COMPUTATIONAL ASPECTS OF THE HYPERIMMUNE-FREE DEGREES

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ABSTRACT. We explore the computational strength of the hyperimmune-free Turing degrees. In particular we investigate how the property of being dominated by recursive functions interact with classical computability notions such as the jump operator, relativization and effectively closed sets.

1. Introduction

The relative computational power between sets of natural numbers has traditionally been measured by Turing reducibility: If $A \leq_T B$ we think of B as containing at least as much algorithmic information as A. There have been various other well-studied methods of calibrating computational power; for instance, by examining the effective enumerations of A, by looking at the algorithmic randomness content of A, and by investigating the rate of growth of functions computable from A. These studies have all yielded deep results relating Turing reducibility with different aspects of computation. This paper is concerned with the last of these: The rate of growth of functions computed by A. Hence a set A can be viewed as being computationally powerful if it is able to compute functions which grow fast enough to dominate a certain other class of functions.

Domination properties have been studied extensively in the literature, and many relationships between domination and Turing degrees have been found. For instance it is easy to see that if B is r.e. in A then $A \geq_T B'$ iff A computes a function dominating every B-partial recursive function. In some cases a class of degrees is first defined in terms of a domination property and subsequently results are then obtained about its computational properties; the almost everywhere dominating degrees is an example of such a class. It is more common to go the other way, for a class of degrees to be first introduced without mentioning domination and then subsequently characterized in terms of a domination property. For example, Martin [9] characterized the high Turing degrees as the degrees which compute a function dominating every recursive function; the class of array non-recursive degrees introduced by Downey, Jockusch and Stob [3] was shown in [4] to be the same as the class of degrees $\mathbf{a}$ where every ω-r.e. function fails to dominate some $\mathbf{a}$-recursive function.

Key words and phrases. Hyperimmune-free, Turing degrees, effectively closed sets.

Ng's research was partially supported by NTU reserach grant No. M4080797.110. Stephan was partially supported by NUS research grant WBS R 252-000-420-112. Yang was partially supported by NUS research grant WBS R 146-000-159-112 and NSFC (No. 11171031). Yu was partially supported by NSFC grant No. 11071114.

The class studied in this paper is the class of hyperimmune-free (HIF) degrees (i.e. the degrees which contain no hyperimmune set). We recall that a set A is hyperimmune iff A is infinite and there is no disjoint strong array of finite sets each of which has non-empty intersection with A. The study of hyperimmune sets can be traced back to Post and attempts to solve Post's problem. Post [12] introduced the notion of a simple and a hypersimple r.e. set (A is hypersimple if $\overline{A}$ is hyperimmune) and it turned out that each hypersimple set is wtt-incomplete, but not necessarily Turing incomplete. Indeed Dekker [1] showed that every non-recursive r.e. degree is hyperimmune (i.e. not HIF), while Miller and Martin [10] showed that every degree $\mathbf{b}$ satisfying $\mathbf{a} < \mathbf{b} < \mathbf{a}'$ for some $\boldsymbol{a}$ is hyperimmune. Dekker and Myhill [2] showed that every non-recursive degree contains an immune set, hence it was rather surprising that Miller and Martin [10] were able to construct a non-recursive HIF degree. Indeed they characterized the HIF degrees using a domination property: $\boldsymbol{a}$ is HIF iff every function recursive in $\boldsymbol{a}$ is dominated by some recursive function. This property asserts that $\boldsymbol{a}$ is "almost recursive"', in that $\boldsymbol{a}$ computes no fast-growing function (relative to the class of recursive functions). However the fact that no non-recursive Δ^0_2 degree possesses this property indicates that this notion of computational feebleness is intrinsically hard to understand.

The main aim of this paper is to shed some light on this class by investigating how the domination related property of HIF degrees is related to the other more traditional methods of measuring computational strength. It is easy to see that each HIF degree is of array recursive degree and is generalized low_2, hence each HIF degree cannot be computationally strong in this sense. On the other hand a HIF degree can be PA-complete.

