Finitary reducibility on equivalence relations

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Abstract

We introduce the notion of finitary computable reducibility on equivalence relations on the domain ω. This is a weakening of the usual notion of computable reducibility, and we show it to be distinct in several ways. In particular, whereas no equivalence relation can be Π^0_{n+2}-complete under computable reducibility, we show that, for every n, there does exist a natural equivalence relation which is Π^0_{n+2}-complete under finitary reducibility. We also show that our hierarchy of finitary reducibilities does not collapse, and illustrate how it sharpens certain known results. Along the way, we present several new results which use computable reducibility to establish the complexity of various naturally defined equivalence relations in the arithmetical hierarchy.

1 Introduction

Computable reducibility provides a natural way of measuring and comparing the complexity of equivalence relations on the natural numbers. Like most notions of reducibility on sets of natural numbers, it relies on the concept of Turing computability to rank objects according to their complexity, even when those objects themselves may be far from computable. It has found particular usefulness in computable model theory, as a measurement of the classical property of being isomorphic: if one can computably reduce the isomorphism problem for computable models of a theory T_0 to the isomorphism problem for computable models of another theory T_1, then it is reasonable to say that isomorphism on models of T_0 is no more difficult than on models of T_1. The related notion of Borel reducibility was famously applied this way by Friedman and Stanley in [10], to study the isomorphism problem on all countable models of a theory. Yet computable reducibility has also become the subject of study in pure computability theory, as a way of ranking various well-known equivalence relations arising there.

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Recently, as part of our study of this topic, we came to consider certain reducibilities weaker than computable reducibility. This article introduces these new, finitary notions of reducibility on equivalence relations and explains some of their uses. We believe that researchers familiar with computable reducibility will find finitary reducibility to be a natural and appropriate measure of complexity, not to supplant computable reducibility but to enhance it and provide a finer analysis of situations in which computable reducibility fails to hold.

Computable reducibility is readily defined. It has gone by many different names in the literature, having been called m-reducibility in [1, 2, 11] and FF-reducibility in [7, 8, 9], in addition to a version on first-order theories which was called Turing-computable reducibility (see [3, 4]).

Definition 1.1 *Let E and F be equivalence relations on ω. A* reduction *from E to F is a function $g : \omega \to \omega$ such that*

$$\forall x, y \in \omega \ \ [x \ E \ y \iff g(x) \ F \ g(y)]. \tag{1}$$

We say that E is computably reducible *to F, written $E \leq_c F$, if there exists a reduction from E to F which is Turing-computable. More generally, for any Turing degree $\boldsymbol{d}$, E is $\boldsymbol{d}$*-computably reducible *to F if there exists a reduction from E to F which is $\boldsymbol{d}$-computable.*

There is a close analogy between this definition and that of *Borel reducibility*: in the latter, one considers equivalence relations E and F on the set 2^ω of real numbers, and requires that the reduction g be a Borel function on 2^ω. In another variant, one requires g to be a continuous function on reals (i.e., given by a Turing functional Φ^Z with an arbitrary real oracle Z), thus defining *continuous reducibility* on equivalence relations on 2^ω.

So a reduction from E to F maps every element in the field of the relation E to some element in the field of F, respecting these equivalence relations. Our new notions begin with *binary computable reducibility.* In some situations, while it is not possible to give a computable reduction from E to F, there does exist a computable function which takes each pair $\langle x_0, x_1\rangle$ of elements from the field of E and outputs a pair of elements $\langle y_0, y_1\rangle$ from that of F such that $y_0 F y_1$ if and only if $x_0 E x_1$. (The reader may notice that this is simply an m-reduction from the set E to the set F.) Likewise, an *n-ary computable reduction* accepts n-tuples $\vec{x}$ from the field of E and outputs n-tuples $\vec{y}$ from F with $(x_i E x_j \iff y_i F y_j)$ for all $i < j < n$, and a *finitary computable reduction* does the same for all finite tuples. Intuitively, a computable reduction (as in Definition 1.1) does the same for all elements from the field of E simultaneously.

A computable reduction clearly gives us a computable finitary reduction, and hence a computable n-reduction for every n. Oftentimes, when one builds a computable reduction, one attempts the opposite procedure: the first step is to build a binary reduction, and if this is successful, one then treats the binary reduction as a basic module and attempts to combine countably many basic modules into a single effective construction. Our initial encounter with finitary reducibility arose when we found a basic module of this sort, but realized that it was only possible to combine finitely many such modules together effectively.

At first we did not expect much from this new notion, but we found it to be of increasing interest as we continued to examine it. For example, we found that the standard Π^0_{n+2} equivalence relation defined by equality of the sets $W_i^{\emptyset^{(n)}}$ and $W_j^{\emptyset^{(n)}}$ is complete among Π^0_{n+2} equivalence relations under finitary reducibility. This is of particular interest because, for precisely these classes, no equivalence relation can be complete under computable reducibility (as shown recently in [13]). Extending our study to certain relations from computable model theory, we found that the isomorphism problem $F^{AC}_{\cong}$ for computable algebraically closed fields of characteristic 0, while Π^0_3-complete as a set, fails to be complete under finitary reducibility: it is complete for 3-ary reducibility, but not for the 4-ary version. This confirms one's intuition that isomorphism on algebraically closed fields, despite being Π^0_3-complete as a set, is not an especially difficult problem, requiring only knowledge of the transcendence degree of the field. In contrast, the isomorphism problem $F^{\mathrm{alg}}_{\cong}$ for algebraic fields of characteristic 0, while only Π^0_2, does turn out to be complete at that level under finitary reducibility.

This paper proceeds much as our investigations proceeded. In Section 2 we present the equivalence relations on ω which we set out to study. We derive a number of results about them, and by the time we reach Proposition 2.8, it should seem clear to the reader how the notion of finitary reducibility arose for us, and why it seems natural in this context. The exact definitions of n-ary and finitary reducibility appear as Definition 3.1. In Sections 3 and 4, we study finitary reducibility in its own right. We produce the natural Π^0_{n+2} equivalence relations described above, defined by equality among Σ^0_n sets, which are complete under finitary reducibility among all Π^0_{n+2} equivalence relations. Subsequently we show that the hierarchy of n-ary reducibilities does not collapse, and that several standard equivalence relations on ω witness this non-collapse for certain n.

2 Background in Computable Reducibility

The purpose of this section is twofold. First, for the reader who is not already familiar with the framework and standard methods used in its study, it introduces some examples of results in computable reducibility, with proofs. The examples, however, are not intended as a broad outline of the subject; they are confined to one very specific subclass of equivalence relations (those which, as sets, are Π^0_4), rather than offering a survey of important results in the field. In fact the results we prove here are new, to our knowledge. They use computable reducibility to establish the complexity of various naturally defined equivalence relations in the arithmetical hierarchy. In doing so, we continue the program of work already set in motion in [6, 2, 11, 5, 1, 13] and augment their results. However, the second and more important purpose of these results is to help explain how we came to develop the notion of finitary reducibility and why we find that notion to be both natural and useful. By the end of the section, the reader will have an informal understanding of finitary reducibility, which is then formally defined and explored in the ensuing two sections.

The following definition introduces several natural equivalence relations which we will consider in this section. Here, for a set $A \subseteq \omega$, we write $A^{[n]} = \{x : \langle x, n\rangle \in A\}$ for the n-th column of A when ω is viewed as the two-dimensional array ω^2 under the standard computable pairing function $\langle \cdot, \cdot \rangle$ from ω^2 onto ω.

Definition 2.1 *First we define several equivalence relations on 2^ω.*

- $E_{perm} = \{\langle A, B\rangle \mid (\exists \text{ a permutation } p : \omega \to \omega)(\forall n) A^{[n]} = B^{[p(n)]}\}$.
- $E_{Cof} = \{\langle A, B\rangle \mid \text{For every } n,\ A^{[n]} \text{ is cofinite iff } B^{[n]} \text{ is cofinite}\}$.
- $E_{Fin} = \{\langle A, B\rangle \mid \text{For every } n,\ A^{[n]} \text{ is finite iff } B^{[n]} \text{ is finite}\}$.

Each of these relations induces an equivalence relation on ω, by restricting to the c.e. subsets of ω and then allowing the index e to represent the set W_e, under the standard indexing of c.e. sets. The superscript "ce" denotes this, so that, for instance,

$$E^{ce}_{perm} = \{\langle i, j\rangle \mid (\exists \text{ a permutation } p : \omega \to \omega)(\forall n) W_i^{[n]} = W_j^{[p(n)]}\}.$$

Similarly we define E^{ce}_{Cof} and E^{ce}_{Fin}, and also the following two equivalence relations on ω (where the superscripts denote oracle sets, so that $W_i^D = dom(\Phi_i^D)$):

- $E^n_= = \{(i, j) \mid W_i^{\emptyset^{(n)}} = W_j^{\emptyset^{(n)}}\}$, *for each* $n \in \omega$.
- $E^n_{max} = \{(i, j) \mid \max W_i^{\emptyset^{(n)}} = \max W_j^{\emptyset^{(n)}}\}$, *for each* $n \in \omega$.

In E^n_{max}, for any two infinite sets $W_i^{\emptyset^{(n)}}$ and $W_j^{\emptyset^{(n)}}$, this defines $\langle i, j\rangle \in E^n_{max}$, since we consider both sets to have the same maximum $+\infty$.

2.1 Π^0_4 equivalence relations

Here we will clarify the relationship between several equivalence relations occurring naturally at the Π^0_4 level. Recall the equivalence relations E_3, E_{set}, and Z_0 defined in the Borel theory. Again the analogues of these for c.e. sets are relations on the natural numbers, defined using the symmetric difference $\triangle$:

$$\begin{aligned}
i\ E^{ce}_3\ j \quad &\iff \quad \forall n\ [|(W_i)^{[n]} \triangle (W_j)^{[n]}| < \infty] \\
i\ E^{ce}_{\mathrm{set}}\ j \quad &\iff \quad \{(W_i)^{[n]} \mid n \in \omega\} = \{(W_j)^{[n]} \mid n \in \omega\} \\
i\ Z^{ce}_0\ j \quad &\iff \quad \lim_n \frac{|(W_i \triangle W_j) \upharpoonright n|}{n} = 0
\end{aligned}$$

The aim of this section is to show that the situation in the following picture holds for computable reducibility.

$$\begin{array}{c} E^{ce}_{\text{set}} \equiv_c E^{ce}_{\text{perm}} \equiv_c E^{ce}_{\text{Cof}} \equiv_c E^2_{=} \\ | \\ E^{ce}_3 \equiv_c Z^{ce}_0 \end{array}$$

Hence all these classes fall into two distinct computable-reducibility degrees, one strictly below the other. Even though no Π^0_4 class is complete under $\leq_c$, we will show that each of these classes is complete under a more general reduction.

The three classes $E^{ce}_3, E^{ce}_{\text{set}}$ and Z^{ce}_0 are easily seen to be Π^0_4. This is not as obvious for E^{ce}_{perm}.

Lemma 2.2 *The relation $E^{ce}_{perm}$is Π^0_4, being defined on pairs $\langle e, j\rangle$ by:*

$$\forall k \forall n_0 < \cdots < n_k \ \exists \textit{ distinct } m_0, \ldots, m_k \ \forall i \leq k \ (W_e^{[n_i]} = W_j^{[m_i]}),$$

in conjunction with the symmetric statement with W_j and W_e interchanged.

Proof. Since "$W_e^{[n_i]} = W_j^{[m_i]}$" is Π^0_2, the given statement is Π^0_4, as is the interchanged version. The statements clearly hold for all $\langle e, j\rangle \in E^{ce}_{\text{perm}}$. Conversely, if the statements hold, then each c.e. set which occurs at least k times as a column in W_e must also occur at least k times as a column in W_j, and vice versa. It follows that every c.e. set occurs equally many times as a column in each, allowing an easy definition of the permutation p to show $\langle e, j\rangle \in E^{ce}_{\text{perm}}$. ■

Theorem 2.3 *E^{ce}_{perm} and E^{ce}_{set} are computably bireducible. (We write $E^{ce}_{perm} \equiv_c E^{ce}_{set}$ to denote this.)*

Proof. For the easier direction $E^{ce}_{\text{set}} \leq_c E^{ce}_{\text{perm}}$, given a c.e. set A, define uniformly the c.e. set $\widehat{A}$ by setting (for each e, i, x) $x \in \widehat{A}^{[\langle e,i\rangle]}$ iff $x \in A^{[e]}$. That is, we repeat each column of A infinitely many times in $\widehat{A}$. Then $A \ E_{\text{set}} \ B$ iff $\widehat{A} \ E_{\text{perm}} \ \widehat{B}$. (Since the definition is uniform, there is a computable function g which maps each i with $W_i = A$ to $g(i)$ with $W_{g(i)} = \widehat{A}$. This g is the computable reduction required by the theorem, with $i \ E^{ce}_{\text{set}} \ j$ iff $g(i) \ E^{ce}_{\text{perm}} \ g(j)$ for all i, j.)

We now turn to $E^{ce}_{\text{perm}} \leq_c E^{ce}_{\text{set}}$. Fix a c.e. set A. We describe a uniform procedure to build $\widehat{A}$ from A. We must do this in a way where for any pair of c.e. sets W, V, $W E^{ce}_{\text{perm}} V$ iff $\widehat{W} E^{ce}_{\text{set}} \widehat{V}$. The computable function q that gives $W_{q(i)} = \widehat{W}_i$ will then be a witness for the reduction $E^{ce}_{\text{perm}} \leq_c E^{ce}_{\text{set}}$.

