dumped n which has been already enumerated into Z_e gets extracted from A. If this happens then P_e *acts* as follows: it commits itself to never act again, and restrains n from being dumped in the future by lower priority requirements. Clearly if such a number n is found then the requirement is met as $n \in W_e \smallsetminus \Gamma(A)$. On the other hand we will see in the later section on interactions between strategies that if no such n is found then it cannot be $\Gamma(A) = W_e$, as otherwise it would be $A = Z_e$ contrary to the fact that A is not c.e.. So, whatever the case, P_e is met, and it acts at most once.

Strategy for N_e. The strategy for N_e is also simple. We enumerate in stages a c.e. set V_e such that if $A = \Psi_e(\Gamma(A))$ then $A = V_e$, which would be a contradiction since A is not c.e.. Basically, we enumerate stage by stage in V_e the numbers $y \in A \cap \Psi_e(\Gamma(A))$, and once a number y has entered V_e we strive to keep it in $\Psi_e(\Gamma(A))$: so if we see at some stage s that $\psi_e(n, s-1) \downarrow$ but $\psi_e(n, s-1)$ gets extracted from A (we may assume $s > 0$ as $A^0 = \omega$), then we dump $\psi_e(n, s-1)$, with the immediate effect that we dump n into $\Psi_e(\Gamma(A))$, so that $n \in A$ if $A = \Psi_e(\Gamma(A))$. We keep doing this until a number n is found for which we can argue that $A(n) \neq \Psi_e(\Gamma(A))(n)$. We will show in the verification that N_e is satisfied, and acts only finitely many times.

Interactions between strategies. Observe that the N-strategies (the only ones which dump numbers through their actions) ensure that any number y is dumped *only after* y *leaves* A, *and not before.* This is important for the P requirements to be satisfied in the presence of dumping. Indeed, given a requirement P_e, we can argue from this that $\Gamma(A) \neq W_e$. Assume first that there is some $n \in Z_e \smallsetminus A$. Notice that if we have enumerated n into Z_e then we have done so at a stage at which n was non-dumped and $n \in \Gamma(A) \cap W_e$. If n leaves A then P_e acts (if it has not already taken a similar action on some other number) and initializes all lower priority requirements; by this re-initialization we permanently get $n \in W_e \smallsetminus \Gamma(A)$, contrary to the assumption $\Gamma(A) = W_e$, as no lower priority requirement N will ever dump n since n has already left A and thus will not be the use of any number m relative to any lower priority N_k, i.e. $n \neq \psi_k(m, t)$ for every $N_k > P_e$, and every stage t following the action of P_e when N_k (and consequently ψ_k) was re-initialized; on the other hand, the higher priority requirements N will never dump n since after this P_e's action, which takes place after the last initialization of P_e, they do not act any longer. If there is no n as above, then $Z_e \subseteq A$. But then, since A is not c.e. and thus $A \neq Z_e$, there is some $n \in A \smallsetminus Z_e$: if we have never enumerated n into Z_e this is because either n never became realized (call a number *n realized at s if $n \in W_{e,s}$*), in which case we get $n \in \Gamma(A) \smallsetminus W_e$, or when n became realized we had that n was dumped already, but this cannot be as we dump numbers only if they have left A, so $n \notin A$ since we work with a Π^0_1 approximation to A. This shows that $\Gamma(A) \neq W_e$.

4.1. **The construction.** At each stage s we define finite approximations $Z_{e,s}, V_{e,s}$ to the c.e. sets Z_e, V_e, respectively, a computable approximation Γ_s to Γ (which gives also the approximation $Y^s = \Gamma_s(A^s)$ to $\Gamma(A)$), a parameter $\psi_e(n,s)$ giving, if defined, the use of Ψ_e for n at stage s. Unless otherwise modified, at stage $s+1$ each parameter retains the same value as at the previous stage. To *initialize* strategy R at stage s means: to declare R *active* at s (R will then stay active until when it will be declared, if ever, *inactive* at some later stage), and to set $Z_{e,s} = \emptyset$, if $R = P_e$ for some e; or if $R = N_e$ for some e, it means to set $\psi_e(n,s)$ undefined for every n, and $V_{e,s} = \emptyset$. We also define

$$n(e,s) = \mu n[n \in A^s \smallsetminus V_{e,s}] :$$

clearly $n(e,s)$ is a total function.

