EFFECTIVE PACKING DIMENSION AND TRACEABILITY

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Abstract. We study the Turing degrees which contain a real of effective packing dimension one. Downey and Greenberg [DG] showed that a c.e. degree has effective packing dimension one iff it is not c.e. traceable. In this paper we show that this characterization fails in general. We construct a real $A \leq_T \emptyset''$ which is hyperimmune-free and not c.e. traceable, such that every real $\alpha \leq_T A$ has effective packing dimension 0. We also construct $B \leq_T \emptyset'$ with the same properties.

1. Introduction

The concern of this paper is with effective packing dimension. This concept can be traced back to the work of Borel and Lebesgue who studied measure as a way of specifying the size of sets. Carathéodory later generalized Lebesgue measure to the n-dimensional Euclidean space, and this was taken further by Hausdorff [Hau19] who generalized the notion of s-dimensional measure to include non-integer values for s in any metric space. In the Cantor space with the clopen topology, this can be viewed as a scaling of the usual Lebesgue measure by a factor of s, in the sense of

$$\mu_s([\sigma]) = 2^{-s|\sigma|},$$

where $[\sigma]$ is the clopen set generated by σ, and $0 \leq s \leq 1$. This gave rise to the concept of classical Hausdorff dimension, which provided a way of classifying different sets of measure zero, based on the intuition that not all null sets are created equal.

There appeared many other related classical notions of fractional dimensions, such as box-counting dimension and packing dimension. The study of effective notions of randomness and their relationship with the Turing degrees was initiated by the early work of de Leeuw, Moore, Shannon and Shapiro [dLMSS56]. The effective versions of these various notions of fractional dimensions have been studied in connection with randomness. The best known examples of such effective notions are the effective Hausdorff, and effective packing dimensions.

Hausdorff measure talks about covering the set by open balls from the exterior, while packing measure considers filling up a set from the interior. One can effectivize these two notions by looking at covering with Σ^0_1 open sets in the Cantor space with s-measure. This work took a new direction when various authors Lutz [Lut90, Lut03], Staiger [Sta93], Mayordomo [May02], Artheya et al [AHLM04], and Reimann [Rei04] showed that there were simple characterizations of effective Hausdorff and packing dimensions using Kolmogorov complexity. Indeed, the effective Hausdorff dimension of a real A can be written as

$$dim_H(A) = \liminf_{n\to\infty} \frac{K(A\upharpoonright_n)}{n},$$

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while its dual notion, the effective packing dimension is

$$dim_p(A) = \limsup_{n\to\infty} \frac{K(A\restriction_n)}{n}.$$

We also refer the reader to Lutz [Lut00] for a characterization in terms of martingales. We mention here that there is a natural way to define the effective dimension of any countable collection of reals, by looking at the lim sup of the effective dimensions of its members. In particular one can talk about the effective dimension of a Turing degree (or a lower cone with respect to Turing reducibility).

Effective packing dimension is a very natural notion of effective dimension to study; indeed the reals of effective packing dimension 1 can be described as one where "measure meets category". In particular this property is shared by both the Martin-Löf random reals, as well as reals which were sufficiently generic (for instance 2-generic). Consequently the class of reals having effective packing dimension one is both co-meager and of measure 1.

Unlike effective Hausdorff dimension, the notion of effective packing dimension is much more tractable. Fortnow, Hitchcock, Aduri, Vinochandran and Wang [FHA+06], proved that the dimension extraction property was true for effective packing dimension with respect to weak truth table reducibility:

Theorem 1.1 (Fortnow et al [FHA+06])**.** *For every $\varepsilon > 0$ and every A, if $dim_P(A) > 0$, then there is $B \equiv_{wtt} A$ such that $dim_P(B) > 1-\varepsilon$.*

Hence their result gives a 0-1 law on the effective packing dimension of wtt degrees - this can be only 0 or 1. In contrast, Miller [Mil] recently solved a long-standing question on "broken Hausdorff dimension", where he constructed a Δ^0_2 degree with effective Haudorff dimension $\frac{1}{2}$, but does not compute any real of a higher Hausdorff dimension.

It is still open if every degree of effective packing dimension one contains a real of effective packing dimension one, and this seems to be a difficult problem. Our task at hand is less ambitious; we are interested in answering a more general question: which Turing degrees are of effective packing dimension 1? Downey and Greenberg gave a classification in the case of c.e. degrees:

Theorem 1.2 (Downey and Greenberg [DG])**.** *A c.e. degree contains a real with positive effective packing dimension iff it is array non-computable.*

Recall that the array computable degrees were the degrees **a** such that there is some $f \leq_{wtt} \emptyset'$ which dominates every **a**-computable function. A degree is array non-computable if it is not array computable. Their result was related to a theorem of Kummer [Kum96], where he proved a gap phenomenon in the growth of C-complexity. In particular he showed that every c.e. array non-computable degree contains a set which has infinitely many segments of maximal C-complexity. On the other hand every c.e. array computable set has initial segments with C-complexity as close to $\log n$ as we want. Downey and Greenberg's classification reinforces the fact that array (non-)computability was intimately related to Kolmogorov complexity.