For each x let $F(x)$ be the number of columns $y \leq x$ such that $A^{[x]} = A^{[y]}$. There is a natural computable guessing function $F_s(x)$ such that for every s, $F_s(x) \leq x$ and $F(x) = \limsup_s F_s(x)$.

Associated with x are the c.e. sets $C[x,n]$ for each $n>0$ and $D[x,i,j]$ for each $i>0, j\in\omega$, defined as follows. $D[x,i,j]$ is the set D such that

$$D^{[k]}=\begin{cases} A^{[x]}, & \text{if } k=0,\\ \{0,1,\cdots,j-1\}, & \text{if } k=i,\\ \emptyset, & \text{otherwise.}\end{cases}$$

and $C[x,n]$ is the set C such that

$$C^{[k]}=\begin{cases} A^{[x]}, & \text{if } k=0,\\ \{t:(\exists s\geq t)(F_s(x)\geq n)\}\,, & \text{if } k=n,\\ \emptyset, & \text{otherwise.}\end{cases}$$

Now let $\widehat{A}$ be obtained by copying all the sets $C[x,n]$ and $D[x,i,j]$ into the columns. That is, let $\widehat{A}^{[2\langle x,n\rangle]}=C[x,n]$ and $\widehat{A}^{[2\langle x,i,j\rangle+1]}=D[x,i,j]$. Now suppose that $A\ E_{\rm perm} B$. We verify that $\widehat{A}\ E_{\rm set}\widehat{B}$, writing $C[A,x,n]$, $C[B,x,n]$, $D[A,x,i,j]$, and $D[B,x,i,j]$ to distinguish between the columns of $\widehat{A}$ and $\widehat{B}$.

Fix x and consider $D[A,x,i,j]$. Since there is some y such that $A^{[x]}=B^{[y]}$ it follows that $D[A,x,i,j]=D[B,y,i,j]$ for every i,j. Now we may pick y such that $F(A,x)=F(B,y)$. It then follows that $C[A,x,n]=C[B,y,n]$ for every $n\leq F(A,x)$, and for $n>F(A,x)$ we have $C[A,x,n]=D[B,y,n,j]$ for some appropriate j. Hence every column of $\widehat{A}$ appears as a column of $\widehat{B}$. A symmetric argument works to show that every column of $\widehat{B}$ is a column of $\widehat{A}$.

Now suppose that $\widehat{A}\ E_{\rm set}\ \widehat{B}$. We argue that $A\ E_{\rm perm}\ B$. Fix x and n such that there are exactly n many different numbers $z\leq x$ with $A^{[z]}=A^{[x]}$. We claim that there is some y such that $A^{[x]}=B^{[y]}$ and there are at least n many $z\leq y$ such that $B^{[z]}=B^{[y]}$.

The column $C[A,x,n]$ of $\widehat{A}$ is the set C such that $C^{[0]}=A^{[x]}$ and $C^{[n]}=\omega$. Now $C[A,x,n]$ cannot equal $D[B,y,i,j]$ for any y,i,j since D-sets have every column finite except possibly for the $0^{\rm th}$ column. So $C[A,x,n]=C[B,y,n]$ for some y. It follows that $A^{[x]}=(C[B,y,n])^{[0]}=B^{[y]}$, and we must have $\limsup_s F_s(B,y)\geq n$. So each $A^{[x]}$ corresponds to a column $B^{[y']}$ of B with $F(B,y')=F(A,x)$. Again a symmetric argument follows to show that each $B^{[y]}$ corresponds to a column $A^{[x]}$ of A with $F(A,x)=F(B,y)$. Hence A and B agree up to a permutation of columns. ■

Theorem 2.4 $E^{ce}_{Cof}\equiv_c E^{ce}_{set}\equiv_c E^2_=$.

Proof. We first show that $E^{ce}_{\rm set}\leq_c E^2_=$. There is a Σ^0_3 predicate $R(i,x)$ which holds iff $\exists n(W_x^{[n]}=W_i)$. Let $f(x)$ be a computable function such that $R(i,x)$ iff $i\in W^{\emptyset''}_{f(x)}$. It is then easy to verify that $x\ E^{ce}_{\rm set}\ y\Leftrightarrow f(x)\ E^2_=\ f(y)$.

Next we show $E^2_=\ \leq_c E^{ce}_{\rm Cof}$. There is a single Σ^0_3 predicate R such that for every a,x, we have $a\in W^{\emptyset''}_x\Leftrightarrow R(a,x)$. Since every Σ^0_3 set is 1-reducible to the set ${\rm Cof}=\{n:W_n={\rm dom}(\varphi_n)\text{ is cofinite}\}$, let g be a computable function so that $a \in W_x^{\emptyset''} \Leftrightarrow W_{g(a,x)}$ is cofinite. Now for each x we produce the c.e. set $W_{f(x)}$ such that for each $a \in \omega$ we have $W_{f(x)}^{[a]} = \mathrm{dom}(\varphi_{g(a,x)})$. Hence f is a computable function witnessing $E_{=}^2 \leq_c E_{\mathrm{Cof}}^{ce}$.

Finally we argue that $E_{\mathrm{Cof}}^{ce} \leq_c E_{\mathrm{set}}^{ce}$. Given a c.e. set A, and i, n, we let $C(i,n) = [0,i] \cup [i+2, i+M+2]$, where M is the smallest number $\geq n$ such that $M \notin A^{[i]}$. Hence the characteristic function of $C(i,n)$ is a string of $i+1$ many 1's, followed by a single 0, and followed by $M+1$ many 1's. Since the least element not in a c.e. set never decreases with time, $C(i,n)$ is uniformly c.e. Note that if $i \neq i'$ then $C(i,n) \neq C(i',n')$. Now let $D(a,b) = [0,a] \cup [a+2, a+b+1]$.

Now let $\widehat{A}$ be a c.e. set having exactly the columns $\{C(i,n) \mid i,n \in \omega\} \cup \{D(a,b) \mid a,b \in \omega\}$. We verify that $A\ E_{\mathrm{Cof}}\ B$ iff $\widehat{A}\ E_{\mathrm{set}}\ \widehat{B}$. Again we write $C(A,i,n), C(B,i,n)$ to distinguish between the different versions. Suppose that $A\ E_{\mathrm{Cof}}\ B$. Since $D(a,b)$ appear as columns in both $\widehat{A}$ and $\widehat{B}$, it suffices to check the C columns. Fix $C(A,i,n)$. If this is finite then it must equal $D(i,b)$ for some b, and so appears as a column of $\widehat{B}$. If $C(A,i,n)$ is infinite then it is in fact cofinite and so every number larger than n is eventually enumerated in $A^{[i]}$. Hence $B^{[i]}$ is cofinite and so $C(B,i,m)$ is cofinite for some m. Hence $C(A,i,n) = C(B,i,m) = \omega - \{i+1\}$ appears as a column of $\widehat{B}$. A symmetric argument works to show that each column of $\widehat{B}$ appears as a column of $\widehat{A}$.

Now assume that $\widehat{A}\ E_{\mathrm{set}}\ \widehat{B}$. Fix i such that $A^{[i]}$ is cofinite. Then $C(A,i,n) = \omega - \{i+1\}$ for some n. This is a column of $\widehat{B}$. Since each $D(a,b)$ is finite $C(A,i,n) = C(B,j,m)$ for some j. Clearly $i = j$, which means that $B^{[i]}$ is cofinite. By a symmetric argument we can conclude that $A\ E_{\mathrm{Cof}}\ B$. ■

Theorem 2.5 $E_3^{ce} \equiv_c Z_0^{ce}$.

Proof. $E_3^{ce} \leq_c Z_0^{ce}$ was shown in [5, Prop. 3.7]. We now prove $Z_0^{ce} \leq_c E_3^{ce}$. Let $F_s(i,j,n) = \frac{|(W_{i,s} \triangle W_{j,s}) \upharpoonright n|}{n}$. Note that for each i,j,n, $F_s(i,j,n)$ changes at most $2n$ times. The triangle inequality holds in this case, that is, for every s,x,y,z,n, we have $F_s(x,z,n) \leq F_s(x,y,n) + F_s(y,z,n)$.

Given i,j,n,p where $i < j < n$ and $p > 3$ we describe how to enumerate the finite c.e. sets $C_{i,j,n,p}(k)$ for $k \in \omega$. We write $C(k)$ instead of $C_{i,j,n,p}(k)$. For each k, $C(k)$ is an initial segment of ω with at most $n^2(n+1)$ many elements.

If $k \geq n$ we permanently let $C(k) = \emptyset$. We enumerate $C(0), \cdots, C(n-1)$ simultaneously. Each set starts off being empty, and we assume that $F_0(i,j,n) < 2^{-p}$. At each stage, for every $k < n$, $C(k)$ will be equal to either $[0,M]$ or $[0,M+1]$, where M is such that $C(i) = [0,M]$. At stage $s > 0$ we act only if $F_s(k_0,k_1,n)$ has changed for some $k_0 < k_1 < n$. Assume s is such a stage. Suppose $C(i) = [0,M-1]$. We then set $C(i) = [0,M]$, and we will make every $C(k)$ equal to either $[0,M]$ or $[0,M+1]$; this is possible as at the previous stage $C(k) = [0,M-1]$ or $[0,M]$.

If $F_s(i,j,n) < 2^{-p}$ we set every $C(k)$ to be equal $[0,M]$. Otherwise suppose that $F_s(i,j,n) \geq 2^{-p}$. Set $C(j) = [0,M+1]$, and for each $k \neq i,j$ we need to decide if $C(k) = [0,M]$ or $[0,M+1]$.

To decide this, consider the graph $G_{i,j,n,p,s}$ with vertices labelled $0,\dots,n-1$. Vertices k and k' are adjacent iff $F_s(k,k',n) < 2^{-(p+k+k'+1)}$, i.e. if $W_k \upharpoonright n$ and $W_{k'} \upharpoonright n$ are "close" in terms of the Hamming distance. It follows easily from the triangle inequality that i and j must lie in different components (since $F_s(i,j,n) \geq 2^{-p}$). If k is in the same component as j we increase $C(k) = [0, M+1]$ and otherwise keep $C(k) = [0, M]$. This ends the description of the construction.

It is clear that $C_{i,j,n,p}(k)$ is an initial segment of ω with at most $2n\binom{n}{2} = n^2(n+1)$ many elements. For each k, define the set $\widehat{W}_k$ by letting $\widehat{W}_k^{[\langle i,j,p\rangle]} = C_{i,j,j+1,p}(k) \star C_{i,j,j+2,p}(k) \star C_{i,j,j+3,p}(k) \star \cdots$ on column $\langle i,j,p\rangle$, where $i < j$ and $p > 3$. Here $C_{i,j,j+1,p}(k) \star C_{i,j,j+2,p}(k)$ denotes the set X where $X(z) = C_{i,j,j+1,p}(k)(z)$ if $z \leq (j+1)^2(j+2)$ and $X(z+(j+1)^2(j+2)+1) = C_{i,j,j+2,p}(k)(z)$. Essentially this concatenates the sets, with $C_{i,j,j+2,p}(k)$ after the set $C_{i,j,j+1,p}(k)$. The iterated $\star$ operation is defined the obvious way (and $\star$ is associative). We call the copy of $C_{i,j,n,p}(k)$ in $\widehat{W}_k^{[\langle i,j,p\rangle]}$ the n^{th} block of $\widehat{W}_k^{[\langle i,j,p\rangle]}$.

We now check that the reduction works. Suppose $W_x\ Z_0^{ce}\ W_y$, where $x < y$. Hence we have $\limsup_n F(x,y,n) = 0$. Fix a column $\langle i,j,p\rangle$. We argue that for almost every n, $C_{i,j,n,p}(x) = C_{i,j,n,p}(y)$. There are several cases.

(i) $\{i,j\} = \{x,y\}$. There exists $n_0 > i,j$ such that for every $n \geq n_0$ we have $F(x,y,n) < 2^{-p}$. Hence $C_{i,j,n,p}(x) = C_{i,j,n,p}(y)$ for all large n.

(ii) $|\{i,j\} \cap \{x,y\}| = 1$. Assume $i = x$ and $j \neq y$; the other cases will follow similarly. There exists $n_0 > i,j,y$ such that for every $n \geq n_0$ we have $F(x,y,n) < 2^{-(p+x+y+1)}$ and so x,y are adjacent in the graph $G_{i,j,n,p,s}$ where s is such that $F_s(x,y,n)$ is stable. Since j cannot be in the same component as x, we have $C_{i,j,n,p}(x) = C_{i,j,n,p}(y)$.

(iii) $\{i,j\} \cap \{x,y\} = \emptyset$. Similar to (ii). Since x,y are adjacent in the graph $G_{i,j,n,p,s}$ then we must have $C_{i,j,n,p}(x) = C_{i,j,n,p}(y)$.

Hence we conclude that $\widehat{W}_x\ E_3\ \widehat{W}_y$. Now suppose that $\widehat{W}_x\ E_3\ \widehat{W}_y$ for $x < y$. Fix $p > 2$ and we have $\widehat{W}_x^{[\langle x,y,p\rangle]} =^* \widehat{W}_y^{[\langle x,y,p\rangle]}$. So there is $n_0 > y$ such that $C_{x,y,n,p}(x) = C_{x,y,n,p}(y)$ for all $n \geq n_0$. We clearly cannot have $F(x,y,n) \geq 2^{-p}$ for any $n > n_0$ and so $\limsup_n F(x,y,n) \leq 2^{-p}$. Hence we have $W_x\ Z_0^{ce}\ W_y$. ■

Theorem 2.6 $E_{set}^{ce} \not\leq_c E_3^{ce}$.