The main difficulty that we face is how to translate between the information contained in a fast-growing function and the ability to code into a given set. The proofs given in this paper give various ways of doing this.

In section 2 we study the distribution of HIF degrees in Π^0_1 classes. The so-called "HIF basis theorem" of Jockusch and Soare [5] asserts that every non-empty Π^0_1 class contains a member of HIF degree. We construct an uncountable Π^0_1 class in which every member is generalized low (GL_1) and of HIF degree. This Π^0_1 class we construct will necessarily have recursive members (in fact, isolated paths).

In section 3 we investigate when a degree can be HIF relative to another. We introduce the notion of being HIF relative to $\mathbf{0}^n$, for $n > 0$. We show that there are uncountably many sets which are simultaneously HIF and HIF relative to $\mathbf{0}''$, but surprisingly we discover that no non-recursive set is both HIF and HIF relative to $\emptyset'$. On the other hand, we construct a perfect closed set of reals which are simultaneously HIF and HIF relative to every low r.e. set. We also obtain another characterization of the K-trivial sets as the Δ^0_2 sets A where some HIF set is A-random.

In section 4 we study the degrees which are the jump of some HIF degree. From folklore it is known that each degree above $\mathbf{0}''$ is the double jump of a HIF degree. However the degrees which are the jump of a HIF degree is not at all well-understood. Kučera and Nies [8] showed that each degree r.e. in and strictly above $\mathbf{0}'$ computes $\mathbf{a}'$ for some HIF degree $\mathbf{a}$, while it follows from Jockusch and Stephan [6] that the jump of each HIF degree cannot be PA-complete relative to $\mathbf{0}'$. We will show that for each 2-generic degree $\mathbf{c}$, there is a HIF degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{c} \cup \mathbf{0}'$. We conjecture that this is in fact a characterization of the degrees which are the jump of a HIF degree.

2. HIF and closed sets

Nies and Miller (unpublished) observed that no real can simultaneously be GL_1, HIF and of diagonally non-recursive (DNR) degree, although any combination of two are possible. It is a natural question to ask to what extent can these properties be reflected in Π^0_1 classes. We first show that there is an uncountable Π^0_1 class where every non-isolated path is GL_1 and of DNR degree. Hence GL_1 and DNR can be simultaneously realized by every non-recursive path in an uncountable Π^0_1 class. We note that the isolated paths are necessary, since every perfect Π^0_1 class contains a path of high degree, and clearly no set if HIF degree can be high.

Lemma 2.1. *Given any tree $T \leq_T \emptyset'$ there exists a recursive tree Q such that every path of T is Turing equivalent to a non-isolated path of Q, and vice versa.*

Proof. Let $T = \lim_s T_s$ for a recursive sequence $\{T_s\}$ of recursive trees. Define the partial recursive function $f(\sigma)$ to be the first stage s such that $\sigma \upharpoonright i \in T_s$ for every $i \leq |\sigma|$. Now define the Turing functional Ψ^X to output $X(0)2^{f(X\upharpoonright 1)}X(1)2^{f(X\upharpoonright 2)}\cdots$. Here 2^s is the symbol 2 repeated s many times. If $f(X \upharpoonright k)$ is partial for some k then the functional outputs a sequence with a tail of 2s, otherwise Ψ^X is a ternary sequence with X coded. If A is on T then $f(A \upharpoonright n)$ is convergent for all n, so clearly $A \equiv_T \Psi^A$.

Now let $Q = \Psi$ applied to 2^ω. Clearly Q is a recursive tree of rank 1. If A is on T then clearly Ψ^A is a non-isolated path of Q. On the other hand if A is not on T then let $\sigma \subset A$ be minimal such that $\sigma \notin T$. For all large enough s and every $\eta \supseteq \sigma$ of length s, $f(\eta)\uparrow$. Thus every infinite branch of Q extending Ψ^σ is isolated. □

Theorem 2.2. *There is an uncountable Π^0_1 class P such that every nonrecursive path of P is GL_1 and computes a DNR function.*