One would naturally conjecture that the above characterization of Downey and Greenberg holds in general. Unfortunately this tempting guess does not work out because there are array computable random degrees (any random hyperimmune-free degree is an example), so the array non-computable degrees fail to give a characterization. Recall that a set Z is of hyperimmune-free degree, if every function computable from Z is dominated by a computable function. In fact the array non-computable degrees also fail to give a characterization within the Δ^0_2 degrees because any superlow random real is also array computable.

Greenberg and Downey observed that it was easy to generalize Kummer's Gap Theorem to a notion called c.e. traceability, which is akin to array computability. Recall that a degree **a** is c.e. traceable if there is some computable, non-decreasing and unbounded function h such that for all $f \leq_T \mathbf{a}$ there is a unifomly c.e. sequence $\{T_x\}$ such that for all x, $|T_x| \leq h(x)$ and $f(x) \in T_x$. This has been studied by Zambella [Zam90], Terwijn and Zambella [TZ01] and also Ishmukhametov [Ish97] who showed that in c.e. degrees, array computability coincided with c.e. traceability. Greenberg and Downey observed that every c.e. traceable set has effective packing dimension 0.

One might now hope that the weaker notion of being not c.e. traceable would give a characterization. The degrees which were not c.e. traceable contain all random degrees, and so the obvious counter-examples for array non-computability are not relevant. In this paper we show that this feeble conjecture fails. In Theorem 2.1 we first construct a hyperimmune-free and Δ^0_3 example:

Theorem 2.1. *There is a Δ^0_3 real A which is of hyperimmune-free degree and not c.e. traceable, such that every real $\alpha \leq_T A$ has effective packing dimension 0.*

Since the degrees containing no real of packing dimension 1 may be thought of as having low algorithmic information content, one might hope to be able to relate this concept with some known lowness class arising in algorithmic randomness. Theorem 2.1 says that the low for Schnorr random reals fail to give a characterization even amongst the hyperimmune-free degrees, because the low for Schnorr random reals are all computably traceable and is therefore too strong a notion. It is not clear if there is any relationship between the degrees of positive effective packing dimension, and the degrees containing a low for Kurtz random (i.e. the hyperimmune-free and non-dnc degrees).

In Theorem 3.1 we show that the property of being not c.e. traceable fail to give a characterization amongst the sets computable from $\emptyset'$:

Theorem 3.1. *There is a real $A \leq_T \emptyset'$ which is not c.e. traceable, such that every real $\alpha \leq_T A$ has effective packing dimension 0.*

Finally we ask if there is a combinatorial characterization of the degrees in PD_1. By combinatorial we mean a definition which does not mention K-complexity, nor any randomness notion.

We associate finite strings with code numbers for them. If σ is a finite string of positive length then σ^- denotes the predecessor of σ. We use $|\sigma|$ to denote the length of a finite string, and $\#$ to denote the cardinality of a finite set. The rest of our notations are generally standard, and follow Soare [Soa87].

2. A Δ^0_3 and hyperimmune-free example

Theorem 2.1. *There is a Δ^0_3 real A which is of hyperimmune-free degree and not c.e. traceable, such that every real $\alpha \leq_T A$ has effective packing dimension 0.*

Proof. By Theorem 1.1, we only need to ensure that $dim_P(\alpha) \leq \frac{1}{2}$ for every $\alpha \leq_T A$. We build the set A of HIF degree by an oracle construction and we define a total function $g = \Gamma^A$ satisfying the requirements

$$\mathcal{P}_e \quad : \quad g(x) \notin V^e_x \text{ for some } x.$$
$$\mathcal{N}_e \quad : \quad \text{if } \Phi^A_e \text{ is total, then } K(\Phi^A_e\restriction_x) \leq x/2 \text{ for almost all } x.$$

We let $\{V^e_x\}_x$ be the e^{th} c.e. trace such that $\#V^e_x < x$ for every e, x. We observe that there are plenty of reals which are of hyperimmune-free degree, but not computably traceable. For instance, any HIF random real will do, but random reals all have effective packing dimension 1. On the other hand, the standard construction of a

Δ^0_3 real of HIF degree also makes it computably traceable, so one has to go out of the way to construct such a real directly (see Terwijn's thesis [Ter98]). The basic idea there is to work in a tree T where every node at level x has x branches or successors (one could work in the Cantor space since T is a homeomorphic copy, but it will be notationally more cumbersome). We have to define the total functional $g = \Gamma^X$ externally, by letting $\Gamma^\sigma(|\sigma|) = \sigma$, which will be clearly total along every path in $[T]$. We want to defeat all possible traces for g, and since T has enough splits at each level we could kill off enough branches at a certain level in order to diagonalize against the e^{th} trace. This is reminiscent of a "bushy tree" type construction used to construct minimal dnc degrees. At the same time we will be able obtain an upperbound for the possible values of $\Phi^X_e(n)$ (to force HIF), just by reading it off the tree.