Proof. Suppose there is a computable function witnessing $E_{\text{set}}^{ce} \leq_c E_3^{ce}$, and which maps (the index for) a c.e. set X to (the index for) $\widehat{X}$, so that $X\ E_{\text{set}}\ Y$ iff $\widehat{X}\ E_3\ \widehat{Y}$. Given (indices for) c.e. sets X and Y, define

$$F_s(X,Y) = \begin{cases} \max\{z < x : X_s(z) \neq Y_s(z)\}, & \text{if } x \text{ enters } X \cup Y \text{ at stage } s, \\ \max\{z < s : X_s(z) \neq Y_s(z)\}, & \text{otherwise.} \end{cases}$$

Here we assume that at each stage s at most one new element is enumerated in $X \cup Y$ at stage s (for the function F to be well-defined), and we take $\max \emptyset = 0$.

One readily verifies that $F_s(X,Y)$ is a total computable function in the variables involved, with $X =^* Y$ iff $\liminf_s F_s(X,Y) < \infty$.

We define the c.e. sets A, B and $C_0, C_1, \cdots$ by the following. Let $A^{[0]} = \omega$ and for $k > 0$ let $A^{[k]} = [0, k-1]$. Let $B^{[k]} = [0, k]$ for every k. Finally for each i define $C_i^{[k]}$ to be

$$\begin{cases} [0,j], & \text{if } k = 2j+1, \\ \omega, & \text{if } k = 2j \text{ and } \exists^\infty s \left(F_s(\widehat{B}^{[i]}, \widehat{C}_i^{[i]}) = j\right), \\ \left[0, \max\{s : F_s(\widehat{B}^{[i]}, \widehat{C}_i^{[i]}) = j\}\right], & \text{if } k = 2j \text{ and } \forall^\infty s \left(F_s(\widehat{B}^{[i]}, \widehat{C}_i^{[i]}) \neq j\right). \end{cases}$$

By the recursion theorem we have in advance the indices for $C_0, C_1, \cdots$ so the above definition makes sense. Fix i. If $\liminf_s F_s(\widehat{B}^{[i]}, \widehat{C}_i^{[i]}) = \infty$ then every column of C_i is a finite initial segment of ω and thus we have $C_i \; E_{\text{set}} \; B$. By assumption we must have $\widehat{C}_i \; E_3 \; \widehat{B}$ and thus the two sets agree (up to finite difference) on every column. In particular $\liminf_s F_s(\widehat{B}^{[i]}, \widehat{C}_i^{[i]}) < \infty$, a contradiction. Hence we must have $\liminf_s F_s(\widehat{B}^{[i]}, \widehat{C}_i^{[i]}) = j$ for some j. The construction of C ensures that $C_i \; E_{\text{set}} \; A$ which means that $\widehat{C}_i \; E_3^{ce} \; \widehat{A}$ and so $\widehat{C}_i^{[i]} =^* \widehat{A}^{[i]}$. Since $\liminf_s F_s(\widehat{B}^{[i]}, \widehat{C}_i^{[i]}) < \infty$ we in fact have $\widehat{B}^{[i]} =^* \widehat{C}_i^{[i]} =^* \widehat{A}^{[i]}$. Since this must be true for every i we have $\widehat{B} \; E_3 \; \widehat{A}$ and so $B \; E_{\text{set}} \; A$, which is clearly false since B has no infinite column. ■

The result of Theorem 2.6 was something of a surprise. We could see how to give a basic module for a computable reduction from E_{set}^{ce} to E_3^{ce}, in much the same way that Proposition 3.9 in [5] serves as a basic module for Theorem 3.10 there. In the situation of Theorem 2.6, we were even able to combine finitely many of these basic modules, but not all ω-many of them. The following propositions express this and sharpen our result. One the one hand, Propositions 2.8 and **??** and the ultimate Theorem 3.3 show that it really was necessary to build infinitely many sets to prove Theorem 2.6. On the other hand, Theorem 2.6 shows that in this case the proposed basic modules cannot be combined by priority arguments or any other methods.

Before proceeding further we introduce a technical convention. Details about it appear in [18, Thm. IV.3.2].

Remark 2.7 *Given a computable approximation $\{X_s\}$ to a Π_2^0 set or predicate X, we have $\limsup_s X_s(n) = X(n)$. At stage s, we say that (the Π_2^0 approximable fact) "$n \in X$"* receives a chip *if $X_s(n) = 1$. Hence $n \in X$ holds if and only if it receives infinitely many chips. We can even fix a uniform assignment of chips in which at most one n receives a chip at each stage.*

Proposition 2.8 *There exists a binary reduction from E_{set}^{ce} to E_3^{ce}. That is, there exist total computable functions f and g such that, for every $x, y \in \omega$, $x \; E_{set}^{ce} \; y$ iff $f(x,y) \; E_3^{ce} \; g(x,y)$.*

Proof. We begin with a uniform computable "chip" function h (see Remark 2.7), such that, for all i and j, $W_i = W_j$ iff $\exists^\infty s \; h(s) = \langle i, j \rangle$; that is, the predicate

"$W_i = W_i$" gets an h-chip at infinitely many stages s. Next we show how to define f and g.

First, for every $k \in \omega$, $W_{f(x,y)}$ contains all elements of every even-numbered column $\omega^{[2k]}$. To enumerate the elements of $W_{g(x,y)}$ from this column, we use h. At each stage $s+1$ for which there is some c such that $h(s)$ is a chip for the sets $W_x^{[k]}$ and $W_y^{[c]}$ (i.e. the k-th and c-th columns of W_x and W_y, respectively, identified effectively by some c.e. indices for these sets), we take it as evidence that these two columns may be equal, and we find the c-th smallest element of $\overline{W^{[2k]}_{g(x,y),s}}$ and enumerate it into $W_{g(x,y),s+1}$.

The result is that, if there exists some c such that $W_x^{[k]} = W_y^{[c]}$, then $W^{[2k]}_{g(x,y)}$ is cofinite, since the c-th smallest element of its complement was added to it infinitely often, each time $W_x^{[k]}$ and $W_y^{[c]}$ received a chip. (In the language of these constructions, the c-th marker was moved infinitely many times; for instance, we refer the reader to Soare [18, IV.3] for more details on the "movable markers" type constructions). Therefore $W^{[2k]}_{g(x,y)} =^* \omega = W^{[2k]}_{f(x,y)}$ in this case. Conversely, if for all c we have $W_x^{[k]} \neq W_y^{[c]}$, then $W^{[2k]}_{g(x,y)}$ is coinfinite, since for each c, the c-th marker was moved only finitely many times, and so $W^{[2k]}_{g(x,y)} \neq^* \omega = W^{[2k]}_{f(x,y)}$. Thus $W^{[2k]}_{g(x,y)} =^* W^{[2k]}_{f(x,y)}$ iff there exists c with $W_x^{[k]} = W_y^{[c]}$.

Likewise, $W_{g(x,y)}$ contains all elements of each odd-numbered column $\omega^{[2k+1]}$, and whenever $h(s)$ is a chip for $W_y^{[k]}$ and $W_x^{[c]}$, we adjoin to $W_{f(x,y),s+1}$ the c-th smallest element of the column $\omega^{[2k+1]}$ which is not already in $W_{f(x,y),s}$. This process is exactly symmetric to that given above for the even columns, and the result is that $W^{[2k]}_{f(x,y)} =^* W^{[2k]}_{g(x,y)}$ iff there exists c with $W_y^{[k]} = W_x^{[c]}$. So we have established that

$$x \ E^{ce}_{\text{set}} \ y \iff f(x,y) \ E^{ce}_3 \ g(x,y)$$

exactly as required. ■

In Theorem 3.3, we will extend this idea, showing that, instead of merely having functions f and g to address two natural numbers x and y, we could address x, y, and z, simultaneously, or even $x_0, \ldots, x_n$. That is (in the ternary case, with x, y, and z), we can construct total computable functions f, g, and h such that, for all $x, y, z \in \omega$:

$$\begin{aligned}
&x \ E^{ce}_{\text{set}} \ y \text{ iff } f(x,y,z) \ E^{ce}_3 \ g(x,y,z),\\
&y \ E^{ce}_{\text{set}} \ z \text{ iff } g(x,y,z) \ E^{ce}_3 \ h(x,y,z), \text{ and}\\
&x \ E^{ce}_{\text{set}} \ z \text{ iff } f(x,y,z) \ E^{ce}_3 \ h(x,y,z).
\end{aligned}$$

We will refer to the triple (f, g, h) as a *ternary reduction* from E^{ce}_{set} to E^{ce}_3. Definition 3.1 will generalize this notion to arbitrary arity, and we will prove in Theorem 3.3 that the n-ary reducibility holds uniformly, despite its failure (cf. Theorem 2.6) to extend to a single reduction on all $x \in \omega$ simultaneously.

3 Introducing Finitary Reducibility

Here we formally begin the study of finitary reducibility, building on the concepts introduced in Proposition 2.8. In Theorem 3.3, we will sketch the proof that this construction can be generalized to any finite arity n. That is, we will show that E^{ce}_{set} is n-arily reducible to E^{ce}_3, under the following definition.

Definition 3.1 *An equivalence relation E on ω is* n-arily reducible *to another equivalence relation F, written $E \leq^n_c F$, if there exists a computable total function $f : \omega^n \to \omega^n$ (called an* n-ary reduction *from E to F) such that, whenever $f(x_0, \ldots, x_{n-1}) = (y_0, \ldots, y_{n-1})$ and $i < j < n$, we have*

$$x_i \; E \; x_j \iff y_i \; F \; y_j.$$

If such functions exist uniformly for all $n \in \omega$, then E is finitarily reducible *to F, written $E \leq^{<\omega}_c F$. Thus a finitary reduction is just a function from $\omega^{<\omega}$ to $\omega^{<\omega}$, mapping n-tuples $\vec{x}$ to n-tuples $\vec{y}$, with the above property.*

An n-ary reduction is sometimes expressed as an n-tuple of n-ary ω-valued functions, such as (f, g, h) when $n = 3$ (at the end of the preceding section). We note that one can consider the "non-uniform" version of finitary reducibility, where E is n-arily reducible to F for all n, but the reductions do not necessarily exist uniformly. We do not know if this implies that E is finitarily reducible to F. However we prefer to focus on the uniform version because, as mentioned in the paragraph before Remark 2.7, our main motivation for considering finitary reducibility was due to the observation that in order to construct a computable reduction between two relations, one can sometimes form a basic module and iterate it uniformly to obtain a computable reduction.

The following properties are immediate.

Proposition 3.2 *Whenever $E \leq^{n+1}_c F$, we also have $E \leq^n_c F$. Finitary reducibility implies all n-reducibilities, and computable reducibility $E \leq_c F$ implies finitary reducibility $E \leq^{<\omega}_c F$.*

Proof. If $E \leq^{n+1}_c F$ via h, then $g(\vec{x}) = (h(\vec{x}, 0))\restriction n$ is an n-reduction. If $E \leq_c F$ via f, then $(x_0, \ldots, x_{n-1}) \mapsto (f(x_0), \ldots, f(x_{n-1}))$ is a finitary reduction. ∎

Unary reducibility is completely trivial, and binary reducibility $E \leq^2_c F$ is exactly the same concept as m-reducibility on sets $E \leq_m F$, with E and F viewed as subsets of ω via a natural pairing function. For $n > 2$, however, we believe n-ary reducibility to be a new concept. To our knowledge, E^{ce}_{set} and E^{ce}_3 form the first example of a pair of equivalence relations on ω proven to be finitarily reducible but not computably reducible. A simpler example appears below in Proposition 4.1.

Theorem 3.3 *E^{ce}_{set} is finitarily reducible to E^{ce}_3 (yet $E^{ce}_{set} \not\leq_c E^{ce}_3$, by Theorem 2.6).*

Proof. We begin by giving a full ternary reduction (f, g, h). The proof of the theorem is not by induction, but we believe that this concrete case is the best way to introduce the ideas and the (rather cumbersome) notation. Afterwards we explain how to modify the proof, uniformly in n, to build an n-ary reduction.

To simplify matters, we lift the relation "E_{set}" to a partial order $\leq_{set}$, defined on subsets of ω by:

$$A \leq_{set} B \iff \text{ every column of } A \text{ appears as a column in } B.$$

So $A\ E_{set}\ B$ just if $A \leq_{set} B$ and $B \leq_{set} A$.

As in Proposition 2.8, we describe the construction of individual columns of the sets $W_{f(x,y,z)}$, $W_{g(x,y,z)}$, and $W_{h(x,y,z)}$, using a uniform chip function for equality on columns. First, for each pair $\langle i, j\rangle$, we have a column designated L^x_{ij}, the column where we consider x on the left for i and j. This means that we wish to guess, using the chip function, whether the column $W^{[i]}_x$ occurs as a column in W_y, and also whether it occurs as a column in W_z. We make $W_{f(x,y,z)}$ contain all of this column right away. For every c, we move the c-th marker in the column L^x_{ij} in both $W_{g(x,y,z)}$ and $W_{h(x,y,z)}$ whenever either:

- the c-th column of W_y receives a chip saying that it may equal $W^{[i]}_x$; or
- the c-th column of W_z receives a chip saying that it may equal $W^{[j]}_x$.

Therefore, these columns in $W_{g(x,y,z)}$ and $W_{h(x,y,z)}$ are automatically equal, and they are cofinite (i.e. $=^* W_{f(x,y,z)}$ on this column) iff either $W^{[i]}_x$ actually does equal some column in W_y or $W^{[j]}_x$ actually does equal some column in W_z.

The result, on the columns L^x_{ij} for all i and j collectively, is the following.