Proof. Let $T \leq_T \emptyset'$ be a tree containing only 2-random reals. By Lemma 2.1 there exists a recursive tree P such that every path of P is either isolated or Turing equivalent to a 2-random. P is clearly uncountable, and every 2-random is DNR and GL_1. □

Next we argue that there can be no Π^0_1 class where every non-recursive path is HIF and DNR

Theorem 2.3. *Suppose P is a Π^0_1 class where every path is HIF. Then P cannot contain a path of DNR degree.*

Sketch of proof. Suppose $A \in P$ and A is of DNR degree. By Kjos-Hanssen, Merkle and Stephan [7], there exists a function $f \leq_T A$ such that $C(f(n)) \geq n$ for every n. Since A is HIF, $f \leq_{tt} A$. The set

$$Q = \{X \in 2^\omega \mid \exists n C(f^X(n)) < n\}$$

is an open set. Observe that $P - Q$ is a non-empty Π^0_1 class as it contains A. Every path of $P - Q$ is of HIF degree and non-recursive. Applying the Low Basis Theorem gives a contradiction. $\square$

Finally we turn to the apparently most difficult combination. We show that there is a rank 1 uncountable Π^0_1 class such that every path is GL_1 and HIF. Again rank 1 is the best possible, since the isolated paths are necessary. We also note that every path in P has a strong minimal cover.

Theorem 2.4. *There is a rank* 1 *uncountable* Π^0_1 *class* P *such that every member of* P *is* GL_1 *and of HIF degree.*

We sketch the proof here. The requirements are

- R_e: For all X in P, there exists recursive function h such that Φ^X_e is dominated by h, provided it is total.

Strategy for a single requirement We use R_0 as an example to illustrate the strategy in isolation.

We start with $T_{-1} = 2^{<\omega}$ – the full binary tree. Recursive in k, we define $h(k)$ and modify the tree. The modification tree includes: Trimming (creating some dead ends), restrict/unrestrict certain portion of the tree. Temporarily let's refer k as level k and we process level by level.

Level 0: At stage s, check whether there exists a node $\sigma \in T_s$ such that $|\sigma| \leq s$ and $\Phi^\sigma_0(0)\downarrow$. If no, go to next stage; otherwise, let σ be the left most one, and define $h(0) = \Phi^\sigma_0(0)+1$ and define $T_{s+1} = \{\tau \in T_s : \tau$ is compatible with $\sigma\}$. Declare all other nodes dead, i.e., all nodes α on $T_s - T_{s+1}$ which have length s become dead ends. Level 0 is finished.

Level 1: When we finish level 0, we have had a node σ. We look for two nodes $\tau_0 \supseteq \sigma\hat{}\langle 0\rangle$ and $\tau_1 \supseteq \sigma\hat{}\langle 1\rangle$ such that $\Phi^{\tau_0}_0(1)\downarrow$ and $\Phi^{\tau_1}_0(1)\downarrow$. We search them one by one. Temporarily, let's refer them as cycles. We have cycle 0 and cycle 1.

In cycle 0, first *isolate* $\sigma\hat{}\langle 1\rangle$. The precise meaning of isolating a node α is: For each stage t, $\alpha\hat{}\langle 0^{t-|\alpha|}\rangle$ is the only node extending α of length t on T_t. (Need to state it in the context of isolating a node on a given tree.) Focus on the basic open set indexed by $\sigma\hat{}\langle 0\rangle$ (informally referred as current playground).

If we never find any node $\tau_0 \supseteq \sigma\hat{}\langle 0\rangle$ such that $\Phi^{\tau_0}_0(1)\downarrow$, then Φ^X_0 is partial for all X in that open set. Because we isolate $\sigma\hat{}\langle 1\rangle$, which gives rise to a recursive path. Thus we satisfied R_0 globally. We often refer the discovery of an open set in which for all X, Φ^X_0 is partial as a Σ^0_2-outcome for R_0.