At stage $s+1$ a requirement R *requires attention* if either

1. $R = P_e$, R is active at the end of stage s, and there exists a non-dumped $n \in Z_{e,s}$ such that $n \in A^s \smallsetminus A^{s+1}$; or
2. $R = N_e$, and
   (a) $n(e,s) \in A^{s+1}$, but now we see that $n(e,s)$ has already been dumped into $\Psi_e(\Gamma(A))$ or we have $\psi_e(n(e,s), s+1)\downarrow$, or
   (b) there exists $y \in V_{e,s}$ such that y has not been dumped already into $\Psi_e(\Gamma(A))$ and $\psi_e(y,s) \in A^s \smallsetminus A^{s+1}$.

Stage 0. Initialize all strategies. Set $\Gamma_0 = \{\langle n, \{n\}\rangle : n \in \omega\}$: thus, at stage 0, $Y^0 = \omega$.

Stage $s+1$. If there is no requirement that requires attention at $s+1$ then go directly to the housekeeping actions, described below. If there is a requirement that requires attention at $s+1$ then take the least such R and distinguish the two possible cases:

1. if $R = P_e$ for some e then (*action*) declare R *inactive*;
2. if $R = N_e$ then
   (a) if it requires attention through (2a) then let $V_{e,s+1} = V_{e,s} \cup \{n(e,s)\}$;
   (b) if N_e requires attention through (2b), then dump $\psi_e(y,s)$ for every $y \in V_{e,s}$ for which $\psi_e(y,s)\downarrow$ but $\psi_e(y,s) \in A^s \smallsetminus A^{s+1}$

Go to housekeeping actions, described below.

Housekeeping actions. Consider all requirements R with $R = P_e$ for some e, and $\Gamma_{s+1}(A^{s+1}) \neq \emptyset$:

- enumerate in $Z_{e,s+1}$ all non-dumped $n \in \Gamma_{s+1}(A^{s+1})$.

If some R acts then it initialize all $R' > R$, and we now go to next stage.

Just an observation before moving to the verification. The actions in (2b) have the purpose to keep in $\Psi_e(\Gamma(A))$ all the numbers y which have been enumerated in V_e although this may cost dumping various $\psi_e(y)$ which leave A: if y is a number such that the convergence of the use $\psi_e(y)$ at some earlier stage has caused us in (2a) to enumerate $y \in V_e$, but at some stage $s+1$ we have

$\psi_e(y, s) \in A^s \smallsetminus A^{s+1}$, then $\psi_e(y, s)$ must be dumped exactly at $s + 1$, as explained in the section about interactions of strategies. This is why in (2b) we consider at $s + 1$ all $y \in V_{e,s}$ having use which leaves A exactly at $s+1$. Notice also that after dumping a number y into $\Psi_e(\Gamma(A))$, we have that $\psi_e(y)$ becomes permanently undefined so that N_e will not require attention again through any clause requesting the convergence of $\psi_e(y)$.

4.2. **The verification.** The verification relies on the following lemmata.

Lemma 4.2. *Each strategy R_i requires attention and acts only finitely often, and is eventually satisfied.*

Proof. By induction on i. Suppose that the claim is true of every $j < i$, and distinguish the following two cases, where we suppose that s_0 is the last stage, if any, at which some $R' < R$ requires attention, and $s_0 = 0$ otherwise:

(1) $R_i = P_e$. Let $Z_e = \bigcup_{s>s_0} Z_{e,s}$. After stage s_0, P_e requires attention and acts at most once, since this may happen only at the least stage $t + 1 > s_0$, if any, such that we have already enumerated some n in Z_e at a stage $s_0 < s \leqslant t$ when n was in A^s and not dumped, but $n \in A^t \smallsetminus A^{t+1}$, as a necessary condition for a number to be dumped is for that number to have left A: then at stage $t + 1$ P_e is the least requirement to require attention, and thus it acts. If no such n exists, then P_e will never requires attention and will never act again. Whatever the case, the housekeeping axioms construct a c.e. set Z_e, and as explained in the informal description of the strategies and their interactions, this guarantees, as $A \neq Z_e$, the existence of a number $n \in W_e \smallsetminus \Gamma(A)$. In conclusion, P_e requires attention and acts only finitely often and it is eventually satisfied.