1. $W_{g(x,y,z)}$ and $W_{h(x,y,z)}$ are always equal to each other on these columns.
2. If $W_x \leq_{set} W_y$, then $W_{f(x,y,z)}$, $W_{g(x,y,z)}$, and $W_{h(x,y,z)}$ are all cofinite on each of these columns.
3. If $W_x \leq_{set} W_z$, then again $W_{f(x,y,z)}$, $W_{g(x,y,z)}$, and $W_{h(x,y,z)}$ are all cofinite on each of these columns.
4. If there exist i and j such that $W^{[i]}_x$ does not appear as a column in W_y and $W^{[j]}_x$ does not appear as a column in W_z, then on that particular column L^x_{ij}, $W_{g(x,y,z)}$ and $W_{h(x,y,z)}$ are coinfinite (and equal), hence $\neq^* W_{f(x,y,z)} = \omega$.

This explains the name L^x: these columns collectively ask whether either $W_x \leq_{set} W_y$ or $W_x \leq_{set} W_z$. We have similar columns L^y_{ij} and L^z_{ij}, for all i and j, doing the same operations with the roles of x, y, and z permuted.

We also have columns R^z_{ij}, for all $i, j \in \omega$, asking about W_z on the right – that is, asking whether either $W_x \leq_{set} W_z$ or $W_y \leq_{set} W_z$. The procedure here, for a fixed i and j, sets both $W_{f(x,y,z)}$ and $W_{g(x,y,z)}$ to contain the entire column R^x_{ij}, and enumerates elements of this column into $W_{h(x,y,z)}$ using the chip function.

Whenever the column $W_x^{[i]}$ receives a chip indicating that it may equal $W_z^{[c]}$ for some c, we move the c-th marker in column R_{ij}^x in $W_{h(x,y,z)}$. Likewise, whenever the column $W_y^{[j]}$ receives a chip indicating that it may equal $W_z^{[c]}$ for some c, we move the c-th marker in R_{ij}^x in $W_{h(x,y,z)}$. The result of this construction is that the column R_{ij}^x in $W_{h(x,y,z)}$ is cofinite (hence $=^* \omega = W_{f(x,y,z)} = W_{g(x,y,z)}$ on this column) iff at least one of $W_x^{[i]}$ and $W_y^{[j]}$ appears as a column in W_z.

Considering the columns R_{ij}^z for all i and j together, we see that:

1. $W_{f(x,y,z)}$ and $W_{g(x,y,z)}$ are always equal to ω on these columns.

2. If $W_x \leq_{set} W_z$, then $W_{f(x,y,z)}$, $W_{g(x,y,z)}$, and $W_{h(x,y,z)}$ are all cofinite on each of these columns.

3. If $W_y \leq_{set} W_z$, then again $W_{f(x,y,z)}$, $W_{g(x,y,z)}$, and $W_{h(x,y,z)}$ are all cofinite on each of these columns.

4. If there exist i and j such that neither $W_x^{[i]}$ nor $W_y^{[j]}$ appears as a column in W_z, then on that particular column R_{ij}^z, $W_{h(x,y,z)}$ is coinfinite, hence $\neq^* \omega = W_{f(x,y,z)} = W_{g(x,y,z)}$.

Once again, in addition to the columns R_{ij}^z, we have columns R_{ij}^x and R_{ij}^y for all i and j, on which the same operations take place with the roles of x, y, and z permuted.

We claim that the sets $W_{f(x,y,z)}$, $W_{g(x,y,z)}$, and $W_{h(x,y,z)}$ enumerated by this construction satisfy the proposition. Consider first the question of whether every column of W_x appears as a column in W_z. This is addressed by the columns labeled L^x and those labeled R^z (which are exactly the ones whose construction we described in detail.) If every column of W_x does indeed appear in W_z, then the outcomes listed there show that all three of the sets $W_{f(x,y,z)}$, $W_{g(x,y,z)}$, and $W_{h(x,y,z)}$ are cofinite on every one of these columns.

On the other hand, suppose some column $W_x^{[i]}$ fails to appear in W_z. Suppose further that $W_x^{[i]}$ also fails to appear in W_y. Then the column L_{ii}^x has the negative outcome: on this column, we have

$$W_{f(x,y,z)} \neq^* \omega = W_{g(x,y,z)} = W_{h(x,y,z)}.$$

This shows that $\langle f(x,y,z), h(x,y,z)\rangle$ (and also $\langle f(x,y,z), g(x,y,z)\rangle$) fail to lie in E_3^{ce}, which is appropriate, since $\langle x,z\rangle$ (and $\langle x,y\rangle$) were not in E_{set}^{ce}.

The remaining case is that some column $W_x^{[i]}$ fails to appear in W_z, but does appear in W_y. In this case, some column $W_y^{[j]}$ (namely, the copy of $W_x^{[i]}$) fails to appear in W_z, and so the negative outcome on the column R_{ij}^z holds:

$$W_{h(x,y,z)} \neq^* \omega = W_{f(x,y,z)} = W_{g(x,y,z)}.$$

This shows that $\langle f(x,y,z), h(x,y,z)\rangle$ (and also $\langle g(x,y,z), h(x,y,z)\rangle$) fail to lie in E_3^{ce}, which is appropriate once again, since $\langle x,z\rangle$ (and $\langle y,z\rangle$) were not in E_{set}^{ce}.

Thus, the situation $W_x \not\leq_{set} W_z$ caused $W_{f(x,y,z)}$ and $W_{h(x,y,z)}$ to differ infinitely on some column, whereas if $W_x \leq_{set} W_z$, then they were the same on all of the columns L^x and R^z. Moreover, if they were the same, then $W_{g(x,y,z)}$ was also equal to each of them on these columns. If they differed infinitely, but $W_x \leq_{set} W_y$, then $W_{g(x,y,z)}$ was equal to $W_{f(x,y,z)}$ on all those columns; whereas if they differed infinitely and $W_y \leq_{set} W_z$, then $W_{g(x,y,z)}$ was equal to $W_{h(x,y,z)}$ on all those columns.

The same holds for each of the other five situations: for instance, the columns L^y and R^x collectively give the appropriate outcomes for the question of whether $W_y \leq_{set} W_x$, while not causing $W_{h(x,y,z)}$ to differ infinitely from either $W_{f(x,y,z)}$ or $W_{g(x,y,z)}$ on any of these columns unless (respectively) $W_z \not\leq_{set} W_x$ or $W_y \leq_{set} W_z$. Therefore, the requirements are satisfied by this construction, and we have a ternary reduction.

To address the general case of an n-ary reduction from $E^{\text{ce}}_{\text{set}}$ to E_3^{ce}, we broaden these ideas. The columns L^x can be viewed as a way of asking whether X has anything else in its equivalence class. With $n = 3$, a negative answer meant that $W_x \not\leq_{set} W_y$ and $W_x \not\leq_{set} W_z$, clearly implying that neither $\langle x, y\rangle$ nor $\langle x, z\rangle$ lies in $E^{\text{ce}}_{\text{set}}$. A positive answer, on the other hand, could fail to imply the $\leq_{set}$ relations, if $W_y \leq_{set} W_x$, for instance. With $n = 3$, such other cases were handled by L^y or similar columns. Here we will give a full argument about the possible equivalence classes into which E_{set} partitions the n given c.e. sets.

For any fixed n, consider each possible partition P of the c.e. sets $A_1, \ldots, A_n$ (given by (arbitrary) indices $m_0, \ldots, m_{n-1}$, with $A_k = m_{k-1}$) into equivalence classes. If P is consistent with E_{set} (that is, if every E_{set}-class is contained in some P-class), then for each i, j with $\langle A_i, A_j\rangle \notin P$, we have two possible relations: either $A_i \not\leq_{set} A_j$ or $A_j \not\leq_{set} A_i$. We consider every possible conjunction of one of these possibilities for each such pair $\langle i, j\rangle$.

We illustrate with an example: suppose $n = 5$ and P has classes $\{A_1, A_2\}$, $\{A_3, A_4\}$, and $\{A_5\}$. One possible conjunction explaining this situation is:

$$\begin{aligned}&A_1 \not\leq_{set} A_3 \;\&\; A_1 \not\leq_{set} A_4 \;\&\; A_2 \not\leq_{set} A_3 \;\&\; A_2 \not\leq_{set} A_4 \;\&\\ &A_1 \not\leq_{set} A_5 \;\&\; A_2 \not\leq_{set} A_5 \;\&\; A_3 \not\leq_{set} A_5 \;\&\; A_4 \not\leq_{set} A_5.\end{aligned}$$

Another possibility is:

$$\begin{aligned}&A_1 \not\geq_{set} A_3 \;\&\; A_1 \not\leq_{set} A_4 \;\&\; A_2 \not\leq_{set} A_3 \;\&\; A_2 \not\geq_{set} A_4 \;\&\\ &A_1 \not\geq_{set} A_5 \;\&\; A_2 \not\leq_{set} A_5 \;\&\; A_3 \not\geq_{set} A_5 \;\&\; A_4 \not\geq_{set} A_5.\end{aligned}$$

For this n and P there are 2^8 such possibilities in all, since there are 8 pairs $i < j$ with $\langle A_i, A_j\rangle \notin P$. If this P is consistent with E_{set}, then at least one of these 2^8 possibilities must hold.

Now, for every partition P of $\{A_1, \ldots, A_n\}$ and for every such possible conjunction (with k conjuncts, say), we have an infinite set of columns used in building the sets $\widehat{A}_1, \ldots, \widehat{A}_n$. These columns correspond to elements of ω^k. In the second possible conjunction in the example above, the column for $\langle i_1, \ldots, i_k\rangle$

corresponds to the question of whether the following holds.

$$(\exists c\ A_1^{[c]} = A_3^{[i_1]}) \text{ or } (\exists c\ A_1^{[i_2]} = A_4^{[c]}) \text{ or } (\exists c\ A_2^{[i_3]} = A_3^{[c]}) \text{ or } (\exists c\ A_2^{[c]} = A_4^{[i_4]}) \text{ or}$$
$$(\exists c\ A_1^{[c]} = A_5^{[i_5]}) \text{ or } (\exists c\ A_2^{[i_6]} = A_5^{[c]}) \text{ or } (\exists c\ A_3^{[c]} = A_5^{[i_7]}) \text{ or } (\exists c\ A_4^{[c]} = A_5^{[i_8]}).$$

As before, a negative answer implies that P is consistent with E_{set} on these sets. Conversely, if P is consistent with E_{set}, then at least one of these 2^8 disjunctions (in this example) must fail to hold.

With this framework, the actual construction proceeds exactly as in the case $n = 3$. A uniform chip function guesses whether any of these eight existential (really Σ_3^0) statements holds. If any one does hold, then all sets $\widehat{A}_i$ are cofinite in the column for this P and this conjunction and for $\langle i_1, \ldots, i_k \rangle$. If the entire disjunction (as stated here) is false, then $\widehat{A}_i =^* \widehat{A}_j$ on this column iff $\langle A_i, A_j \rangle \in P$. So, if P is consistent with E_{set}, then we have not caused $\widehat{A}_i\ E_3\ \widehat{A}_j$ to fail for any $\langle i, j \rangle$ for which $A_i\ E_{set}\ A_j$, but we have caused $\widehat{A}_i\ E_3\ \widehat{A}_j$ to fail whenever $\langle A_i, A_j \rangle \notin P$. (Also, if P is inconsistent with E_{set}, then every disjunction has a positive answer, so every $\widehat{A}_i$ is cofinite on each of the relevant columns, and thus they are all $=^*$ there.)

Of course, one of the finitely many possible equivalence relations P on $\{A_1, \ldots, A_n\}$ is actually equal to E_{set} there. This P shows that, whenever $\langle A_i, A_j \rangle \notin E_{set}$, we have $\langle \widehat{A}_i, \widehat{A}_j \rangle \notin E_3$; while the argument above shows that whenever $A_i\ E_{set}\ A_j$, neither this P nor any other causes any infinite difference between any of the columns of $\widehat{A}_i$ and $\widehat{A}_j$, leaving $\widehat{A}_i\ E_3\ \widehat{A}_j$. So we have satisfied the requirements of finitary reducibility, in a manner entirely independent of n and of the choice of sets $A_1, \ldots, A_n$. ∎

A full understanding of this proof reveals that it was essential for each disjunction to consider every one of the sets $A_1, \ldots, A_n$. If the disjunction caused $\widehat{A}_1 \neq^* \widehat{A}_2$ on a particular column, for example, by making $\widehat{A}_2$ coinfinite on that column, then the value of $\widehat{A}_p$ (for $p > 2$) on that column will be either $\neq^* \widehat{A}_1$ or $\neq^* \widehat{A}_2$, and this decision cannot be made at random. In fact, one cannot even just guess from A_p whether or not the relevant column $A_1^{[i]}$ which fails to appear in A_2 appears in A_p; in the event that it does not appear, $\widehat{A}_p$ may need to be not just coinfinite but actually $=^* \widehat{A}_2$ on that column. Since A_p is included in the disjunction (and in the partition P which generated it), we have instructions for defining $\widehat{A}_p$: either we choose at the beginning to make it $= \widehat{A}_1 (= \omega)$ on this column, or we choose at the beginning to keep it $= \widehat{A}_2$ there. The partition P is thus essential as a guide. For a finite number n of sets, there are only finitely many P to be considered, but on countably many sets $A_1, A_2, \ldots$ (such as the collection $W_0, W_1, \ldots$ of all c.e. sets), there would be 2^ω-many possible equivalence relations. Even if we restricted to the Π_4^0 partitions P (which are the only ones that could equal E_{set}^{ce}), we would not know, for a given P, whether $\widehat{A}_p$ should be kept equal to $\widehat{A}_1$ or to $\widehat{A}_2$, since a Π_4^0 relation is too complex to allow effective guessing about whether it contains $\langle 1, p \rangle$ or $\langle 2, p \rangle$.