Suppose we now want to combine the above construction with the requirement $\mathcal{N}_e$. Note that $\mathcal{N}_e$ requires us not only to have to keep the initial segment complexity of A small (which is easy), but rather we need to keep the complexity of Φ^A_e small. This creates an additional difficulty, because in general there is no effective relationship between the length of a segment σ, and the length of the use which produces that segment (i.e. τ such that $\Phi^\tau_e = \sigma$). In particular, we could have very long segments τ such that $\Phi^{\tau^\frown i}_e$ are all different for different values of i, such that $|\Phi^{\tau^\frown i}_e|$ is relatively short. Remember that we have to keep enough successors of τ left on the tree for diagonalization, so we might have to end up issuing many descriptions witnessing $K(\Phi^{\tau^\frown i}_e) \leq \frac{1}{2}|\Phi^{\tau^\frown i}_e|$ for many different i. The number of different i could be too large relative to $|\Phi^{\tau^\frown i}_e|$. The obvious thing to try might be for instance, to choose a longer τ so that $|\Phi^{\tau^\frown i}_e|$ is longer, and hence cost less to describe, but remember that generally at level $|\tau|$ we have to keep at least $|\tau|$ many successors of τ left on the tree (since g is, and in fact has to be, defined externally). So generally looking for a longer τ doesn't help, since this also corresponds to having even more possibilities of $|\Phi^{\tau^\frown i}_e|$ for which we have to issue descriptions.

Our solution to this is to gather a *majority vote*, or consensus. We start with a tree T which has, at level x, a large number of successors say $x2^{L(x)}$ many (for some $L(x)$ to be determined). We will define a computable subtree T_e of T by the following. We may assume that for every x and above every string σ on T we can always force convergence, i.e. find some $\eta \supset \sigma$ on T such that $\Phi^\eta_e(x)\downarrow$. (otherwise we can just take T_e to be the full subtree above some node). First pick a level x_0 which will be the first level in T_e for which we will put up splits. We then pick a length $L(x_0)$ which is very large relative to x_0, and search for strings $\sigma_1 \supset 1^{x_0}1, \sigma_2 \supset 1^{x_0}2, \cdots$ such that $\Phi^{\sigma_i}_e\upharpoonright_{L(x_0)}\downarrow$ for all i. Since there are $x_0 2^{L(x_0)}$ many different σ_i, and only at most $2^{L(x_0)}$ many possibilities for $\Phi^{\sigma_i}_e\upharpoonright_{L(x_0)}$, it follows there is some τ such that $\Phi^{\sigma_i}_e\upharpoonright_{L(x_0)}= \tau$ for at least x_0 many different σ_i. Leave 1^{x_0} on T_e, as well as the extensions σ_i which voted for the majority, and kill all other incomparable nodes. We then move on to the next level x_1. This ensures that the tree T_e still has enough splits at infinitely many levels (so that we can proceed with diagonalization for other $\mathcal{P}$ requirements), but yet we are able to restrict the possibilities for Φ_e.

Suppose $\sigma_{i_1}, \cdots, \sigma_{i_{x_0}}$ were the extensions of 1^{x_0} which survived on T_e. After we pick x_1 we will repeat the "majority vote" strategy separately above each σ_{i_j} to ensure we have enough splits left on T_e at level x_1, and kill off all other incomparable nodes. We now have $\eta_1, \eta_2, \cdots, \eta_{x_0} \supset \Phi^{\sigma_{i_0}}_e\upharpoonright_{L(x_0)}$ where η_j was voted by the majority of nodes extending σ_{i_j}. To satisfy $\mathcal{N}_e$ we need to issue descriptions for all possible segments for Φ_e. There is one segment of length $L(x_0)$, x_0 many segments of length $L(x_1)$, x_0x_1 many segments of length $L(x_2)$ and so on, for us to describe. As long

L is chosen such that $x_0 \cdots x_k 2^{-L(x_k)/2}$ is small, then we will be fine. The exact details are supplied in the formal construction.

Formal construction. For each $x \in \mathbb{N}$ and rational $0 < \delta < 1$, we define $\ell(x, \delta)$ to be the least number larger than $4 - 2\log\frac{\delta}{x^x}$, so that $x^x 2^{-\frac{1}{2}\ell(x,\delta)+2} < \delta$ holds. This seemingly bizzare choice for ℓ will become clear later; it is simply a huge number that bounds everything we need. Define the computable sequence of functions $L_1, L_2, \cdots$ inductively by

$$L_1(x) = \ell(x, 2^{-x}), \quad L_{n+1}(x) = \ell(x2^{L_1(x)+\cdots+L_n(x)}, 2^{-x}),$$

for all positive $x \in \mathbb{N}$. It is a simple exercise to show that $L_n(x)$ is increasing in both variables. In this proof, a tree is defined to be a partial computable function $T : \omega^{<\omega} \mapsto \omega^{<\omega}$, such that $\sigma \subset \tau \ \wedge \ T(\tau)\downarrow \Rightarrow T(\sigma)\downarrow \subset T(\tau)$, and incomparable strings map to incomparable strings. A tree T is said to be *crowded* if it satisfies the following

(1) $T(\emptyset)\downarrow$.
(2) if $T(\sigma)\downarrow$, then $T(\sigma^\frown i)\downarrow$ for all $i = 1, \cdots, x2^{L_1(x)+\cdots+L_{|\sigma|+1}(x)}$, where $x = |T(\sigma)|$.
(3) if $i \neq j$ then $T(\sigma^\frown i)\upharpoonright_{1+|T(\sigma)|} \neq T(\sigma^\frown j)\upharpoonright_{1+|T(\sigma)|}$.
(4) if $T(\sigma)\downarrow$ and $T(\tau)\downarrow$ and $|\sigma| = |\tau|$, then $|T(\sigma)| = |T(\tau)|$.
(5) T is defined nowhere else, and is being built up from $\emptyset$ by applying rules (1) to (4).