(2) $R_i = N_e$. Let $V_e = \bigcup_{s>s_0} V_{e,s}$. Since $A \neq V_e$ (being A not c.e.) we can pick the least n such that $A(n) \neq V_e(n)$. Hence (using the fact that we work with a Π^0_1 approximation to A) there is a least stage s_1 such that $n(e, s) = n$ for all $s \geqslant s_1$. If $n \in V_e \smallsetminus A$ then there is a stage $s + 1 > s_0$ at which we have enumerated n in V_e, having seen either that n has been already dumped into $\Psi_e(\Gamma(A))$, or that $\psi_e(n, s + 1) \downarrow$. In the former case, we already trivially have $n \in \Psi_e(\Gamma(A))$; in the latter case we make sure that $n \in \Psi_e(\Gamma(A))$ because if later $\psi_e(n, s + 1)$ leaves A then we dump it as per (2b) of N_e in the construction, with the result of dumping n into $\Psi_e(\Gamma(A))$. In either case N_e is satisfied. On the other hand, if $n \in A \smallsetminus V_e$ then $n \notin \Psi_{e,s+1}(\Gamma_s(A^{s+1}))$ at all $s + 1 > s_1$ as otherwise action (2a) of N_e would enumerate $n \in V_e$ at some stage $s + 1 > s_1$. Therefore, again, N_e is satisfied as $n \notin \Psi_e(\Gamma(A))$. Finally we observe that in either case N_e requires attention and acts finitely often: this claim follows from the fact that V_e is finite since N_e enumerates in V_e only numbers of the form $n(e, t)$ (and thus, finitely many numbers, as $n(e, t)$ has limit), and the fact that N_e requires attention through (2b) at most once for every $y \in V_e$.

□

Lemma 4.3. $\Gamma(A)$ *is* Π^0_1.

Proof. We show that $\{Y^s : s \in \omega\}$ (where $Y^s = \Gamma_s(A^s)$) is a Π^0_1 approximation to $\Gamma(A)$. The predicate "$n \in Y^s$" is clearly decidable. Given a number n, either $n \in \Gamma_s(A^s)$ for every s because n never leaves A (in this case the value $Y^s(n) = 1$ never changes), or there is a unique stage $t + 1$ such that $n \in A^t \smallsetminus A^{t+1}$: now either we never dump n (so that $n \notin Y^s$, for every $s \geqslant t + 1$: in this case the value $Y^s(n)$ changes exactly once, being $Y^s(n) = 1$ for every $s \leqslant t$, and $Y^s(n) = 0$ for every $s \geqslant t+1$), or we dump n exactly at stage $t+1$ thus getting $n \in Y^{t+1}$, and therefore $n \in Y^s$ for every $s \geqslant t+1$, and in this case the value $Y^s(n) = 1$ never changes. □

This completes the proof of the theorem. □

Corollary 4.4. *There is no minimal s_2-degree in the Π^0_1 s_2-degrees (and thus, a fortiori, there is no Π^0_1 s_2-degree can be minimal). The first order theory of the Δ^0_2 s_2-degree differs from the first order theory of the Π^0_1 s_2-degrees.*

Proof. The claim follows from the fact that by Theorem 2.1 there exists a minimal Δ^0_2 s_2-degree, whereas there is no minimal element in the Π^0_1 s_2-degrees. □

5. Downwards density in the Π^0_1 s_1-degrees and in the c.e. Q_1-degrees, and non-immunity properties

Theorem 5.1 below (or, rather, its version for Q_1-reducibility given by the isomorphism between Q_1-reducibility and s_1-reducibility) shows that no c.e. non-hyperhypersimple set can have minimal degree in the c.e. Q_1-degrees. This, together with the above cited Fact 1.5, due to Chitaia, stating that no c.e. Q_1-degree below a hyperhypersimple Q_1-degree can be minimal in the c.e. Q_1-degrees, yields as an immediate corollary that there is no c.e. Q_1-degree which is minimal in the c.e. Q_1-degrees (and a fortiori in the Q_1-degrees).

Theorem 5.1. *If A is a non-c.e. non-hyperhyperimmune Π^0_1 set then there exists a non-c.e. Π^0_1 set Y such that $Y <_{s_1} A$.*

Proof. Let A be a non-c.e. Π^0_1 set. We will work with some fixed Π^0_1 approximation for A as the one described at the beginning of the proof of Theorem 4.1.