The concept of n-ary reducibility could prove to be a useful measure of how close two equivalence relations E and F come to being computably reducible. The higher the n for which n-ary reducibility holds, the closer they are, with finitary reducibility being the very last step before actual computable reducibility $E \leq_c F$. The example of E^{ce}_{set} and E^{ce}_3 is surely quite natural, and shows that finitary reducibility need not imply computable reducibility. At the lower levels, we will see in Theorem 4.2 that there can also be specific natural differences between n-ary and $(n+1)$-ary reducibility, at least in the case $n = 3$. Another example at the Π^0_2 level will be given in Proposition 4.1. Right now, though, our first application is to completeness under these reducibilities.

Working with Ianovski and Nies, we showed in [13, Thm. 3.7 & Cor. 3.8] that no Π^0_{n+2} equivalence relation can be complete amongst all Π^0_{n+2} equivalence relations under computable reducibility. However, we now show that, under finitary reducibility, there is a complete Π^0_{n+2} equivalence relation, for every n. Moreover, the example we give is very naturally defined. We consider, for each n, the equivalence relation $E^n_= = \{(i,j) \mid W_i^{\emptyset^{(n)}} = W_j^{\emptyset^{(n)}}\}$. Clearly $E^n_=$ is a Π^0_{n+2} equivalence relation. We single out this relation $E^n_=$ because equality amongst c.e. sets (and in general, equality amongst Σ^0_{n+1} sets) is indisputably a standard equivalence relation and, as n varies, permits coding of arbitrary arithmetical information at the Σ^0_{n+1} level.

We begin with the case $n = 0$.

Theorem 3.4 *The equivalence relation $E^0_=$ (also known as $=^{ce}$) is complete amongst the Π^0_2 equivalence relations with respect to the finitary reducibility.*

Proof. Fix a Π^0_2 equivalence relation R. We must produce a computable function $f(k, \vec{x})$ such that $f(k, \cdot) : \omega^k \to \omega^k$ gives the k-ary reduction from R to $E^0_=$. We will define $f(k, \cdot) = (f_{k,0}, \dots, f_{k,k-1})$ as a k-tuple of functions from ω^k to ω. Note that the case $k = 2$ follows trivially from the fact that $E^0_=$ is Π^0_2-complete as a set. However the completeness of $E^0_=$ under $\leq^k_c$ for $k > 2$ does not follow trivially from this. Nevertheless we will mention the strategy for $k = 2$ since it will serve as the basic module.

$k = 2$: The strategy for $k = 2$ is simple. We monitor the stages at which the pair (m_0, m_1) gets a new chip in R. Each time we get a new chip we make $W_{f_{2,0}(m_0,m_1)} = [0, s]$ and $W_{f_{2,1}(m_0,m_1)} = [0, s+1]$ where s is a fresh number. Clearly $m_0 R m_1$ iff $W_{f_{2,0}(m_0,m_1)} = W_{f_{2,1}(m_0,m_1)} = \omega$. This will serve as the basic module for the pair (m_0, m_1).

$k = 3$: We fix the triple m_0, m_1, m_2. For ease of notation we rename these as $0, 1, 2$ instead. We must build, for $i < 3$, the c.e. set $A_i = W_{f_{3,i}(0,1,2)}$. Each A_i will have $\binom{3}{2} = 3$ columns, which we denote as $A_i^{a,b}$ for $0 \leq a < b < 3$. That is, $A_i^{[0]} = A_i^{0,1}, A_i^{[1]} = A_i^{1,2}, A_i^{[2]} = A_i^{0,3}$ and $A_i^{[j]} = \emptyset$ for $j > 2$. We assume that at each stage, at most one pair (i, i') gets a new chip.

Each time we get a $(0,1)$-chip we must play the $(0,1)$-game, i.e., we set $A_0^{0,1} = [0, s]$ and $A_1^{0,1} = [0, s+1]$ for a new large number s. Of course $A_2^{0,1}$ must decide what to do on this column; for instance if there are infinitely many $(0,2)$-chips then we must make $A_2^{0,1} = A_0^{0,1}$ and if there are infinitely many

$(1,2)$-chips then we must make $A_2^{0,1} = A_1^{0,1}$. At the next stage where we get an $(i,2)$-chip we make $A_2^{0,1} = A_i^{0,1}$. This can be done by padding the shorter column with numbers to match the longer column, and if $A_0^{0,1}$ is made longer then we need to also make $A_1^{0,1}$ longer to keep $A_0^{0,1} \neq A_1^{0,1}$ at every finite stage.

If there are only finitely many $(0,2)$-chips and finitely many $(1,2)$-chips then $\neg 0R2$ and $\neg 1R2$ and we do not care if $A_2^{0,1} = A_0^{0,1}$ or $A_2^{0,1} = A_1^{0,1}$. Of course A_2 has to be different from both A_0 and A_1 but this will be true at the appropriate columns: the strategy will ensure that $A_2^{0,2} \neq A_0^{0,2}$ and $A_2^{1,2} \neq A_1^{1,2}$. At some point when the $(i,2)$-chips run out we will stop changing the columns $A_0^{0,1}$ and $A_1^{0,1}$ due to having to ensure the correctness of A_2. Hence the outcome of the $(0,1)$-game will be correctly reflected in the columns $A_0^{0,1}$ and $A_1^{0,1}$.

If on the other hand there are infinitely many $(0,2)$-chips and only finitely many $(1,2)$-chips then we have $0R2$ and $\neg 1R2$. We would have ensured that $A_2^{0,1} = A_0^{0,1}$ (which is important as we must make $A_2 = A_0$). Again we do not care if $A_2^{0,1}$ equals $A_1^{0,1}$.

Lastly if there are infinitely many $(i,2)$-chips for each $i < 2$ then the interference of A_2 will force both columns $A_0^{0,1}$ and $A_1^{0,1}$ to be ω. This is acceptable, because $0R1$ must hold (unless R is not an equivalence relation) and so the $(0,1)$-game would be played at infinitely many stages anyway.

$k = 4$: Again we fix the elements $0,1,2,3$ and build $A_i^{a,b}$ for $i < 4$ and $0 \leq a < b < 4$. There are now $\binom{4}{2} = 6$ columns in each A_i. The strategy we used above would seem to suggest in this case that every time we get a (i,j)-chip we play the (i,j)-game and match columns $A_i^{a,b}$ and $A_j^{a,b}$ whenever $\{a,b\} \cap \{i,j\} = 1$. At $n = 4$, however, it is clear that this will not be enough. For instance we could have the equivalence classes $\{0\},\{1\},\{2,3\}$. It could well be that the final $(0,2)$-chip came after the final $(1,2)$-chip, while the final $(1,3)$-chip came after the final $(0,3)$-chip. Then $A_2^{0,1}$ would end up equal to $A_0^{0,1}$ while $A_3^{0,1}$ would end up equal to $A_1^{0,1}$. Since $A_0^{0,1} \neq A_1^{0,1}$ this makes $A_2 \neq A_3$, which is not good.

Thus every time (i,j) gets a chip we have to to match columns $A_i^{a,b}$ and $A_j^{a,b}$ for every pair a,b except the pair (i,j). In the above scenario this new rule would force $A_0^{0,1}$ and $A_1^{0,1}$ to increase when a $(2,3)$-chip is obtained. The only way this can happen infinitely often is when $2R3$, and either ($0R2$ and $1R3$) or ($1R2$ and $0R3$). This cycle means that $0R1$ must also be true, and so the $(0,1)$-game would be played infinitely often anyway.

Arbitrary $k \geq 2$: We now fix $k \geq 2$, and fix c.e. sets $A_0, \dots, A_{k-1}$. We describe how to build $A_i^{a,b}$ for $i < k$ and $0 \leq a < b < k$. At every stage every column $A_i^{a,b}$ is just a finite initial segment of ω. We assume at each stage, at most one chip is obtained. At the beginning enumerate 0 into $A_b^{a,b}$ for every $a < b$. At a particular stage in the construction, if no chip is obtained, do nothing. Otherwise suppose we have an (i,j)-chip. We play the (i,j)-game, i.e. set $A_i^{i,j} = [0,s]$ and $A_j^{i,j} = [0,s+1]$ for a fresh number s. For each pair a,b such that $(a,b) \neq (i,j)$ we match the columns $A_i^{a,b}$ and $A_j^{a,b}$. What this means is to do nothing if they are currently equal, and if they are unequal, say $|A_i^{a,b}| < |A_j^{a,b}|$, we fill up $A_i^{a,b}$ with enough numbers to make it equal $A_j^{a,b}$.

Furthermore if $a = i$ then $A_b^{a,b}$ should also be topped up to have one more element than $A_i^{a,b}$. This ends the construction of the columns $A_i^{a,b}$ and of the sets A_i.

We now verify that the construction works. It is easy to check that at every stage of the construction, and for every $a < b$ and i, we have $|A_a^{a,b}| + 1 = |A_b^{a,b}|$ and $|A_i^{a,b}| \leq |A_b^{a,b}|$. Now suppose that iRj. Then there are infinitely many (i,j)-chips obtained during the construction and each time we play the (i,j)-game and match every other column of A_i and A_j. Hence $A_i = A_j$. Now suppose that $\neg iRj$. We verify that $A_i^{i,j} \neq A_j^{i,j}$. Suppose they are equal, so that they both have to be ω. Let t_0 be the stage where the last (i,j)-chip is issued. Hence $A_i^{i,j} = [0, s]$ and $A_j^{i,j} = [0, s+1]$ for some fresh number s, and so we have $|A_l^{i,j}| \leq |A_i^{i,j}|$ for every $l \neq j$. Let $t_1 > t_0$ be the least stage such that either $A_i^{i,j}$ or $A_j^{i,j}$ is increased.

Claim 3.5 *If $A_l^{i,j}$ is increased to equal $A_j^{i,j}$ for some $l \neq j$ at some stage $t > t_0$, then at t some (l,c)-chip or (c,l)-chip is obtained with $A_c^{i,j} = A_j^{i,j}$.*

Proof. At t suppose a (i_0, j_0)-chip was issued. At t we have three different kind of actions:

(i) The (i_0, j_0)-game is played, affecting columns $A_{i_0}^{i_0,j_0}$ and $A_{j_0}^{i_0,j_0}$.

(ii) For each $(a,b) \neq (i_0, j_0)$, the smaller of the two columns $A_{i_0}^{a,b}$ or $A_{j_0}^{a,b}$ is increased to match the other.

(iii) $A_b^{i_0,b}$ is increased in the case $a = i_0$ and $A_{i_0}^{i_0,b}$ is smaller than $A_{j_0}^{i_0,b}$, or $A_b^{j_0,b}$ is increased in the case $a = j_0$ and $A_{j_0}^{j_0,b}$ is smaller than $A_{i_0}^{j_0,b}$.

At t the column $A_l^{i,j}$ is increased due to an action of type (i), (ii) or (iii). (i) cannot be because otherwise we have $i_0 = i$ and $j_0 = j$, but we have assumed that no more (i,j)-chips were obtained. It is not possible for (iii) because otherwise $l = j$. Hence we must have (ii) which holds for some $a = i, b = j$. Furthermore $l \in \{i_0, j_0\}$, and letting c be the other element of the set $\{i_0, j_0\}$ we have the statement of the claim. ■

At t_1 we cannot have an increase in $A_j^{i,j}$ without an increase in $A_i^{i,j}$, due to the fact that the two always differ by exactly one element. Hence at t_1 we know that $A_i^{i,j}$ is increased. It cannot be increased by more than one element because the (i,j)-game can no longer be played and we have already seen that $|A_l^{i,j}| \leq |A_j^{i,j}|$ for every l. Hence at t_1, $A_i^{i,j}$ (and also $A_j^{i,j}$) is increased by exactly one element. Now apply the claim successively to get a sequence of distinct indices $c_0 = i, c_1, c_1, c_2, \cdots, c_N = j$ such for every x, at least one (c_x, c_{x+1})- or (c_{x+1}, c_x)-chip is obtained in the interval between t_0 and t_1. Hence we have a new cycle of chips beginning with i and ending with j.

Note that at t_1, $A_i^{i,j}$ was increased to match $A_c^{i,j}$. Thus the construction at t_1 could not have increased the column $A_l^{i,j}$ for any $l \notin \{i,j\}$. Hence after the action at t_1 we again have the similar situation at t_0, that is, we again have $|A_l^{i,j}| \leq |A_i^{i,j}|$ for every $l \neq j$. If $t_1 < t_2 < t_3 < \cdots$ are exactly the stages where $A_i^{i,j}$ or $A_j^{i,j}$ is again increased, we can repeat the claim and the argument above to show that between two such stages we have a new cycle of chips starting with i and ending with j. Since there are only finitely many possible cycles, there is a cycle which appears infinitely often, contradicting the transitivity of R.

The construction produces computable functions $f_{k,i}(\vec{x})$ giving the k-ary reduction from the Π_2^0 relation R to $E_=^0$. Since the construction is uniform in k, finitary reducibility follows. ∎

Next we relativize this proof to an oracle. This will give Π_{n+2}^0 equivalence relations which are complete at that level under finitary reducibility, and will also yield the striking Corollary 3.9 below, which shows that finitary reductions can exist even when full reductions of arbitrary complexity fail to exist.