Condition (5) compacts the tree in the sense that we eliminate the situtation where we have $T(0)\downarrow$ and $T(2)\downarrow$ but $T(1)\uparrow$. A crowded tree generalizes simultaneously, to the non-binary case, both the ideas of having a "perfect" tree, as well as having enough branches at infinitely many levels.

Generate the tree T by letting $T(\emptyset) = 1$, and inductively if $T(\sigma)\downarrow$ then let $T(\sigma^\frown i) = T(\sigma)^\frown i$ for $i = 1, \cdots, x2^{L_1(x)+\cdots+L_x(x)}$ where $x = |\sigma| + 1$. T is in some sense the identity tree, and is clearly crowded. We say that T is the *full crowded tree.* Since $Ran(T)$ is computably homeomorphic to the Cantor space, we will construct A as an infinite path through $Ran(T)$. If T is a tree, we let $[T]$ be the set of all infinite strings X such that there are infinitely many $\tau \in Ran(T)$ such that $\tau \subset X$. Equivalently, $[T] = \{X : \exists Y \forall n T(Y\upharpoonright_n) \subset X\}$. If P and Q are trees, then we say that $P \subseteq Q$ if for every σ such that $P(\sigma)\downarrow$, we have some η such that $P(\sigma) = Q(\eta)$.

The functional Γ is defined as follow: $\Gamma^\sigma(x)\downarrow = \sigma$ if σ is on T and $|\sigma| = x$. Clearly Γ^X is total for any path $X \in [T]$. During the construction, at each stage $s + 1$ we will define a crowded subtree $T_{s+1} \subset T_s$, and let $A = \cup_s T_s(\emptyset)$. For each s and k note that $T_s(1^k)$ is always convergent, and if T_s is crowded, then $|T_s(\sigma)| = |T_s(1^k)|$ for every convergent $|\sigma| = k$. If P is a tree we say that σ *is on* P, or equivalently $\sigma \in P$ to mean that $\sigma \subseteq P(\eta)$ for some η. If P is crowded, then $Ran(P)$ is computable as a set of nodes, so that the relation $\sigma \in P$ is a computable relation (given an index for P).

If P is a crowded tree and σ is on P, then we define $\tilde{P}$ as the *crowded subtree of P above σ* by the following. Look for the minimal η such that $P(\eta) \supseteq \sigma$. Let $\tilde{P}(\tau) = P(\eta^\frown\tau)$ for all τ, and then we chop off the superflous branches, i.e. restrict the domain of $\tilde{P}$ sufficiently so as to satisfy condition (2). It is clear that $\tilde{P}$ is crowded as well. This is where we use the idea that crowded trees are "perfect" in some sense; at any point in the construction we can just extract P above any σ and still end up with a crowded tree, and this makes no sense if, for instance, P contains dead ends.

The construction. At stage $s = 0$ we let $T_0 = T$. At stage $s = 3e > 0$, we satisfy $\mathcal{N}_e$. Ask if it is the case that $(\forall x \forall \sigma \in T_{s-1})(\exists \tau \supset \sigma)(\tau \in T_{s-1} \ \wedge \ \Phi_e^\tau(x)\downarrow)$. If the answer is no, find a counter example σ on T_{s-1}, and let T_s be the crowded subtree of T_{s-1} above σ. If the answer is yes, we will define both T_s and a computable tree P_s by the following. The idea is that $[P_s]$ is a Π^0_1 class (with very few splits) and containing all possibilities for Φ^A_e.

First let $\eta \neq \emptyset$ be the first string found such that $\Phi_e^{T_{s-1}(\eta)}(0)\downarrow$ and set $T_s(\emptyset) = T_{s-1}(\eta)$. Next, assume that $T_s(\sigma)$ has been defined up till level k, i.e. for all relevant $|\sigma| \leq k$. Let $N := \{\sigma : |\sigma| = k \ \wedge \ T_s(\sigma)\downarrow\}$. Assume that inductively we have the properties

- T_s is crowded so far
- for every $\sigma \in N$, $T_s(\sigma)$ has at least $x2^{L_1(x)+\cdots+L_{k+2}(x)}$ many successors on T_{s-1}, where $x = |T_s(1^k)|$
- for every $\sigma \in N$, $P_s(\sigma^-)\downarrow$ and $\Phi_e^{T_s(\sigma)} \supseteq P_s(\sigma^-)$.