The case A not immune. We first assume that A is non-immune. The proof in this case is similar to that of Theorem 4.1. We indicate only the modifications needed to adapt that proof to the present case. Given our non-c.e. non-immune Π^0_1 set A we want to build an s_1-operator Γ such that the set $Y = \Gamma(A)$ satisfies the claim. We have the same requirements but of course each s-operator involved in the proof must now be an s_1-operator Ψ, for which no axiom of the form $\langle x, \varnothing\rangle \in \Psi$ is allowed: axioms of this type were used in the proof of Theorem 4.1 to *dump* numbers n by defining axioms of the form $\langle n, \varnothing\rangle \in \Gamma$ so as to permanently get $n \in \Gamma(A)$. What corresponds to dumping here? Since we start with an infinite set A which is not immune there is some infinite decidable subset $X \subseteq A$: *dumping* a number n at stage s (so as to permanently get $n \in \Gamma(A)$) means now to add an axiom of the form $\langle n, \{x\}\rangle \in \Gamma$ for some $x \in X$ where x is *fresh*, i.e. no axiom $\langle m, \{x\}\rangle \in \Gamma$ has been added before stage s. Notice that such an axiom is consistent with Γ being an s_1-operator. Moreover we need a more dynamic definition of Γ, which in the previous proof was assumed to contain from the very beginning $\langle n, \{n\}\rangle \in \Gamma$, for every $n \in \omega$; this is now not consistent with making Γ an s_1-operator, because there could be distinct n, x with $x \in X$ such that by dumping we enumerate an axiom $\langle n, \{x\}\rangle \in \Gamma$ which, together with $\langle x, \{x\}\rangle \in \Gamma$ would violate injectivity of Γ. Thus in this new context at stage 0 we only put in Γ axioms of the form $\langle n, \{n\}\rangle \in \Gamma$ for all $n \notin X$. At the end of stage $s+1$, if $s \in X$ then we pick the least $x \in X$ such that x is fresh, and we add the axiom $\langle s, \{x\}\rangle \in \Gamma$: this ends up with having $s \in \Gamma(A)$. Finally, for requirement N_e, *dumping* a number $\psi_e(n, s)$ at $s+1$ is not achieved of course by adding $\langle \psi_e(n, s), \varnothing\rangle \in \Gamma$, but (accordingly to the new current meaning of dumping) by adding an axiom of the form $\langle \psi_e(n, s), \{x\}\rangle \in \Gamma$, for some *fresh* $x \in X$. Finally we now say that a number n *has already been dumped into* $\Psi_e(\Gamma(A))$ *at the* *beginning of stage* $s+1$ if there is an axiom $\langle n, \{y\}\rangle \in \Psi_{e,s+1}$ and y is dumped; during stage $s+1$ we say that we *dump* n *into* $\Psi_e(\Gamma(A))$ if at this stage we dump y, for some y such that there is an axiom $\langle n, \{y\}\rangle \in \Psi_{e,s+1}$. Notice also that we need to modify the definition of $\psi_e(n,s)$ by defining $\psi_e(n,s)\downarrow$ if at stage s, the number n has not been already dumped into $\Psi_e(\Gamma(A))$, but there exists an axiom $\langle n, \{y\}\rangle \in \Psi_{e,s}$ with $y \in \Gamma_{s-1}(A^s)$, thus $y \in A^s$, and in case of convergence $\psi_e(n,s)$ equals the least such y.

Modulo these modifications, the proof of Theorem 4.1 goes through, and $Y = \Gamma(A)$ is our desired set. Finally we observe that $\Gamma(A) \in \Pi^0_1$. For this, we consider a suitable Π^0_1 approximation $\{Y^s : s \in \omega\}$ to $\Gamma(A)$: this approximation starts up with $Y^0 = \omega$; if $n \in X$ then $n \in Y^s$ for every s (we know that $n \in \Gamma(A)$, as achieved at stage $n+1$); if $n \notin X$, then we can argue as in Lemma 4.3 that the value $Y^s(n)$ changes at most once.

The case A immune, but not hyperhyperimmune. We now tackle the case when A is immune, but not hyperhyperimmune. Let $\{W_{f(i)} : i \in \omega\}$ be a weak disjoint array of finite sets witnessing that A is not hyperhyperimmune: thus, for all $m, n \in \omega$ each $W_{f(n)}$ is finite, $W_{f(m)} \cap W_{f(n)} \neq \varnothing$ if $m \neq n$, and $W_{f(n)} \cap A \neq \varnothing$. The P-requirements take the form

$$P_e : \Gamma(A) = W_e \Rightarrow Z_e \text{ is an infinite c.e. subset of } A,$$

where Z_e is built by us; the requirement N_e is as before.