Corollary 3.6 *For each $X \subseteq \omega$, the equivalence relation $E_=^X$ defined by*

$$i \; E_=^X \; j \qquad \Longleftrightarrow \qquad W_i^X = W_j^X$$

is complete amongst all Π_2^X equivalence relations with respect to the finitary reducibility.

Proof. Essentially, one simply relativizes the entire proof of Theorem 3.4 to the oracle X. The important point to be made is that the reduction f thus built is not just X-computable, but actually computable. Since every set W_e^X in question is now X-c.e., the program $e = f(i, k, \vec{x})$ is allowed to give instructions saying "look up this information in the oracle," and thus to use an X-computable chip function for an arbitrary Π_2^X relation R, without actually needing to use X to determine the program code e. ∎

By setting $X = \emptyset^{(n)}$, we get Π_n^0-complete equivalence relations (under finitary reducibility) right up through the arithmetical hierarchy.

Corollary 3.7 *Each equivalence relation $E_=^n$ is complete amongst the Π_{n+2}^0 equivalence relations with respect to the finitary reducibility.* ∎

This highlights the central role $E_=^n$ plays amongst the Π_{n+2}^0 equivalence relations; it is complete with respect to the finitary reducibility. A wide variety of Π_{n+2}^0 equivalence relations arise naturally in mathematics (for instance, isomorphism problems for many common classes of computable structures), and all of these are finitarily reducible to $E_=^n$. In particular, every Π_4^0 equivalence relation considered in this section is finitarily reducible to $E_=^2$. Indeed, E_3^{ce} is complete amongst Π_4^0 equivalence relations with respect to the finitary reducibility, even though $E_=^2 \not\leq_c E_3^{ce}$.

Corollary 3.8 *E_3^{ce} is complete amongst the Π_4^0 equivalence relations with respect to the finitary reducibility.*

Proof. By Theorem 2.4, $E^2_= \leq_c E^{ce}_{\text{set}}$, and by Theorem 3.3, $E^{ce}_{\text{set}} \leq^{<\omega}_c E^{ce}_3$. The corollary then follows from Corollary 3.7 and Proposition 3.2. ∎

Allowing arbitrary oracles in Corollary 3.6 gives a separate result. Recall from Definition 1.1 the notion of $\boldsymbol{d}$-computable reducibility.

Corollary 3.9 *For every Turing degree* $\boldsymbol{d}$*, there exist equivalence relations* E *and* F *on* ω *such that* E *is finitarily reducible to* F *(via a computable function, of course), but there is no* $\boldsymbol{d}$*-computable reduction from* E *to* F*.*

Proof. We again recall from [13] that there is no Π^0_2-complete equivalence relation under $\leq_c$. The proof there relativizes to any degree $\boldsymbol{d}$ and any set $D \in \boldsymbol{d}$, to show that no Π^D_2 equivalence relation on ω can be complete among Π^D_2 equivalence relations even under $\boldsymbol{d}$-computable reducibility. (The authors of [13] use this relativization to show that there is no Π^0_3-complete equivalence relation, for example, by taking $D = \emptyset'$, but their proof really shows that for every Π^0_3 equivalence relation, there is another one which is not even $\mathbf{0}'$-computably reducible to the first one.)

Therefore, there exists some Π^D_2 equivalence relation E such that $E \not\leq_{\boldsymbol{d}} E^D_=$. However, Corollary 3.6 shows that E does have a finitary reduction f to $E^D_=$ (with f specifically shown to be computable, not just $\boldsymbol{d}$-computable). ∎

4 Further Results on Finitary Reducibility

4.1 Π^0_2 equivalence relations

Recall the Π^0_2 equivalence relations $E^{ce}_{\min}$ and $E^{ce}_{\max}$, which were defined by

$$i \ E^{ce}_{\min} \ j \iff \min(W_i) = \min(W_j) \qquad i \ E^{ce}_{\max} \ j \iff \max(W_i) = \max(W_j).$$

(Here the empty set has minimum $+\infty$ and maximum $-\infty$, by definition, while all infinite sets have the same maximum $+\infty$.) It was shown in [5] that $E^{ce}_{\max}$ and $E^{ce}_{\min}$ are both computably reducible to $E^{ce}_= = E^0_=$, and that $E^{ce}_{\max}$ and $E^{ce}_{\min}$ are incomparable under $\leq_c$. The proof given there that $E^{ce}_{\max} \not\leq_c E^{ce}_{\min}$ seemed significantly simpler than the proof that $E^{ce}_{\min} \not\leq_c E^{ce}_{\max}$, but no quantitative distinction could be expressed at the time to make this intuition concrete. Now, however, we can use finitary reducibility to distinguish the two results rigorously.

Proposition 4.1 E^{ce}_{max} *is not binarily reducible to* E^{ce}_{min}*. However* E^{ce}_{min} *is finitarily reducible to* E^{ce}_{max}*.*

Proof. To show $E^{ce}_{\max}$ is not binarily reducible to $E^{ce}_{\min}$, let f be any computable total function. We build the c.e. sets W_i, W_j and assume by the recursion theorem that the indices i, j are given in advance. At each stage, $W_{i,s}$ and $W_{j,s}$ will both be initial segments of ω, with $W_{i,0} = W_{j,0} = \emptyset$. Whenever $\max(W_{i,s}) = \max(W_{j,s})$ and $\min(W_{f(0,i,j),s}) = \min(W_{f(1,i,j),s})$, we add the least available element to $W_{i,s+1}$, making the maxima distinct at stage $s + 1$.

Whenever $\max(W_{i,s}) \neq \max(W_{j,s})$ and $\min(W_{f(0,i,j),s}) \neq \min(W_{f(1,i,j),s})$, we add the least available element to $W_{j,s+1}$, making the maxima the same again. Since the values of $\min(W_{f(0,i,j),s})$ and $\min(W_{f(1,i,j),s})$ can only change finitely often, there is some s with $W_i = W_{i,s}$ and $W_j = W_{j,s}$, and our construction shows that these are both finite initial segments of ω, equal to each other iff $\min(W_{(f(0,i,j)}) \neq \min(W_{f(1,i,j)})$. Thus f was not a binary reduction.

To show that $E^{ce}_{\min}$ is finitarily reducible to $E^{ce}_{\max}$, we must produce a computable function $f(k,i,\vec{x})$ such that $f(k,-,-)$ gives the k-ary reduction from $E^{ce}_{\min}$ to $E^{ce}_{\max}$. Fixing $k \geq 2$ and indices $m_0, \cdots, m_k$ we describe how to build $W_{f(k,i,\vec{m})}$ for each $i < k$. We denote $A_i = W_{f(k,i,\vec{m})}$. We begin with $A_i = \emptyset$ for all i. Each time at a stage s we find a new element enumerated into some $W_{m_i}[s]$ below its current minimum we set $A_j = [0, t + \min W_{m_j}[s]]$ for every $j < k$, where t is a fresh number.

There are only finitely many m_i, so A_j is modified only finitely often. So there exists t such that for every $j < k$, $A_j = [0, t + \min W_{m_j}]$. Hence $\min W_{m_i} = \min W_{m_j}$ iff $\max A_i = \max A_j$. ■

This tells us that $E^{ce}_{\min} \leq_c E^{ce}_{\max}$ is a lot closer to being true than $E^{ce}_{\max} \leq_c E^{ce}_{\min}$. Surprisingly, we found that the Π^0_2 relation $E^{ce}_{\max}$ is complete for the ternary reducibility but not for 4-ary reducibility.

Theorem 4.2 *E^{ce}_{max} is complete for ternary reducibility $\leq^3_c$ among Π^0_2 equivalence relations, but not so for 4-ary reducibility $\leq^4_c$.*

Proof. By Theorem 3.4, we may use the relation $E^0_=$ of equality of c.e. sets (also known as $=^{ce}$), needing only to show that $E^0_= \leq^3_c E^{ce}_{\max}$ and that $E^0_= \not\leq^4_c E^{ce}_{\max}$. First we address the former claim, building a computable 3-reduction $f(n,i,j,k)$ as follows.

For any $i, j, k \in \omega$ and any stage s, let

$$m_{ij,s} = \begin{cases} s, & \text{if } W_{i,s} = W_{j,s}; \\ \min(W_{i,s} \triangle W_{j,s}), & \text{else.} \end{cases}$$

Thus $W_i \neq W_j$ iff $\lim_s m_{ij,s} < \infty$. We define $m_{ik,s}$ and $m_{jk,s}$ similarly for those pairs of sets, and set $f(0,i,j,k)$, $f(1,i,j,k)$ and $f(2,i,j,k)$ to be c.e. indices of the three corresponding sets $\widehat{W}_i$, $\widehat{W}_j$, and $\widehat{W}_k$ built by the following construction.

At each stage s, $\widehat{W}_{i,s}$, $\widehat{W}_{j,s}$, and $\widehat{W}_{k,s}$ will each be a distinct finite initial segment of ω. Each time the sets W_i and W_j get a chip (i.e. appear to be equal), we lengthen each of these initial segments to be longer than $\widehat{W}_k$ (but still distinct from each other), so that $\widehat{W}_i = \widehat{W}_j = \omega$ iff $W_i = W_j$, and otherwise they have distinct maxima. Similar arguments apply for i and k, and also for j and k.

Let $\widehat{W}_{i,0} = \{0,1\}$, $\widehat{W}_{j,0} = \{0\}$, and $\widehat{W}_{k,0} = \emptyset$. At each stage $s+1$, set $\hat{m}_s = \max(\widehat{W}_{i,s}, \widehat{W}_{j,s}, \widehat{W}_{k,s})$. We first act on behalf of i and j, checking whether $m_{ij,s+1} \neq m_{ij,s}$. If so, then we make $\widehat{W}_i = [0, \hat{m}_s + 3]$ and $\widehat{W}_j = [0, \hat{m}_s + 2]$, so that both are longer than they were before, and if also either $m_{ik,s+1} \neq m_{ik,s}$

or $m_{jk,s+1} \neq m_{jk,s}$, then we set $\widehat{W}_{k,s+1} = [0, \hat{m}_s + 1]$. (Otherwise $\widehat{W}_k$ stays unchanged at this stage.)

If $m_{ij,s+1} = m_{ij,s}$, then we check whether $m_{ik,s+1} \neq m_{ik,s}$. If so, then we make $\widehat{W}_i = [0, \hat{m}_s + 3]$ and $\widehat{W}_k = [0, \hat{m}_s + 2]$, and if also $m_{jk,s+1} \neq m_{jk,s}$, then we set $\widehat{W}_{j,s+1} = [0, \hat{m}_s + 1]$. (Otherwise $\widehat{W}_j$ stays unchanged at this stage.)

Lastly, if $m_{ij,s+1} = m_{ij,s}$ and $m_{ik,s+1} = m_{ik,s}$, then we check whether $m_{jk,s+1} \neq m_{jk,s}$. If so, then we make $\widehat{W}_j = [0, \hat{m}_s + 3]$ and $\widehat{W}_k = [0, \hat{m}_s + 2]$, with $\widehat{W}_i$ staying unchanged. This completes the construction.

Notice first that if $W_i = W_j$, then $\widehat{W}_i$ and $\widehat{W}_j$ were both lengthened at infinitely many stages, so that $\max(\widehat{W}_i) = \max(\widehat{W}_j) = +\infty$. The same holds for W_i and W_k, and also for W_j and W_k, (even though in those cases some of the lengthening may have come at stages at which we acted on behalf of W_i and W_j). On the other hand, if $W_i \neq W_j$, then at least one of these must be distinct from W_k as well. If $W_i \neq W_k$, then $\widehat{W}_i$ was lengthened at only finitely many stages; likewise for $\widehat{W}_j$ if $W_j \neq W_k$. So, if two of these sets were equal but the third was distinct, then the two equal ones gave rise to sets with maximum $+\infty$ and the third corresponded to a finite set. And if all three sets were distinct, then after some stage s_0 none of $\widehat{W}_i$, $\widehat{W}_j$, and $\widehat{W}_k$ was ever lengthened again, in which case they are the three distinct initial segments built at stage s_0, with three distinct (finite) maxima. So we have defined a ternary reduction from $E^0_=$ to $E^{ce}_{\max}$.

However, no 4-ary relation exists. We prove this by a construction using the Recursion Theorem, supposing that f were a 4-ary reduction and using indices i, j, k, and l which "know their own values." We write $\widehat{W}_i$ for $W_{f(0,i,j,k,l)}$, $\widehat{W}_j$ for $W_{f(1,i,j,k,l)}$, and so on as usual, having first waited for f to converge on these four inputs. If it converges on them all at stage s, we set $W_{i,s+1} = \{0\}$, $W_{j,s+1} = \{0, 2\}$, $W_{k,s+1} = \{1\}$, and $W_{l,s+1} = \{1, 3\}$.

Thereafter, at any stage $s + 1$ for which $W_{i,s} \neq W_{j,s}$ and $\max(\widehat{W}_{i,s}) \neq \max(\widehat{W}_{j,s})$, we add the next available even number to $W_{i,s+1}$, leaving $W_{i,s+1} = W_{j,s+1} = W_{j,s}$. At any stage $s + 1$ for which $W_{i,s} = W_{j,s}$ and $\max(\widehat{W}_{i,s}) = \max(\widehat{W}_{j,s})$, we add the next available even number to $W_{j,s+1}$, leaving $W_{i,s+1} = W_{i,s} \subsetneq W_{j,s+1}$. Similarly, at any stage $s + 1$ for which $W_{k,s} \neq W_{l,s}$ and $\max(\widehat{W}_{k,s}) \neq \max(\widehat{W}_{l,s})$, we add the next available odd number to $W_{k,s+1}$, leaving $W_{k,s+1} = W_{l,s+1} = W_{l,s}$. At any stage $s + 1$ for which $W_{k,s} = W_{l,s}$ and $\max(\widehat{W}_{k,s}) = \max(\widehat{W}_{l,s})$, we add the next available odd number to $W_{l,s+1}$, leaving $W_{k,s+1} = W_{l,s} \subsetneq W_{l,s+1}$. This is the entire construction.