If at some stage t, we find a node $\tau_0 \supseteq \sigma\hat{}\langle 0\rangle$ such that $\Phi^{\tau_0}_0(1)\downarrow$, then we isolate τ_0 and shift the playground to the open set indexed by $\sigma' = \sigma\hat{}\langle 1\rangle\hat{}\langle 0^{t-|\sigma|-1}$. This means we no long isolate $\sigma\hat{}\langle 1\rangle$. And we look for $\tau_1 \supseteq \sigma'$ such that $\Phi^{\tau_1}_0(1)\downarrow$. As argued before, if we never see such a convergence, then we win R_0 in a Σ^0_2-way. Now suppose we find such a τ_1 at $t' > t$, then we modify the tree by defining $T_t = \{\alpha : \alpha$ is compatible with either $\tau_0\hat{}\langle 0^{t'-|\tau_0|}\rangle$ or $\tau_1\}$, and define $h(1) = \max\{\Phi^{\tau_0}_0(1), \Phi^{\tau_1}_0(1)\} + 1$. (We made one more Π^0_2 instance true.)

Level $k+1$: In general, suppose we have completed k Π^0_2 instances and have defined $h(0), h(1), \cdots, h(k)$. We then need to look for $N = 2^{k+1}$ incompatible strings $\tau_0, \tau_1, \cdots, \tau_{N-1}$ such that $\Phi_0^{\tau_i}(k+1)\downarrow$. We use the same strategy as above, except we now have N cycles. There are two outcomes:

- We stuck at finding some convergent computation $\Phi_0^{\tau_i}(k+1)$. Then all $N-1$ other τ_j's are isolated forever; and the playground will be the open set indexed by τ_i (possible extended by certain zeros). Since Φ_0^X is partial for all X is this open set, we win R_0.
- Otherwise, we could complete N cycles eventually and we are able to define $h(k+1)$ and obtain a size 2^k perfect tree (so to speak).

Thus the eventual outcomes for R_0 (after completing all levels) are as follows:

- We stuck at the i-th cycle in some k-th level. Let us use (k, i) to indicate this outcome. The final tree looks like a perfect tree which is the open set index by some τ_i, together with $2^k - 1$ many isolated paths.
- Or succeed in all levels, we then get a perfect subtree T_0 and a recursive function h which uniformly dominate all Φ_0^X for all $X \in [T_0]$.

Interaction between two strategies Consider now two requirements R_0 and R_1, we will have different versions of R_1.

If R_0 has Σ^0_2-outcome (k, i), then eventually only the R_1 which guesses (k, i) correctly is active. This R_1 will work on a perfect tree (the open neighborhood indexed by certain σ) which is its playground; and work in a similar fashion as R_0 above. There is one extra caution though. We do not want the R_1 shift the root of the tree (in other words, we don't want the combined effort of R_e to eventually trim the tree into a single branch, even though for any fixed e, we have a perfect tree surviving.) Therefore, we artificially fix a split for R_0 and have two copies of R_1: $R_{1,0}$ and $R_{1,1}$, $R_{1,0}$ works on the left subtree of T_0 (the tree produced by R_0) and produces $h_{1,0}$ possibly and $R_{1,1}$ works on the right subtree in a similar fashion. This will reduce the uniformity of the dominating function h. By breaking the uniformity of h and adding extra splits, it is easier to argue the uncountability of the resulting Π^0_1-class.

Back to the discussion on interactions, suppose that R_0 has Π_2-outcome, then R_1 would receive a perfect tree (piecewise, level by level) from R_0. Then R_1 can exert its power to that tree, e.g., isolating certain nodes and treat some open set as its playground. Note that R_1's action will have an impact on the (Π_2-strategy of) R_0. R_0 may hand to R_1 a size 2^k perfect tree and R_1 may turn it into "an active playground" plus a few isolated paths, let T^* temporarily denote the resulting damaged tree. (The Π_2) R_0 has to work on T^* instead of the small perfect tree which it passes to R_1. The interaction is the reason that we have to use stage by stage construction instead of forcing by Π^0_1-classes.