For each $\sigma \in N$, do the following. Since σ has at least $x2^{L_1(x)+\cdots+L_{k+2}(x)}$ many successors on T_{s-1}, we label these successors by $T_s(\sigma)^\frown n_1, T_s(\sigma)^\frown n_2, \cdots$. For each $i \leq x2^{L_1(x)+\cdots+L_{k+2}(x)}$ find the first string $\sigma_i \supset T_s(\sigma)^\frown n_i$ on T_{s-1}, such that $\Phi_e^{\sigma_i}\restriction_{L_{k+2}(x)}\downarrow\supset P_s(\sigma^-)$. There must be some τ of length $L_{k+2}(x)$ such that $\tau \supset P_s(\sigma^-)$ and we have at least $x2^{L_1(x)+\cdots+L_{k+1}(x)}$ many values of i such that $\Phi_e^{\sigma_i}\restriction_{L_{k+2}(x)}= \tau$. Take the first $x2^{L_1(x)+\cdots+L_{k+1}(x)}$ many such i, and arrange them in increasing order $(i_1 < i_2 < \cdots)$ and for each $j \leq x2^{L_1(x)+\cdots+L_{k+1}(x)}$ we set $T_s(\sigma^\frown j) = T_{s-1}(\nu)$ where ν is some string such that $T_{s-1}(\nu) \supset \sigma_{i_j}$. Also set $P_s(\sigma) = \tau$. Repeat for each $\sigma \in N$, and we may assume (by choosing a longer ν) that $|T(\sigma^\frown j)|$ is constant for all $\sigma \in N$ and all j, and that $T_s(\sigma^\frown j)$ has enough successors on T_{s-1}.

Now assume we are at stage $s = 3e+1$, and T_{s-1} is crowded. We want to satisfy $\mathcal{P}_e$. Pick the least $i \leq 1 + |T_{s-1}(\emptyset)|$ such that $T_{s-1}(i)\restriction_{1+|T_{s-1}(\emptyset)|} \notin V^e_{1+|T_{s-1}(\emptyset)|}$, and let T_s be the crowded subtree of T_{s-1} above $T_{s-1}(i)\restriction_{1+|T_{s-1}(\emptyset)|}$.

Finally assume we are at stage $s = 3e+2$. We run the usual hyperimmune-free strategy. Ask if $(\forall x \forall \sigma \in T_{s-1})(\exists \tau \supset \sigma)(\tau \in T_{s-1} \ \wedge \ \Phi^\tau_e(x)\downarrow)$. If the answer is no, find a counter example σ on T_{s-1}, and let T_s be the crowded subtree of T_{s-1} above σ. If the answer is yes, we define T_s by the following. Set $T_s(\emptyset) = T_{s-1}(\emptyset)$. Suppose $T_s(\sigma)$ has been defined for all σ of length $\leq k$, and that $T_s(\sigma)$ has at least $x2^{L_1(x)+\cdots+L_{|\sigma|+1}(x)}$ many successors where $x = |T_s(1^{|\sigma|})|$ on T_{s-1}. Label them $T_s(\sigma)^\frown n_0, T_s(\sigma)^\frown n_1, \cdots$. For each $|\sigma| = k$ and $i \leq x2^{L_1(x)+\cdots+L_{k+1}(x)}$, we set $T_s(\sigma^\frown i)$ to be $T_{s-1}(\eta)$ for some η such that $T_{s-1}(\eta) \supset T_s(\sigma)^\frown n_i$ and $\Phi_e^{T_{s-1}(\eta)}(k)\downarrow$. Once again we may assume that $|T_s(\sigma^\frown i)|$ are all equal for all combinations of σ and i, by choosing a longer η if necessary.

Verification. The construction produces a sequence of trees $T_0 \supseteq T_1 \supseteq \cdots$, where the sequence of indices for the trees is computable in $\emptyset''$. Hence A is Δ^0_3 and $A \in [T_s]$ for every s. We verify that for every s, T_s is crowded and every infinite path through T_s (in particular A) has the desired properties. Assume that T_{s-1} is crowded, and let $s = 3e > 0$. If the answer to the Π_2-question is no, then T_s is clearly crowded and there is some x such that for every $X \in [T_s]$, we have $\Phi^X_e(x)\uparrow$. Suppose on the other hand that the answer given was yes. Observe that the inductive definition of T_s done level by level is well-defined, and maintains the property of being crowded at each finite level. Observe also that P_s is a tree with the following properties:

(1) $dom(P_s) = dom(T_s)$.
(2) for every σ, $|P_s(\sigma)| = L_{|\sigma|+2}(|T_s(\sigma)|)$.
(3) for every infinite path $X \in [T_s]$, we have Φ^X_e total and is an infinite path through $[P_s]$.