Now if f is indeed a 4-ary reduction, then it must keep adding elements to both $\widehat{W}_i$ and $\widehat{W}_j$, since if either of these sets turns out to be finite, then the construction would have built W_i and W_j to contradict f. So in particular, $W_i = W_j = \{0, 2, 4, \ldots\}$, and $\max(\widehat{W}_i) = \max(\widehat{W}_j) = +\infty$. Similarly, it must keep adding elements to both $\widehat{W}_k$ and $\widehat{W}_l$, and so $W_k = W_l = \{1, 3, 5, \ldots\}$, and $\max(\widehat{W}_k) = \max(\widehat{W}_l) = +\infty$. But then $W_i \neq W_k$, yet $\max(\widehat{W}_i) = \max(\widehat{W}_k) = +\infty$. So in fact f was not a 4-ary reduction. ■

The preceding proof of the lack of any 4-ary reduction can be viewed as the simple argument that, since $E^{ce}_{\max}$ has exactly one Π^0_2-complete equivalence class (and all its other classes are Δ^0_2) while $E^0_=$ has infiinitely many Π^0_2-complete classes, the latter cannot reduce to the former. It requires four distinct elements of the equivalence relation to show this, as evidenced by the first half of the proof. One naturally conjectures that a Π^0_2 equivalence relation with exactly two Π^0_2-complete classes might be complete under $\leq^4_c$, but not under $\leq^5_c$. In Subsection 4.3 we will see that this intuition was not correct.

Corollary 4.3 *Theorem 4.2 relativizes. That is, for every set D, the equivalence relation E^D_{max} defined by*

$$i\ E^D_{max}\ j \iff \max(W^D_i) = \max(W^D_j)$$

is complete for ternary computable reducibility $\leq^3_c$ among Π^D_2 equivalence relations, but not so for 4-ary computable reducibility $\leq^4_c$.

Proof. Notice that relativizing the proof of Theorem 4.2 entirely would give this same result for D-computable ternary and 4-ary reducibility. That would be correct, and it follows that $E^D_{\max}$ is not Π^D_2-complete for 4-ary computable reducibility $\leq^4_c$ either, since certain Π^D_2 relations are not even D-computable 4-arily reducible to it. However, the ternary completeness required is also under *computable* reducibility. Proving it requires the use of the same trick as in Corollary 3.6. Our ternary reduction accepts an input $\langle i, j, k\rangle$ and outputs indices $\hat{i}$, $\hat{j}$, and $\hat{k}$ of oracle Turing programs which enumerate W^D_i, W^D_j, and W^D_j using their own oracles (since those oracles all happen to be D as well), and then execute the same strategy as in Theorem 4.2 for those three sets. ∎

The relations $E^{ce}_{\max}$ and $E^D_{\max}$ are quickly seen to be computably bireducible with the equivalence relations E^{ce}_{card} and E^D_{card} (respectively) defined by:

$$i\ E^{ce}_{\text{card}}\ j \iff |W_i| = |W_j| \qquad i\ E^D_{\text{card}}\ j \iff |W^D_i| = |W^D_j|.$$

So E^{ce}_{card} is are also Π^0_2-complete under ternary reducibility but not under 4-ary reducibility (by Proposition 3.2), and similarly with E^D_{card} for Π^D_2-completeness under these reducibilities. The reason for introducing such a similar relation is that a specific relativized version of it, $E^{\emptyset'}_{\text{card}}$, appears very useful in computable model theory. The discussion above, along with Corollary 4.3, shows that $E^{\emptyset'}_{\text{card}}$ is Π^0_3-complete under ternary $\mathbf{0}'$-computable reducibility but not under 4-ary computable reducibility. We will use this fact in the next subsection.

Proposition 4.4 *The equivalence relation $E^{\emptyset'}_{card}$ is Π^0_3-complete under ternary computable reducibility, but not under 4-ary computable reducibility.* ∎

4.2 Equivalence Relations from Algebra

Having so far considered only equivalence relations from pure computability theory, we now turn briefly to computable model theory, which one naturally expects to be a fertile source of equivalence relations. For background and details relevant to this section, we refer the reader to [14, 15, 16].

Definition 4.5 *Fix a computable presentation K of the algebraic closure $\overline{\mathbb{Q}}$ of the rational numbers. For each e, define the field K_e to be the subfield of K which one gets by closing the c.e. subset W_e of the domain of K under the field operations. The equivalence relation $F_{\cong}^{alg}$ is now defined to represent the isomorphism relation among these fields:*

$$i \ F_{\cong}^{alg} \ j \iff K_i \cong K_j.$$

Since every computable algebraic field has a computable embedding into K, with c.e. image, we know that the sequence $\langle K_e \rangle_{e \in \omega}$ includes representatives of every computable algebraic field, up to computable isomorphism. Notice also that each K_e may be considered, up to computable isomorphism, as a computable field itself, since the domain of K_e (which is c.e., uniformly in e, and infinite) can be pulled back to ω, uniformly in e. In fact, given an e such that φ_e computes the atomic diagram of a computable algebraic field of characteristic 0, one can uniformly find an i and a j such that φ_i is a computable isomorphism from K_j onto that field.

Algebraically closed fields are usually seen as a simpler class of structures than algebraic fields, even when the former are allowed to contain transcendental elements. In fact, though, the isomorphism problem for computable algebraically closed fields of characteristic 0 is Π^0_3-complete, and thus quantifiably more difficult than that for computable algebraic fields. Theorem 4.6 below shows that the gap is not as large as suggested by the raw complexity levels: while $F_{\cong}^{\mathrm{alg}}$ for algebraic fields is Π^0_2-complete for finitary reducibility, $F_{\cong}^{AC}$ for computable algebraically closed fields is not Π^0_3-complete in this way. Rather, it exhibits the same properties as the relation $E^{\emptyset'}_{\mathrm{card}}$ from the preceding subsection: it is Π^0_3-complete under ternary reducibility, but not under 4-ary reducibility.

To make isomorphism on computable algebraically closed fields into an equivalence relation on ω in a natural way, we define the field L_e to have transcendence degree $d_e = |\overline{W_e}|$. Notice that one can construct a computable copy of this field L_e uniformly effectively in e: for each n, we have a field element x_n which appears to be transcendental over the preceding elements $x_0, \ldots, x_{n-1}$, but becomes algebraic over $\mathbb{Q}$ if ever n enters W_e. Conversely, given any computable algebraically closed field F of characteristic 0, we can find an i with $F \cong L_i$, effectively in an index e such that φ_e decides the atomic diagram of F. This is straightforward, since the property of being algebraically independent over all previous elements of the field is Π^0_1. Thus, $L_i \cong L_j$ iff $\overline{W_i}$ and $\overline{W_j}$ have the same size (possibly infinite). This should immediately remind the reader of $E^{\emptyset'}_{\mathrm{card}}$, and indeed, the real content of the following theorem is that the equivalence relation $F_{\cong}^{AC}$ defined by

$$i \ F_{\cong}^{AC} \ j \iff L_i \cong L_j$$

is computably bireducible with $E^{\emptyset'}_{\mathrm{card}}$, while $F_{\cong}^{\mathrm{alg}}$ is bireducible with $E^0_{=}$.

Theorem 4.6 *The equivalence relation $F_{\cong}^{alg}$ on ω, which is Π^0_2-complete as a set (under 1-reducibility), is complete under finitary reducibility $\leq_c^{<\omega}$ among all Π^0_2 equivalence relations. However, the equivalence relation $F_{\cong}^{AC}$ on ω, which is Π^0_3-complete as a set, is only complete under ternary reducibility $\leq_c^3$ among all Π^0_3 equivalence relations; it is incomplete under 4-ary reducibility $\leq_c^4$ there.*

Proof. The $\leq_c^{<\omega}$-completeness result for $F_{\cong}^{\text{alg}}$ follows (using Proposition 3.2 and Theorem 3.4) from the computable reduction f from $E^0_{=}$ to $\leq_c^{<\omega}$ which we now describe. In $\overline{\mathbb{Q}}$, we can effectively find p_n-th roots of 2, where p_n is the n-th prime number in the subring $\mathbb{Z}$. Let q_n be the first element in the domain ω of this presentation of $\overline{\mathbb{Q}}$ satisfying $(q_n)^{p_n} = 2$. Of course, for each n, q_n does not lie in the subfield generated by the set $\{q_m : m \neq n\}$, since this subfield contains no extension of $\mathbb{Q}$ of prime degree p_n. Thus, adjoining any collection W of these q_m's to $\mathbb{Q}$ to form a field will not cause any p_n-th root of 2 with $q_n \notin W$ to appear in that field. Therefore, our computable reduction f simply maps each e to an index $f(e)$ of the c.e. set $\{q_n \in \overline{\mathbb{Q}} : n \in W_e\}$.

On the other hand, we will show that $F_{\cong}^{AC}$ and $E_{\text{card}}^{\emptyset'}$ are computably bireducible. (Recall that $E_{\text{card}}^{\emptyset'}$ is the relation which holds of indices of Σ^0_2 sets which have the same cardinality.) Propositions 4.4 and 3.2 then complete our argument. The computable reduction h from $F_{\cong}^{AC}$ to $E_{\text{card}}^{\emptyset'}$ is easy: just let $W_{h(e)}^{\emptyset'}$ enumerate the elements of $\overline{W_e}$.

For the computable reduction g from $E_{\text{card}}^{\emptyset'}$ to $F_{\cong}^{AC}$, we define $g(e)$ using a fixed total computable chip function $c(e, n, s)$ with $n \in W_e^{\emptyset'}$ iff only finitely many s have $c(e, n, s) = 1$. Build a computable field extension F of $\mathbb{Q}$, starting with elements $x_{n,0}$ (for every n) which do not yet satisfy any algebraic relation over $\mathbb{Q}$. Go through all pairs $\langle n, s\rangle$ in turn, and whenever we find that $c(e, n, s) = 1$, we make the current $x_{n,k}$ algebraic over $\mathbb{Q}$ (in some way consistent with the finite portion of the atomic diagram of F enumerated thus far), and create a new element $x_{n,k+1}$ of F which does not yet satisfy any algebraic relation over the existing elements. As we continue, we fill in all the atomic facts needed to make F into a computable algebraically closed field; details may be found in [17]. Thus, if $n \in W_e^{\emptyset'}$, then x_{n,k_n} will stay transcendental forever over the preceding elements (where k_n is the greatest k for which $x_{n,k}$ ever appears in F); while otherwise all $x_{n,k}$ (for every k) will eventually be made algebraic. Thus $\{x_{n,k_n} : n \in W_e^{\emptyset'}\}$ is a transcendence basis for F, and so the transcendence degree of F is the cardinality of $W_e^{\emptyset'}$. We set $g(e)$ to be an index such that $L_{g(e)}$ is isomorphic to F; this index can be found effectively, as remarked above, and clearly then g is a computable reduction from $E_{\text{card}}^{\emptyset'}$ to $F_{\cong}^{AC}$. ■

On the other hand, there do exist natural Π^0_3 isomorphism problems which are complete under $\leq_c^{<\omega}$ at that level. The example we give here is quick, albeit slightly unnatural, in that the field of the equivalence relation $E^0_{\cong}$ is a proper subset of ω. (An *equivalence structure* is just an equivalence relation on the domain ω.) For details, we refer the reader to [12, 14].

Theorem 4.7 *The isomorphism problem $E^0_{\cong}$ for the class $\mathcal{E}_0$ of computable equivalence structures with no infinite classes is Π^0_3-complete under $\leq_c^{<\omega}$; indeed, $E^1_{=}$ is computably reducible to $E^0_{\cong}$.*

Proof. For the computable reduction, given an index i of a set $W_i^{\emptyset'}$, we build a computable equivalence structure $\mathcal{S}$ with domain ω. $\mathcal{S}$ begins with infinitely many classes of each odd size. Whenever we see an initial segment $\sigma \subseteq \emptyset'_s$ of the stage-s approximation to $\emptyset'$, and an $n \in \omega$ for which $\Phi^{\sigma}_{e,s}(e)\downarrow$, we add a new equivalence class to $\mathcal{S}$, containing $2n+2$ elements. As long as this convergence persists at subsequent stages $t > s$, we keep this class this way. However, if we ever reach a stage $t > s$ with $\emptyset'_t \restriction |\sigma| \neq \emptyset'_s \restriction |\sigma|$, then we add one more element to this class, giving it an odd number of elements. In this case, we start searching again for a new σ for which convergence occurs. This is the entire construction.

It follows that $\mathcal{S}$ has a class of size $2n+2$ iff $n \in W_i^{\emptyset'}$, and that $\mathcal{S}$ has infinitely many classes of each odd size. Hence $E^1_{=} \leq_c E^0_{\cong}$ as required. (Similar constructions show that $E^1_{=} \leq_c E^{\alpha}_{\cong}$ for every $\alpha \leq \omega$, where this is the isomorphism problem for the class $\mathcal{E}_\alpha$ of computable equivalence structures with exactly α-many infinite classes.) ■

We remark that the completeness results about $F^{AC}_{\cong}$ can readily be seen also to hold of computable rational vector spaces, which form an extremely similar class of structures, and could be conjectured to hold for the class of all computable models of any other strongly minimal theory for which the independence relation is Π^0_1 and the spectrum of computable models of that theory contains all countable models of the theory. (In all such classes, the isomorphism relation is determined by the *dimension*, which is the size of a particular subset of the structure, usually a maximal independent set.) On the other hand, it would be natural to investigate other classes for which the isomorphism problem is Π^0_2, and to determine whether their isomorphism problems are also Π^0_2-complete under finitary reducibility, as in Theorem 4.6.