This modified R_0 will take over the finite tree T^* (after all, R_0 still have the highest priority). R_0 will still run cycles, and in each cycle look for convergent computations $\Phi_0^\tau(k+1)$. The cycles now will be ranging over the leaves of T^* (whose number is less than 2^{k+1} most likely). If R_0 is able to complete the whole (modified) cycle, it then defines $h(k+1)$ and passes to R_1 (who will immediately damage it almost surely). If R_0 gets stuck at certain cycle, then R_0 would have Σ_2-outcome, the version of R_1 would be irrelevant.

The full details of the construction will appear in the journal version of the paper.

Corollary 2.5. *There exists a perfect tree $T \leq_T \emptyset''$ with no dead ends such that every path of $[T]$ is HIF and GL_1.*

3. HIF and relativization

Definition 3.1. We say that X is HIF relative to A if every function recursive in $X \oplus A$ is dominated by an A-recursive function. For $n \geq 0$ we call A an $(n+1)$-HIF if every function recursive in $A^{(n)}$ is dominated by a $\emptyset^{(n)}$-recursive function.

Fact 3.2. *A is $(n+1)$-HIF implies that A is HIF relative to $\emptyset^{(n)}$.*

Fact 3.3. *2-HIF is equivalent to being GL_1 and HIF relative to $\emptyset'$.*

Example 3.4. *Every low$_2$ HIF is $(n+3)$-HIF for every $n \geq 0$.*

Proposition 3.5. *There exists uncountably many reals which are both HIF and 3-HIF.*

Proof. For every $C \geq_T \emptyset''$ there is a HIF A such that $A'' \equiv_T C$. Relativize the construction of a HIF real to $\emptyset''$, we get uncountably many reals C which are HIF relative to $\emptyset''$. □

We can show that there are sets which are HIF relative to every low r.e. set:

Theorem 3.6. *There exists uncountably many HIF sets which are HIF relative to every low r.e. set.*

The proof of this constructs an uncountable tree combined with the Robinson's technique for guessing Σ^0_1 facts about low sets. We refer the reader to the full paper for further details.

Lemma 3.7. *Suppose that C is PA-complete and B is r.e so that $C \not\geq_T B$, then $C \oplus B \geq_T \emptyset'$.*

Proof. Suppose that C is PA and B is r.e so that $C \not\geq_T B$. We may assume that for any r.e. set W_e and number n, if $n \in W_{e,s+1} \setminus W_{e,s}$, then $s = 2^e \cdot 3^t$ for some t. Fix a recursive bijection $\langle , \rangle : \omega^2 \to \omega$.

Now define a Π^0_1 set P so that $A \in P$ if and only if

(1) For any n, $n \in A$ implies $n = \langle 2^{e_0} \cdot 3^{m_0}, 2^{e_1} \cdot 3^{m_1} \rangle$; and
(2) For any e_0, m_0 and e_1, m_1, either $\langle 2^{e_0} \cdot 3^{m_0}, 2^{e_1} \cdot 3^{m_1} \rangle \in A$ or $\langle 2^{e_1} \cdot 3^{m_1}, 2^{e_0} \cdot 3^{m_0} \rangle \in A$; and
(3) For any e_0, m_0, e_1, m_1 and e_2, m_2, if $\langle 2^{e_0} \cdot 3^{m_0}, 2^{e_1} \cdot 3^{m_1} \rangle \in A$ and $\langle 2^{e_1} \cdot 3^{m_1}, 2^{e_2} \cdot 3^{m_2} \rangle \in A$, then $\langle 2^{e_0} \cdot 3^{m_0}, 2^{e_2} \cdot 3^{m_2} \rangle \in A$; and
(4) For any e_0, m_0, e_1, m_1 and s, $\langle 2^{e_0} \cdot 3^{m_0}, 2^{e_1} \cdot 3^{m_1} \rangle \in A$ and $m_0 \notin W_{e_0,s}$, then $m_1 \notin W_{e_1,s}$

By (2) and (3), every $A \in P$ codes a linear order. By (4), ω is as an order type of an initial segment of A. Moreover, the initial segment of A is exactly the set $\{2^e \cdot 3^m \mid m \in W_e\}$.