We claim that every infinite path through $[P_s]$ has effective packing dimension at most $\frac{1}{2}$, because P_s is extremely sparse. This is similar to the proof that every Π^0_1 class of measure 0 contains no random path. To see this, we enumerate a Kraft-Chaitin set, where we enumerate axioms $\langle \eta, \frac{1}{2}|\eta|\rangle$ for every string $\eta \supset P_s(\emptyset)$ on P_s. We let $h_k := |T_s(1^k)|$, and $S_k := \{\sigma \in dom(T_s) \mid |\sigma| = k\}$. We can show easily that $\#S_0 = 1$ and $\#S_k \leq (h_{k-1}2^{L_1(h_{k-1})+\cdots+L_k(h_{k-1})})^k$ for any $k > 0$. It follows then that the total cost of all these requests is bounded above (very generously) by

$$\sum_{\eta\in dom(P_s)} \sum_{x\geq|P_s(\eta)|} 2^{-\frac{1}{2}x} \cdot \# \text{ of successors of } P_s(\eta)$$

$$\leq \sum_{\eta\in dom(P_s)} 2^{-\frac{1}{2}|P_s(\eta)|+2} \cdot \# \text{ of successors of } P_s(\eta).$$

Since T_s is crowded, so every level looks the same, hence we can express the above sum as

$$\begin{aligned} & \sum_k \#S_k \cdot 2^{-\frac{1}{2}L_{k+2}(h_k)+2} \cdot h_k 2^{L_1(h_k)+\cdots+L_{k+1}(h_k)} \\ \leq\ & \sum_k 2^{-\frac{1}{2}L_{k+2}(h_k)+2} \cdot (h_k 2^{L_1(h_k)+\cdots+L_{k+1}(h_k)})^{k+1} \\ \leq\ & \sum_k 2^{-h_k} < 1 \text{ (by the choice of } \ell). \end{aligned}$$

This shows that every infinite path through $[P_s]$ has effective packing dimension at most $\frac{1}{2}$.

Next, we let $s = 3e+1$. In the construction we will be able to find the required i, since $\#V^e_{1+|T_{s-1}(\emptyset)|} \leq |T_{s-1}(\emptyset)|$. We have $\Gamma^X(1 + |T_{s-1}(\emptyset)|) = T_{s-1}(i)\restriction_{1+|T_{s-1}(\emptyset)|} \notin V^e_{|T_{s-1}(\emptyset)|+1}$ for every infinite path X through $[T_s]$.

Finally, consider $s = 3e+2$. If the answer to the Π_2-question is no, then again T_s is clearly crowded and for every $X \in [T_s]$, Φ^X_e is not total. If the answer given was yes, then it is clear that T_s is also crowded, and that furthermore for every infinite path X through $[T_s]$, we must have that for every x, $\Phi^X_e(x)$ must converge with use at most $|T_s(1^{x+1})|$. One can then proceed to generate a computable bound for all possible Φ^X_e.

Since $A \in [T_s]$ for every s, it follows that A has all the properties we require. □

3. A Δ^0_2 example

In this section we provide an effective version of the proof of Theorem 2.1. By doing so we prove that:

Theorem 3.1. *There is a Δ^0_2 set A which is not c.e. traceable, such that every real $\alpha \leq_T A$ has effective packing dimension 0.*

Proof. We work in a variation of the Cantor space which is finitely branching. We build a path g in this space by finite extension where $g = \cup_e \sigma_e$. Again we only need to ensure that $dim_P(\alpha) \leq \frac{1}{2}$ for every $\alpha \leq_T g$. We ensure that the following requirements are met:

$$\begin{aligned} \mathcal{P}_e &: \quad g(x) \notin V^e_x \text{ for some } x. \\ \mathcal{N}_e &: \quad \text{if } \alpha_e = \Phi^g_e \text{ is total, then } K(\alpha_e\restriction_x) \leq x/2 \text{ for almost all } x. \end{aligned}$$

Then the set A can be taken to be say, the graph of g. Again we let $\{V^e_x\}_x$ to be the e^{th} c.e. trace, with identity bound. We maintain a sequence of computable trees $T_0 \supset T_1 \supset \cdots$ and build g as a path through $\cap_e T_e$. At every stage s we use $\emptyset'$ as an oracle to search through the tree T_e, and when we discover that the tree is not total we change our mind on T_e. This will resemble a finite injury with oracle $\emptyset'$, and is similar to the way in which Sacks' construction of a minimal degree below $\emptyset'$ is a $\emptyset'$-effective version of Spector's construction.

We retain most of the notations and parameters of the previous Δ^0_3 construction. Like in Sacks' proof, we have to allow our crowded trees to be "partial", in the sense that they may now contain dead ends. To wit, we now declare that a tree is *crowded*, if

(1) If $T(\sigma)\downarrow$, then either $T(\sigma^\frown i)\downarrow$ for all $i=1,\cdots,x2^{L_1(x)+\cdots+L_{|\sigma|+1}(x)}$, where $x=|T(\sigma)|$ or else $T(\sigma^\frown i)\uparrow$ for every i.
(2) If $i\neq j$ then $T(\sigma^\frown i)\restriction_{1+|T(\sigma)|}\neq T(\sigma^\frown j)\restriction_{1+|T(\sigma)|}$ whenever they converge.
(3) T is defined nowhere else, and is built up using the above rules.