5.1. **Defining valid axioms.** Since there is now no infinite decidable subset of A, we need yet a more dynamic approximation $\{\Gamma_s : s \in \omega\}$ to the s_1-operator Γ than in the previous case, so that now, at each stage s, Γ_s is finite. In the end our operator Γ will satisfy that if $\langle n, \{y\}\rangle \in \Gamma$ then $y \in W_{f(n)}$. Let us say that an already existing axiom $\langle n, \{y\}\rangle \in \Gamma$ is *valid* at a stage s, if $y \in A^s$. Suppose now that s is a stage and n is a number such that at the beginning of stage s there is no valid axiom, and we want to define one at this stage. To do this, we search for the first z such that $z \in W_{f(n)} \cap A^s$ (such a z exists since $W_{f(n)} \cap A \neq \varnothing$, and we work with a Π^0_1 approximation to A, so that for every number y we have that $y \in A$ if and only if $y \in A^t$ for every stage t), and we will add the axiom $\langle n, \{z\}\rangle \in \Gamma_s$, which is valid at the end of stage s. We observe that if for n we keep adding axioms of this kind every time we see that the previously defined axioms have ceased to be valid then, as $W_{f(n)}$ is finite and $W_{f(n)} \cap A \neq \varnothing$, we eventually get $n \in \Gamma(A)$ due to an axiom which will stay valid forever.

In the following remark we summarize some of the observations elaborated so far about how the Γ-axioms are defined.

Remark 5.2. Since $\{W_{f(i)} : i \in \omega\}$ is a disjoint array, and $\langle n, \{y\}\rangle \in \Gamma$ is a Γ-axiom only if $y \in W_{f(n)}$ we have that Γ is an s_1-operator. For every n, at each stage s there is at most one valid axiom for n, which in case is the last one appointed for n; as we are working with a Π^0_1 approximation to A, we have that if at some stage there is no valid axioms for a number n, and after that stage we do not add any more axiom for n, then $n \notin \Gamma(A)$. Also, if we keep adding axioms for n (making sure that an axiom added a stage s be valid at s) then eventually we hit an axiom which will stay valid forever guaranteeing that $n \in \Gamma(A)$.

5.2. **Dumping.** What corresponds to dumping in the present case? *Dumping* a number n at stage s (so as to permanently get $n \in \Gamma(A)$) means to *commit* ourselves, from this stage on, to keep appointing enough axioms for n (making sure that an axiom added a stage s be valid at s) until by Remark 5.2 we get $n \in \Gamma(A)$.

We outline the modified strategies for the requirements.

5.3. **Strategy for P_e and its outcomes.** As usual, we say that a witness n for P_e is realized at stage s if $n \in W_{e,s}$. We keep appointing fresh witnesses $n_0, n_1, \ldots$ (so that for each n_i, when we appoint n_i, we have that n_i is unrealized and we also add at this stage a valid axiom for n_i) *until* one of the following two cases holds:

(a) We appoint a witness n that never gets realized. This is a winning outcome, since while waiting for n to become realized we eventually manage to equip n with a valid axiom: if at the beginning of a stage t we have that there is no valid axiom $\langle n, \{y\}\rangle \in \Gamma$ then at this stage we add a valid axiom for n, so that by Remark 5.2, if n stays unrealized then in the end $n \in \Gamma(A)$: this yields $n \in \Gamma(A) \smallsetminus W_e$.

(b) All appointed witnesses become realized and for some such (least) witness n we see $n \notin \Gamma(A)$. This is a winning outcome, since by construction when a witness becomes realized we stop adding new Γ-axioms for it, and on the other hand if $n \notin \Gamma(A)$ at a stage t after becoming realized then each already existing Γ-axiom $\langle n, \{y\}\rangle \in \Gamma$ has ceased to be valid, i.e. $y \notin A^t$. If P_e does not add any more axiom for n, then by Remark 5.2 $n \notin \Gamma(A)$: this relies of course on the fact that, after chosen, n will not be dumped by any lower priority requirement.

We finally describe how the construction ensures that if we never appoint a witness which stays unrealized as in (a) then eventually we hit a realized witness n as in (b). To see this, notice that by Remark 5.2 when at any stage a witness n becomes realized but n is still in $\Gamma(A)$ this means that at that stage n has a unique valid axiom $\langle n, \{y\}\rangle \in \Gamma$: in this case we *neutralize* the witness n by enumerating y into Z_e. Therefore failure to hit (a) or (b) would result in appointing infinitely many neutralized witnesses $n_0, n_1, \ldots$ so that for every j there are a corresponding number y_j and an axiom $\langle n_j, \{y_j\}\rangle \in \Gamma$ which stays valid forever and thus $y_j \in A$, implying that $Z_e = \{y_0, y_1, \ldots\}$ is an infinite c.e. subset of A, which is not possible since A is immune. The claim that in this case each y_i must be in A relies on the fact that, after been chosen, n_i will not be dumped by any lower priority requirement.