Obviously P is not empty. So there is a set $A \in P$ recursive in C.

Now suppose that $B = W_{e_0}$ for some e_0. Were there exist some e_1, m_1 so that $n \in W_{e_0}$ if and only if $\langle 2^{e_0} \cdot 3^n, 2^{e_1} \cdot 3^{m_1} \rangle \in A$, then B would be recursive in A, a contradiction to the assumption.

Now suppose that $\emptyset' = W_{e_1}$. Then for any m, $m \notin W_{e_1}$ if and only if there exists some $n \notin W_{e_0}$ so that $\langle 2^{e_0} \cdot 3^n, 2^{e_1} \cdot 3^m \rangle \in A$. $\omega \setminus \emptyset'$ is r.e. in $B \oplus A$. In other words, $\emptyset' \leq_T A \oplus B \leq_T C \oplus B$. □

Theorem 3.8. *No PA-complete set can be both HIF and HIF relative to some non-recursive r.e. set.*

Proof. Let C be a PA-complete HIF and B be a non-recursive r.e. set.. such that C is HIF relative to B. Since C forms a minimal pair with $\emptyset'$, by Lemma 3.7 we have $C \oplus B \geq_T \emptyset'$. Hence $C \oplus B$ computes the function $c_{\emptyset'}$ where $c_{\emptyset'}(n) =$ least stage s such that $\emptyset'_s \upharpoonright n = \emptyset' \upharpoonright n$. Since C is HIF relative to B, this is dominated by some function $g \leq_T B$. Hence $B \equiv_T \emptyset'$. By Theorem 3.11 we get that C is recursive, a contradiction. □

Lemma 3.9. *If a tree $T \leq_T \emptyset'$ contains a HIF path A then there is a recursive tree Q containing A such that $[Q] \subseteq [T]$.*

Theorem 3.10. *If A is HIF and HIF relative to some PA-complete set $B \leq_T \emptyset'$ then A is recursive.*

Proof. Assume that a non-recursive A and a B exist as above. There exists a uniformly B-recursive sequence $\{B_e\}_{e \in \omega}$ of reals such that for every e, either the e^{th} Π^0_1 class is empty or B_e is a member of the e^{th} Π^0_1 class. Let $f^{A \oplus B}(e)$ be the first x found such that $B_e \upharpoonright x \neq A \upharpoonright x$. Then $f^{A \oplus B}$ is total since $B \leq_T \emptyset'$ and so A cannot be recursive in B. This is majorized by some B-recursive function g^B. It is easy to see that there is a B-recursive and hence $\emptyset'$-recursive tree T containing exactly the paths X such that for every e, $X \upharpoonright g(e) \neq B_e \upharpoonright g(e)$. Clearly T contains the HIF path A and so by Lemma 3.9 there is a recursive tree Q such that $[Q] \subseteq [T]$. Since $[Q]$ is a non-empty Π^0_1 class, examining its index gives a contradiction. □

We obtain the following pleasing corollary, which says that every non-recursive HIF set must not be HIF relative to $\emptyset'$:

Corollary 3.11. *If A is HIF and HIF relative to $\emptyset'$ then A is recursive.*

We now turn to investigating the interactions of HIF and randomness. By the HIF basis theorem, there are random sets of HIF degree. For which sets A are there A-random HIF sets? In the case for $A \leq_T \emptyset'$ we get exactly the class of K-trivial sets, yielding yet another characterization of K-triviality.

Theorem 3.12. *Let $A \leq_T \emptyset'$. Then A is K-trivial iff some HIF set is A-random.*

Proof. Left to right follows trivially from the existence of HIF random reals. Suppose that A is not low for Ω, and some HIF set B is A-random. Then there exists a Π^0_1 class relative to A which contains B and only contains

A-random reals. This class contains no left-r.e. path since A is low for Ω. This contradicts Lemma 3.9. $\square$

Attempts to generalize this globally to obtain a characterization of low for Ω fails. Any HIF set A cannot be low for Ω, yet by the relativized HIF basis theorem, there exists an A-random which is HIF relative to A and hence HIF.