The difference now is that we allow $T(\sigma)\downarrow$ but has no successors. We also have to allow $T(\sigma)$ and $T(\eta)$ to be of different lengths when $|\sigma|=|\eta|$, because we might not be able to find convergent strings densely; as we will see this will have no serious impact on the calculations. If $T(\sigma)\downarrow$ but $T(\sigma^\frown i)\uparrow$ for every i then we say that $T(\sigma)$ *has no successors.*

If T is a crowded tree and σ is on T, we let $Full(T,\sigma)$ be the crowded subtree of T above σ as before. An index for $Full(T,\sigma)$ can be found effectively in σ and an index for T. The second operation is the *majority e-subtree above σ*, denoted as $Maj(e,T,\sigma)$. This is the tree Q defined by the following. We also define a partial computable tree P together with Q:

First let $\gamma\neq\emptyset$ be the first string found such that $T(\gamma)\downarrow\supset\sigma$ and $\Phi_e^{T(\gamma)}(0)\downarrow$. If no such string is found then $Q(\emptyset)\uparrow$, otherwise set $Q(\emptyset)=T(\gamma)$. Next, assume that $Q(\eta)$ has been defined, and that inductively we have the properties

- Q is crowded so far,
- $Q(\eta)=T(\gamma)$ for some $|\gamma|>|\eta|$,
- $P(\eta^-)\downarrow$ and $P(\eta^-)=\Phi_e^{Q(\eta)}\restriction_{L_{|\eta^-|+2}(|Q(\eta^-)|)}$.

We now compute $Q(\eta^\frown i)$ for an appropriate number of i's. Let $x=|Q(\eta)|$. First wait for $T(\gamma^\frown i)\downarrow$ for every $i\leq x2^{L_1(x)+\cdots+L_{|\gamma|+1}(x)}$. For each i find the first string $\gamma_i\supseteq\gamma^\frown i$ such that $T(\gamma_i)\downarrow$ and $\Phi_e^{T(\gamma_i)}\restriction_{L_{|\eta|+2}(x)}\downarrow$. Necessarily we must have $\Phi_e^{T(\gamma_i)}\restriction_{L_{|\eta|+2}(x)}\supset P(\eta^-)$. There must be some τ of length $L_{|\eta|+2}(x)$ such that $\tau\supset P(\eta^-)$ and we have at least $x2^{L_1(x)+\cdots+L_{|\eta|+1}(x)}$ many values of i such that $\Phi_e^{T(\gamma_i)}\restriction_{L_{|\eta|+2}(x)}=\tau$. Take the first $x2^{L_1(x)+\cdots+L_{|\eta|+1}(x)}$ many such i's and define $Q(\eta^\frown i)=T(\gamma_i)$. Observe that the three properties still hold for $Q(\eta^\frown i)$. This ends the definition of Q. Note that we may assume that $|Q(\eta)|\neq|Q(\eta')|$ whenever $|\eta|=|\eta'|$ (by searching further along T).

An index for Q is obtained effectively from e,T and σ. In fact the following holds:

Lemma 3.2. *If T is crowded and σ is on T, then*

(i) *Q is crowded,*
(ii) *every $\alpha=\Phi_e^X$ for $X\in[Q]$ has $dim_P(\alpha)\leq\frac{1}{2}$,*
(iii) *if $Q(\emptyset)\uparrow$ then there is no $X\supset\sigma$ such that $X\in[T]$ and Φ_e^X is total,*
(iv) *if $Q(\eta)\downarrow=T(\gamma)$ but has no successors on Q, then either*
   – *it has no successors on T, or else*
   – *there is some k such that $T(\gamma^\frown k)\downarrow$, and for every $X\supset T(\gamma^\frown k)$, $X\in[T]$, we have Φ_e^X is not defined somewhere below $L_{|\eta|+2}(|Q(\eta)|)$.*

Proof. The others are straightforward, so we only prove (ii). Observe that P is closed under initial segments, and in this case satisfies similar properties as before:

(1) for every η, $P(\eta)\downarrow\Leftrightarrow Q(\eta)\downarrow$ and has successors,

(2) for every η, $|P(\eta)| = L_{|\eta|+2}(|Q(\eta)|)$,
(3) for every infinite path $X \in [Q]$, we have Φ_e^X total and is an infinite path through $[P]$.

Since P is a partial computable tree, hence the set of strings τ on P is a c.e. set. We then enumerate a KC-set $\{\langle \tau, \frac{1}{2}|\tau|\rangle : \tau \supset P(\emptyset) \text{ is on } P\}$. We need to show that the total size of these requests is bounded. Just as before, the size of these requests is bounded above by

$$\sum_{\eta \in dom(P)} 2^{-\frac{1}{2}|P(\eta)|+2} \cdot \# \text{ of successors of } P(\eta) \leq \sum_{\eta \in dom(P)} 2^{-|Q(\eta)|},$$

where $o(\eta) = x2^{L_1(x)+\cdots+L_{|\eta|+1}(x)}$ and $x = |Q(\eta)|$. That is $o(\eta)$ is the number of successors of $Q(\eta)$. Since we assumed that $|Q(\eta)| \neq |Q(\eta')|$ whenever η, η' are of the same length, we can reduce the sum to

$$\sum_k \sum \{2^{-|Q(\eta)|} : |\eta| = k, \eta \in dom(Q)\} \leq \sum_k 2^{-k+1} < \infty.$$