Notice also that any P_e appoints witnesses that are fresh, so that different P-strategies appoint disjoint sets of witnesses. Therefore if n is a witness appointed by a P-requirement, then Γ-axioms for n may be appointed only by this requirement, or by dumping.

In addition to the parameters of the proof of Theorem 4.1, the strategy for P_e will use the following parameter: $\nu_{e,s}$ is the finite set of witnesses currently appointed at stage s; if $\nu_{e,s} \neq \varnothing$ then let $n^{\max}_{e,s} = \max \nu_{e,s}$.

5.4. **Strategy for N_e.** Strategy, outcomes, and parameters are as for N_e in the proof of Theorem 4.1, with the difference that if at some stage s we need to dump $\psi_e(n, s-1)$ then we act as in Section 5.2.

5.5. **Interactions between strategies.** The above strategies interact which each other through a standard finite priority argument, in particular via the usual initialization mechanism: when a requirement R acts, it *initializes* all lower priority requirements R'. To *initialize* R' means that if $R' = P_e$ for some e, then we cancel the currently appointed witnesses for R' by *re-initializing* ν_e, which is set anew as $\nu_e = \varnothing$. If $R' = N_e$ then R' is initialized exactly as in the proof of Theorem 4.1.

5.6. **The construction.** At step s we define a finite approximation Γ_s to Γ, starting with $\Gamma_0 = \emptyset$. We also define a cofinite set Y^s with the aim that $\{Y^s : s \in \omega\}$ is a Π^0_1 approximation to the final set $\Gamma(A)$: the predicate "$x \in Y^s$" is decidable, $Y^0 = \omega$, $Y^s \supseteq Y^{s+1}$ for each s, and $\Gamma(A) = \bigcap_s Y^s$.

Stage 0. Initialize all strategies. Set $\Gamma_0 = \emptyset$, and $Y^0 = \omega$.

Stage $s+1$. Unless otherwise specified, all relevant computations are understood to be approximated as at the end of the previous stage s, which we often omit to mention for the sake of simplicity.

We say that the requirement P_e *requires attention* at stage $s+1$ if at least one of the following cases holds:

(i) $\nu_e \neq \emptyset$, $n_e^{\max}$ is not realized and has currently no valid axiom;
(ii) all elements of ν_e are realized, but there is no $n \in \nu_e$ such that $n \notin \Gamma(A)$;
(iii) there is a realized $n \in \nu_e$ such that the last appointed axiom $\langle n, \{y\}\rangle \in \Gamma$ for n was still valid at s but now $y \notin A^{s+1}$.

The requirement N_e *requires attention* exactly as in the proof of Theorem 4.1.

If there is no requirement P_e, N_e which deserves attention at $s+1$ then go directly to the housekeeping actions, otherwise pick the least requirement R that requires attention, and distinguish the two possible cases:

Case $R = P_e$: If (i), (ii), or (iii) happens then P_e *acts* accordingly through the first case that happens:

(i) Add to Γ a valid axiom for $n_e^{\max}$;
(ii) In this case the action consists in the following: if $\nu_e \neq \emptyset$ then as P_e requires attention it follows by Remark 5.2 that $n_e^{\max}$ has a unique axiom $\langle n_e^{\max}, \{y\}\rangle \in \Gamma$ which is valid: in this case we *neutralize* $n_e^{\max}$ by enumerating y into Z_e. Next (including the case in which $\nu_e = \emptyset$) we appoint a fresh (hence, in particular, still in Y^s) unrealized witness $m > n_e^{\max}$, and add a valid axiom $\langle m, \{z\}\rangle \in \Gamma$ for m. Thus $\nu_{e,s+1} = \nu_e \cup \{m\}$ and $n_{e,s+1}^{\max} = m$.
(iii) In this case the action of R consists uniquely in initializing all $R' > R$: this guarantees that if n is realized and the axiom $\langle n, \{y\}\rangle \in \Gamma$ has ceased to be valid, then by initialization of its ψ_j no lower priority requirement N_j will ever dump n as $\psi_j(m,t) \neq n$ for every m and for every stage $t+1 > s+1$.

Case $R = N_e$: as in the proof of Theorem 4.1.

In either case, (whether R is a P- or an N-requirement) if R acts then R initializes all $R' > R$.