We now study the situation when we replace "random" with "complex". Recall that a set B is complex if there exists a recursive function f such that $C(B \upharpoonright m) > n$ whenever $m > f(n)$. B is A-complex if the same holds for an A-recursive f and C^A. A set B is autocomplex if there is a B-recursive function f such that $C(B \upharpoonright m) > n$ whenever $m > f(n)$. B is A-autocomplex if the same holds for a $A \oplus B$-recursive f and C^A.

Theorem 3.13. *Let $A \leq_T \emptyset'$.*

(i) *If A is K-trivial then some HIF set is A-complex.*
(ii) *If some HIF set is A-complex then A is low.*
(iii) *If A is a low r.e. set then some HIF set is A-autocomplex.*

Proof. (i): Trivial.

(ii): Let B be a HIF A-complex set. By [7] Theorem 2.3 relativized to A, $B \oplus A$ computes a function $\Phi^{B \oplus A}$ which is DNR relative to A, where the functional $\Phi^{X \oplus A}$ converges for every X. The set of all X such that $\Phi^{X \oplus A}(n) = \Phi_n^A(n)$ for some n is $\Sigma_1^0(A)$, so there exists an A-recursive tree T containing B, where for every path X of T, $\Phi^{X \oplus A}$ is an A-DNR function. By Lemma 3.9 T must contain some left-r.e. path. Hence $\emptyset'$ computes an A-DNR function, and by Rupprecht, Miller and Ng [11] implies that A is low.

(iii): Suppose A is a low r.e. set. If A is recursive then we are done, so assume that A is non-recursive. Take B to be any HIF PA-complete set. By Lemma 3.7 we have $B \oplus A \geq_T \emptyset'$. $B \oplus A$ is able to compute for each n, a length $f(n)$ such that no string of length $f(n)$ or more has A-Kolmogorov complexity below n, since it is $\Pi_1^0(A)$ to test each possible length. Hence B is A-autocomplex. $\square$

We remark that by [11], the class of sets $A \leq_T \emptyset'$ where $\emptyset'$ is A-autocomplex is exactly the low sets.

4. HIF and the jump operator

The aim of this section is to investigate the degrees which are the jump of a non-recursive HIF. We first note that every HIF set preserves highness:

Proposition 4.1. *Every HIF set preserves highness. That is, if A is HIF and B is a high set then $(A \oplus B)' \geq_T A''$.*

Let $\mathcal{J}_{\mathcal{H}} = \{C \in 2^\omega \mid C \geq_T \emptyset'$ and there exists a non-recursive HIF A with $A' \equiv_T C\}$. Let $\mathcal{J}_{\mathcal{R}}$ be defined similarly with recursively traceable in place of HIF.

By Folklore, every degree computing $\mathbf{0}''$ is the double jump of a HIF. However the situation for the single jump appears to be much more difficult. It is known (Jockusch and Stephan [5]) that no degree PA-complete relative to $\mathbf{0}'$ is in $\mathcal{J}_H$. By Theorem 3.8 the only sets in $\mathcal{J}_H$ which are HIF relative to $\mathbf{0}'$ are sets of degree $\mathbf{0}'$.

Theorem 4.2 (Kučera, Nies [8])**.** *If $C >_T \emptyset'$ is Σ^0_2 then C computes a set in $\mathcal{J}_H$.*

It is easy to modify their construction to make C compute a set in $\mathcal{J}_R$ (this will also follow from Theorem 4.4 below). However in contrast we show that no degree in $\mathcal{J}_H$ can compute a properly Σ^0_2 set.