This shows that every infinite path through $[P]$ has effective packing dimension at most $\frac{1}{2}$, and shows that (ii) holds. $\square$

Construction of g. We build g by finite extension. At each stage s, $T_e[s]$ denotes the tree which we use to satisfy requirement $\mathcal{N}_e$. Let T be the full crowded tree in Theorem 2.1. By convention $T_{-1} = T$. At stage $s = 0$ we initialize T_e for every e, and let $\sigma_0 = \langle\rangle$. At $s > 0$ we assume that inductively we have the following:

(1) $T \supset T_0[s] \supset \cdots$, and are all crowded,
(2) $\eta_0 \geq \eta_1 \geq \cdots$ such that $\sigma_{s-1} = T_0(\eta_0)[s] = T_1(\eta_1)[s] = \cdots$.

We find the least $e \geq 0$ such that T_e is defined and σ_{s-1} has no successors on T_e. If e exists, then T_e must have been obtained from T_{e-1} by taking the e-majority subtree operation. We claim that $\emptyset'$ can compute some $\rho \supset \eta_{e-1}$ such that $T_{e-1}(\rho)\downarrow$, and for every $X \supseteq T_{e-1}(\rho)$ and $X \in [T_{e-1}]$, Φ_e^X is not total.

First go through each k and ask if there is some string $\rho_k \supseteq \eta_{e-1}{}^\frown k$ such that $T_{e-1}(\rho_k)\downarrow$, and $\Phi_e^{T_{e-1}(\rho_k)}\upharpoonright_{L_{|\eta_e|+2}(|\sigma_{s-1}|)}\downarrow$. If the answer no for some k, then take $\rho = \eta_{e-1}{}^\frown k$. If ρ_k is found for every k, then by Lemma 3.2(iv), there will be some k such that $[T_{e-1}(\rho_k)] \cap [T_{e-1}] = \emptyset$. By compactness we can search for it using $\emptyset'$. Let $\rho = \rho_k$. In any case once ρ is found we let $\sigma_s = T_{e-1}(\rho)$. We keep $T_0, \cdots, T_{e-1}$ and set T_e to be the $Full(T_{e-1}, \sigma_s)$. Initialize all $T_{e+1}, T_{e+2}, \cdots$. Adjust $\eta_0, \cdots, \eta_e$ accordingly.

Suppose on the other hand e does not exist. Let e_0 be the largest such that $T_{e_0}\downarrow$. Hence $T_{e_0}(\eta_{e_0})$ has at least $1 + |\sigma_{s-1}|$ many successors on T_{e_0}; since $T_{e_0}(\eta_{e_0}{}^\frown i)$ are all different at the $|\sigma_{s-1}|^{th}$ bit, we pick some i so that $T_{e_0}(\eta_{e_0}{}^\frown i)(|\sigma_{s-1}|) \notin V^{e_0}_{|\sigma_{s-1}|}$. Let $\rho = T_{e_0}(\eta_{e_0}{}^\frown i)$. Ask if $Maj(e_0+1, T_{e_0}, \rho)(\emptyset)\downarrow$. If the answer is yes, let T_{e_0+1} be $Maj(e_0+1, T_{e_0}, \rho)$ and $\sigma_s = T_{e_0+1}(\emptyset)$. Otherwise the answer is no; we let $T_{e_0+1} = Full(T_{e_0}, \rho)$ and let $\sigma_s = \rho$. Define $\eta_0, \cdots, \eta_{e_0+1}$ appropriately.

Verification. Clearly for every s, $\sigma_{s+1} \supsetneq \sigma_s$. Let $g = \cup\sigma_s$ and hence $g \leq_T \emptyset'$. It is also easy to see that for each e, T_e is initialized finitely often and receives a final definition; let $\tilde{T}_e$ denote this. It is clear that $g \in [\tilde{T}_e]$ for every e. Now fix an e. We claim that $\mathcal{P}_e$ is satisfied. There is a least stage s where T_{e+1} receives a definition; at that stage we ensured that $g(|\sigma_{s-1}|) \notin V^e_{|\sigma_{s-1}|}$. Now we verify that $\mathcal{N}_e$ is satisfied. Suppose that Φ_e^g is total. Let s be the stage where T_e is defined as $\tilde{T}_e$. Suppose at s we found that e is the least such that T_e is defined and σ_{s-1} has no successors on T_e. However σ_s is defined such that any infinite extension $X \supset \sigma_s$, where $X \in [\tilde{T}_{e-1}]$ has the property that Φ_e^X is not total. Since Φ_e^g is total, hence at s the second scenario in the construction applies, where $e_0 + 1 = e$ and ρ is on $\tilde{T}_{e-1}$.

By Lemma 3.2(iii) we must have $Maj(e, \tilde{T}_{e-1}, \rho)(\emptyset)\downarrow$. Hence $\tilde{T}_e = Maj(e, \tilde{T}_{e-1}, \rho)$. By Lemma 3.2(ii), we have $dim_P(\Phi_e^g) \leq \frac{1}{2}$. $\square$

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