Then we go to the housekeeping actions.

Housekeeping actions: We distinguish the following three sets of housekeeping actions at the end of stage $s+1$:

(1) For every requirement P_k, which has not acted during stage $s+1$, such that $\nu_k \neq \emptyset$, $n_k^{\max}$ is not realized, $n_k^{\max}$ had a valid axiom $\langle n_k^{\max}, \{y\}\rangle \in \Gamma$ at s (i.e. $y \in A^s$), but $y \notin A^{s+1}$, then add to Γ a valid axiom at $s+1$ for $n_k^{\max}$. This action for $n_k^{\max}$ still makes $n_k^{\max} \in \Gamma_{s+1}(A^{s+1})$, keeping the value $\Gamma(A)(n_k^{\max}) = 1$ from stage s to $s+1$ although the previous axioms have lost their validity.

(2) If n is dumped but all previous axioms for n have lost their validity by the end of stage s then we add in Γ an axiom for n which is valid at $s+1$. (This makes sure that eventually $n \in \Gamma(A)$.)
(3) Set $n \notin Y^{s+1}$ for all $n \leqslant s$ such that $n \notin \Gamma_{s+1}(A^{s+1})$. For all other numbers n, keep $Y^{s+1}(n) = Y^s(n)$. (Notice that we use the final value of Γ_{s+1} as given by items (1) and (2).)

Go to next stage.

5.7. **The verification.** The verification follows along the usual lines of a typical finite injury argument. By a straightforward inductive argument one sees that each requirement requires attention and acts only finitely often, and thus every requirement R is initialized only finitely often. After its last initialization R is free to pursue its strategy without any interference by higher priority requirements, and is eventually satisfied. To see this, assume that every $R' < R$ requires attention and acts only finitely often. Let s_0 be the last stage at which R is initialized.

Suppose first that $R = P_e$. We begin by observing that after s_0, the set ν_e is never re-initialized, and no $n \in \nu_e$ is ever dumped. Indeed, as long as n is not realized it cannot be dumped because, by (i) of the action of P_e, no lower priority N_j will see n leave $\Gamma(A)$; if n is realized and an axiom $\langle n, \{y\} \in \Gamma$ loses its validity then P_e will not add any more axiom, and by initialization due to action (iii) of P_e no lower priority requirement N_j will ever dump n, as explained in the comment accompanying the description of action (iii) of P_e. Not being involved in any dumping, by Remark 5.2 any $n \in \nu_e$ has at any stage at most a valid axiom, and in case this is of the form $\langle n, \{y\}\rangle \in \Gamma$ for some $y \in W_{f(n)}$. To show that P_e is satisfied, and requires attention only finitely many times, we distinguish the following cases:

(a) At some least stage $s_1 + 1 > s_0$ we appoint a new witness n for P_e such that n stays unrealized forever, and thus $n = n^{\max}_{e,s}$ for all $s \geqslant s_1 + 1$. In this case $n \in \Gamma(A)$ by (i) of the action of P_e, since at each stage at which we see $n \notin \Gamma(A)$ we add a new axiom for n thus eventually implying that $n \in \Gamma(A)$ by Remark 5.2. Therefore P_e is met as $n \in \Gamma(A) \smallsetminus W_e$. Notice that after this, P_e may still requires attention through (iii), but this may happen only finitely many times, as ν_e is finite and we are working with a Π^0_1 approximation to A.
(b) At some least stage $s_1 + 1 > s + 0$ following the last initialization of P_e we see that all witnesses in ν_e are realized, and one of them, let us call it n, has left $\Gamma(A)$: we may assume that n is the least one with this property. Hence neither P_e nor the housekeeping actions add any further axioms for n after stage $s_1 + 1$, and by initialization no lower priority requirement will ever need to dump n, as n will never be in the range of the use ψ_j of any lower priority requirement N_j. Therefore P_e is met as $n \in W_e \smallsetminus \Gamma(A)$. After this, again, P_e may still requires attention through (iii), but this may happen only finitely many times.
(c) Otherwise. As argued in the informal discussion of the strategy for P_e, we show that (c) is not possible. Indeed, (c) implies that ν_e grows up to an infinite c.e. set of realized witnesses such that every $n \in \nu_e$ is neutralized: at the very moment at which n is neutralized we enumerate in Z_e a number y_n such that $\langle n, \{y_n\}\rangle \in \Gamma$ is the unique valid axiom for n at that stage. As n does not witness $n \in W_e \smallsetminus \Gamma(A)$, we have by Remark 5.2 that $y_n \in A$, since no axiom for n which has lost validity may become valid again, and no other axiom will ever be added for n, either by P_e or by a lower priority requirement through dumping. This implies that the elements enumerated in Z_e after last initialization of P_e grow up to a c.e. set $Z_e \subseteq A$. On the other hand Z_e is infinite because $y_n \neq y_m$ if $n \neq m$, since y_n and y_m

come from two distinct sets of the disjoint array $\{W_{f(i)} : i \in \omega\}$. In conclusion, we would get that Z_e is an infinite c.e. subset of A, which is a contradiction since A is immune.