Theorem 4.3. *Suppose A is HIF and $A' \geq_T C$ where C is a Σ^0_2 set. Then $C \leq_T \emptyset'$.*

Proof. Let f be an A recursive function and R a recursive predicate such that for every x, $\lim_s f(x,s) = 1$ iff $(\exists s)(\forall t > s)R(x,t)$. Define $g(x,s)$ to be the first $t > s$ found such that $\neg R(x,t)$ or $f(x,t) = 1$. Then $g(x,s)$ is a total function recursive in A. Let $\tilde{g}$ be a recursive function majorizing g. Let $\tilde{R}(x,s) = \prod_{t=s}^{\tilde{g}(x,s)} R(x,t)$. For each x, $\lim_s \tilde{R}(x,s)$ exists. To see this, suppose that $\tilde{R}(x,s) = 0$ for infinitely many s. Then $x \notin C$ and hence $\lim_s f(x,s) = 0$. Hence for almost every s. there is some $s < t \leq g(x,s)$ for which $\neg R(x,t)$ holds. Hence $\tilde{R}(x,s) = 0$ for almost every s. Finally it is easy to check that $\lim_s f(x,s) = 1$ iff $\lim_s \tilde{R}(x,s) = 1$, and hence $C \leq_T \emptyset'$. $\square$

Theorem 4.4. *Let C be 2-generic. Then there is a recursively traceable set A such that $A' \equiv_T A \oplus \emptyset' \equiv_T C \oplus \emptyset'$.*

Proof. We build a recursive sequence of total recursive functions $T_s : 2^{<\omega} \mapsto 2^{<\omega}$ such that for each s, T_s satisfies the usual definition of a tree and for every s and σ, there is some $\tau \supseteq \sigma$ such that $T_{s+1}(\sigma) = T_s(\tau)$. Provided that each σ is moved finitely often, we get that $T = \lim T_s$ exists and is a Π^0_1 class.

We start with T_0 the identity function. For each s and σ, we say that σ requires attention if there exists some $\tau \supset \sigma$ and $i, j < |\sigma|$ such that $\Phi_i^{T_s(\tau)}(j)\downarrow$ but $\Phi_i^{T_s(\sigma)}(j)\uparrow$. At s pick the lexicographically least σ requiring attention, and let $T_{s+1}(\sigma * \eta) = T_s(\tau * \eta)$ for every $\eta \in 2^{<\omega}$. If there is more than one pair (i,j) we move σ for the sake of the least pair in some fixed ordering of pairs of numbers. If no σ requires attention at s, set $T_{s+1} = T_s$.

Clearly each σ requires attention only finitely often. Hence $T \leq_T \emptyset'$. Let C be 2-generic, and $A = T(C)$. Clearly $A \oplus \emptyset' \equiv_T C \oplus \emptyset'$. It remains to verify that A is recursively traceable and $A' \leq_T C \oplus \emptyset'$. To see the former, fix e, and let $V = \{\sigma \mid |\sigma| > e \text{ and } \Phi_e(i)^{T(\sigma)}\uparrow \text{ for some } i \leq |\sigma|\} \leq_T \emptyset'$. Hence C must meet or strongly avoid V. If C meets V then by construction Φ_e^A is not total. Otherwise there exists $\eta \subset C$ such that $|\eta| > e$ and no extension of η is in C. This means that for every $\sigma \supseteq \eta$, $\Phi_e^{T(\sigma)}(|\sigma|)\downarrow$. Assume that η is never moved again. To compute a trace for $\Phi_e^A(i)$, $i > e$, we run the construction until a stage s is found such that $\Phi_e^{T_s(\sigma)}(i)\downarrow$ for every $\sigma \supseteq \eta$ of length i. There are at most 2^i many such values. Furthermore since T_s is an approximation to a Π^0_1 class, we have that $A \supset T_s(\sigma)$ for one such σ.

Finally to see that $A' \leq_T C \oplus \emptyset'$, note that $e \in A'$ if and only if $\Phi_e^{T(C\upharpoonright e+1)}(e)\downarrow$. $\square$

Question 4.5. *Do the degrees $\boldsymbol{a} \cup \mathbf{0}'$ where $\boldsymbol{a}$ is 2-generic characterize the class $\mathcal{J}_H$?*

Question 4.6. *Is the jump of each HIF degree also the jump of a recursively traceable degree?*

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