For $R = N_e$ the claim that N_e requires attention only finitely many times and is satisfied follows as in the proof of Theorem 4.1.

Finally we prove that $\{Y^t : t \in \omega\}$ is a Π^0_1 approximation to $\Gamma(A)$, i.e. $\Gamma(A) = \bigcap_s Y^s$: in particular it is easy to see that $\Gamma(A) \subseteq Y^s$ for each s, and numbers not lying in $\Gamma(A))$ are eventually extracted from Y by (3) of the housekeeping actions. Finally, for every m, it is easy to see that the value $Y^t(m)$ changes at most once. Indeed, exactly as in the proof of Theorem 4.1 we need not worry about elements m that get dumped. Indeed for such elements we have $Y^t(m)$ for every t: if we dump m at stage $t+1$ then we do so when at the beginning of a stage $t+1$ we see that m had a valid axiom at t, but at stage $t+1$ its previous valid axiom has lost validity; so, by dumping at stage $t+1$ we eventually manage to keep $m \in \Gamma(A)$ forever, even at stage $t+1$. On the other hand, apart from dumping, we may need to enumerate again at some stage in $\Gamma(A)$ a number m if m currently lies in some ν_e, it is not realized, it had a valid axiom at a stage t, but this axiom has lost its validity at $t+1$. However as for dumping, item (1) of housekeeping actions makes sure that the re-enumeration of m in $\Gamma(A)$ be done exactly at $t+1$, keeping at $t+1$ the same value $\Gamma(A)(m) = 1$, as at t. □

Corollary 5.3. *There is no minimal s_1-degree in the Π^0_1 s_1-degrees (and a fortiori there is no Π^0_1 s_1-degree can be minimal). As a consequence, there is no minimal Q_1-degree in the c.e. Q_1-degrees (and a fortiori there is no minimal c.e. Q_1-degree in the full structure of Q_1-degrees). As a consequence, the first order theory of the Δ^0_2 s_1-degree differs from the first order theory of the Π^0_1 s_1-degrees, and the first order theory of the Δ^0_2 Q_1-degree differs from the first order theory of the c.e. Q_1-degrees.*

Proof. The claim about non-minimality follows by the isomorphism of $\leqslant_{Q_1}$ and $\leqslant_{s_1}$ given by complements of sets, together with the following observation: If we put together the previous theorem with Fact 1.5 then we conclude that no non-c.e. Π^0_1 s_1-degree (either hyperhyperimmune or non-hyperhyperimmune) can be minimal in the Π^0_1 s_1-degrees.

The proof of the claim about the first order theories of the s_1-degrees is similar to the proof of Corollary 4.4 given for s_2-reducibility. The claim about Q_1 follows by the isomorphism given by complements of sets between the s_1-degrees and the Q_1-degrees. □

Remark 5.4. By [10, Corollary 5], if A, B are Δ^0_2 sets then $A \leqslant_{Q_1} B$ if and only if there is a weak disjoint array of finite sets $\{W_{f(i)} : i \in \omega\}$ such that

$$(\forall x)[x \in A \Leftrightarrow W_{f(x} \subseteq B].$$

A strictly related reducibility (at least on Δ^0_2 sets) is therefore the so-called *conjunctive reducibility*, denoted by $\leqslant_c$, for which, for every pair of sets A, B, we define $A \leqslant_c B$ if there exists a strong disjoint array of finite sets $\{D_{f(i)} : i \in \omega\}$ such that

$$(\forall x)[x \in A \Leftrightarrow D_{f(x} \subseteq B].$$

Notwithstanding the obvious similarities, the two reducibilities when restricted to the c.e. sets give rise to degree structures which are not elementarily equivalent, as by Corollary 5.3 there is no minimal Q_1-degree in the c.e. Q_1-degrees, but [3, Theorem 2.1] shows that there exist minimal c-degrees in the c.e. c-degrees.

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