4.3 Distinguishing Finitary Reducibilities

Theorem 4.2 implies that 3-ary and 4-ary reducibility are distinct notions, and it is natural to attempt to extend this result to other finitary reducibilities. Above we suggested that one way to do so might be to create Π^0_2 equivalence relations in which only finitely many of the equivalence classes are themselves Π^0_2-complete as sets. (We use the class of Π^0_2-equivalence relations simply because it is the one we found useful in the preceding subsection. The same principle could be applied at the Π^0_p or other levels, for any p.) Theorem 4.12 below will prove this attempt to be in vain, but the suspicion that n-ary reducibilities are distinct for distinct n turns out to be well-founded, as we will see in Theorem **??**.

It is not difficult to create a Π^0_2 equivalence relation E on ω having exactly c distinct Π^0_2-complete equivalence classes. Define $m \; E \; n$ iff:

$$(\exists i < m)[m \equiv n \equiv i \pmod{c} \;\&\; \max(W_{\frac{m-i}{c}}) = \max(W_{\frac{n-i}{c}})].$$

This essentially just partitions ω into c distinct classes modulo c, and then partitions each of those classes further using the relation $E^{ce}_{\max}$. As with $E^{ce}_{\max}$, we intend here that $\max(W) = \max(V)$ iff W and V are both infinite or both empty or else have the same (finite) maximum. For each $i < c$, the class of those $m \equiv i (\text{mod } c)$ with $\frac{m-i}{c} \in \mathbf{Inf}$ is Π^0_2-complete, while every other class is defined by such an i along with a condition of having either a specific finite maximum (which is a Δ^0_1 condition) or being empty (which is Π^0_1).

However, this E is not complete among Π^0_2 equivalence relations under 4-ary reducibility. To build an F with $F \not\leq^4_c E$, one uses infinitely many nonconflicting basic modules, one for each e, showing that no φ_e is a 4-ary reduction from F to E. To do this, assign four specific numbers $w = 4e$, $x = 4e+1$, $y = 4e+2$ and $x = 4e+3$ to this module. Wait until all four of these computations converge: $\varphi_e(1, w, x, y, z)$, $\varphi_e(2, w, x, y, z)$, $\varphi_e(3, w, x, y, z)$, and $\varphi_e(4, w, x, y, z)$. (If any diverges, then φ_e is not total, and we define each of the four inputs to be an F-class unto itself.) If the four outputs are all congruent modulo c, then we use the same process which showed that $E^{ce}_{\max}$ is not 4-arily complete for Π^0_2 equivalence relations, since now there is only one Π^0_2 complete class to which $\varphi_e(w)$ and the rest could belong. On the other hand, if, say, $\varphi_e(1, w, x, y, z) \not\equiv \varphi_e(2, w, x, y, z) \pmod{c}$, then these two values lie in distinct E-classes, so we just make $w \; F \; x$; similarly for the other five possibilities.

Nevertheless, there is a straightforward procedure for building an equivalence relation which is 4-complete but not 5-complete among Π^0_2 equivalence relations, and it generalizes easily to larger finitary reducibilities as well, showing them all to be distinct. This will be discussed in Theorem 4.9.

Recall first the following fact.

Proposition 4.8 *For every $p \geq 0$, there exists a Σ^0_p equivalence relation which is complete under finitary reducibility $\leq^{<\omega}_c$ among Σ^0_p equivalence relations, and a Π^0_p equivalence relation which is complete under $\leq^{<\omega}_c$ among Π^0_p equivalence relations.*

Proof. For $p = 0$, equality on ω is Σ^0_0-complete (equivalently, Π^0_0-complete). For $p > 0$, it is well known that there is an equivalence relation which is Σ^0_p-complete under full computable reducibility: let $\{V_e : e \in \omega\}$ be a uniform list of the Σ^0_p sets, and take the closure of $\{(\langle e, i\rangle, \langle e, j\rangle) : \langle i, j\rangle \in V_e\}$ under reflexivity, symmetry, and transitivity. A Π^0_1-complete equivalence relation under computable reducibility was constructed in [13], and the equivalence relation $\{(i, j) : W^{\emptyset^{(p-2)}}_i = W^{\emptyset^{(p-2)}}_j\}$ is Π^0_p-complete under $\leq^{<\omega}_c$ for each $p > 1$. ■

Theorem 4.9 *For every $p \geq 0$ and every $n \geq 2$, there exists a Σ^0_p equivalence relation which is complete under n-ary reducibility $\leq^n_c$ among Σ^0_p equivalence relations, but fails to be complete among them under $\leq^{n+1}_c$. Likewise, there exists a Π^0_p equivalence relation which is complete under $\leq^n_c$ among Π^0_p equivalence relations, but not under $\leq^{n+1}_c$.*

Proof. The $p = 0$ case is trivial: every computable equivalence relation with exactly n equivalence classes clearly satisfies the theorem. Now we consider $p > 0$. We first illustrate the special case of Π^0_2 equivalence relations:

We first prove that for every $n > 1$, there exists a Π^0_2 equivalence relation E which is Π^0_2-complete under $\leq^n_c$, but not under $\leq^{n+1}_c$. Start with a computable listing $\{(a_{m,0}, \ldots, a_{m,n-1})\}_{m\in\omega}$ of all n-tuples in ω^n, without repetitions. The idea is that E should use the natural numbers $nm, nm+1, \ldots, nm+n-1$ to copy $=^{ce}$ from the m-th tuple. For $i, j \in \omega$, we define $i\ E\ j$ if and only if

$$\exists m[nm \leq i < (n+1)m \ \ \& \ \ nm \leq j < (n+1)m \ \ \& \ \ a_{m,i-mn} =^{ce} a_{m,j-mn}].$$

The last condition just says that $W_{a_{m,i-mn}} = W_{a_{m,j-mn}}$, which is Π^0_2. Of course, for each i, only $m = \lfloor \frac{i}{n} \rfloor$ can possibly satisfy the existential quantifier, so this E really is a Π^0_2 equivalence relation. Moreover, it is immediate that $=^{ce}$ has an n-reduction f to E: for each n-tuple $(x_0, \ldots, x_{n-1}) \in \omega^n$, just find the unique m with $(a_{m,0}, \ldots, a_{m,n-1}) = (x_0, \ldots, x_{n-1})$, and set $f(i, x_0, \ldots, x_{n-1}) = mn + i$. That f is an n-reduction follows directly from the design of E. But every Π^0_2 equivalence relation F has an n-reduction to $=^{ce}$, since $=^{ce}$ is complete under finitary reducibility, and so our E is complete under $\leq^n_c$ among Π^0_2 equivalence relations.

To show that E is not complete under $\leq^{n+1}_c$, we show that $=^{ce} \ \not\leq^{n+1}_c \ E$. This is surprisingly easy. Fix any $e \in \omega$, and define $x_0, \ldots, x_n$ to be the indices of the following programs, using the Recursion Theorem. The programs wait until $\varphi_e(i, x_0, \ldots, x_n)$ has converged for every $i \leq n$, say with $\hat{x}_i = \varphi_e(i, x_0, \ldots, x_n)$. If all of $\hat{x}_0, \ldots, \hat{x}_n$ lie in a single interval $[nm, (n+1)m)$ for some m, then each program x_i simply enumerates i into its set. Thus we have $x_i \neq^{ce} x_j$ for $i < j \leq n$, but some two of $\hat{x}_0, \ldots, \hat{x}_n$ must be equal, by the Pigeonhole Principle, and hence φ_e was not an $(n+1)$-reduction. On the other hand, if there exist $j < k \leq n$ for which $\hat{x}_j$ and $\hat{x}_k$ do not lie in the same interval $[nm, (n+1)m)$, then no program x_i ever enumerates anything. In this case we have $x_j =^{ce} x_k$, since both are indices of the empty set, yet $\langle \hat{x}_j, \hat{x}_k \rangle \notin E$ by the definition of E. Therefore, no φ_e can be an $(n+1)$-reduction, and so $=^{ce} \not\leq^{n+1}_c E$.

Now fix an arbitrary $p > 0$ and consider Σ^0_p equivalence relations. The technique is almost the same as above. Fix the Σ^0_p equivalence relation F which is complete among Σ^0_p equivalence relations under $\leq^{<\omega}_c$, as given in Proposition 4.8. Define $i\ E\ j$ if and only if

$$\exists m[nm \leq i < (n+1)m \ \ \& \ \ nm \leq j < (n+1)m \ \ \& \ \ a_{m,i-nm}\ F\ a_{m,j-mn}],$$

again using an effective enumeration $\{(a_{m,0}, \ldots, a_{m,n-1}) : m \in \omega\}$ of ω^n. Once again we have an n-reduction from F to E: set $f(i, x_0, \ldots, x_{n-1}) = nm + i$, where $(a_{m,0}, \ldots, a_{m,n-1}) = (x_0, \ldots, x_{n-1})$.

The same strategy as for Π^0_2 succeeds in showing that no φ_e can be an $(n+1)$-reduction from F to E, although this must be checked for the different cases. When $p > 0$, for each fixed φ_e, there is a computable reduction to the Σ^0_p-complete equivalence relation F from the Σ^0_p equivalence relation which makes $0, \ldots, n$ all equivalent if all $\varphi_e(x_i)$ converge to values in the same interval $[nm, n(m+1))$, and leaves them pairwise inequivalent otherwise.

Now consider Π^0_p for arbitrary $p > 0$. The same argument also works with Π^0_p in place of Σ^0_p. Our F, defined exactly the same way, is now a Π^0_p equivalence relation, and the n-ary reduction from E is also the same. We claim that again $E \not\leq_c^{n+1} F$. For $p > 1$, our F is equality of the sets $W_i^{\emptyset^{(n)}}$ and $W_j^{\emptyset^{(n)}}$, and so the proof using the Recursion Theorem still works, each c.e. set being also c.e. in $\emptyset^{(n)}$. For $p = 1$, let all the numbers $\leq n$ be equivalent unless, on all of those $(n+1)$ numbers, φ_e converges to values in the same interval $[nm, n(m+1))$, in which case they become pairwise inequivalent. This Π^0_1 equivalence relation must have a computable reduction to the Π^0_1-complete equivalence relation F, which therefore cannot have any $(n+1)$-ary reduction to E. ∎

Corollary 4.10 *For every $n \neq n'$ in ω, n-ary reducibility and n'-ary reducibility do not coincide.* ∎

Finally, we adapt Theorem 4.9 to compare finitary reducibility with full computable reducibility. Of course, it is already known that equality of $\emptyset^{(n)}$-c.e. sets is Π^0_{n+2}-complete under the former, but not under the latter.

Theorem 4.11 *For each $p > 0$, there exists a Σ^0_p equivalence relation E which is complete under finitary reducibility among Σ^0_p equivalence relations, but not under computable reducibility.*

Proof. Again, let F be Σ^0_p-complete under computable reducibility. This time we use an effective enumeration $\{(a_{m,0}, \ldots, a_{m,n_m})\}_{m\in\omega}$ of $\omega^{<\omega}$, and define the computable function g by $g(0) = \langle 0, 0\rangle$, and

$$g(x+1) = \begin{cases} \langle m, i+1\rangle, & \text{if } g(x) = \langle m, i\rangle \text{ with } i < n_m; \\ \langle m+1, 0\rangle, & \text{if } g(x) = \langle m, n_m\rangle. \end{cases}$$

We let $x\ E\ y$ iff there is an m with $g(x) = \langle m, j\rangle$ and $g(y) = \langle m, k\rangle$ and $a_{m,j}\ F\ a_{m,k}$. Since F is Σ^0_p, so is E, and the finitary reduction from F to E is given by $h(i, x_0, \ldots, x_n) = g^{-1}(\langle m, i\rangle)$, where $(x_0, \ldots, x_n) = (a_{m,0}, \ldots, a_{m,n_m})$. With F Σ^0_p-complete under $\leq_c$, this makes E Σ^0_p-complete under $\leq_c^{<\omega}$. But for each computable total function f (which you think might be a full computable reduction from F to E), there would be a computable reduction to E from a particular slice of F (say the c-th slice) on which we wait until $f(\langle c, 0\rangle)$ converges to some number $\langle m, k\rangle$, then wait until f has converged on each of $\langle c, 1\rangle, \ldots, \langle c, 1+n_m\rangle$ as well, and define these $(2+n_m)$ elements to be in distinct F-classes if f maps each of them to a pair of the form $\langle m, j\rangle$ for the same m, or else all to be in the same F-class if not. As usual, this shows that f cannot have been a computable reduction. ∎

So we have answered the basic question. However, the proof did not involve any equivalence relation with only finitely many Π^0_2-complete equivalence classes, as we had originally guessed it would. Indeed, 4-completeness for Π^0_2 equivalence relations turns out to require a good deal more than just two Π^0_2-complete equivalence classes, as we now explain.

Say that a total computable function h is a Π^0_2-approximating function for an equivalence relation E if

$$(\forall x \forall y)[x\ E\ y \iff \exists^\infty s\ h(x,y,s) = 1].$$

(We may assume that h has range $\subseteq \{0,1\}$. Every Π^0_2 equivalence relation has such a function h.) We say that, under this h, a particular E-class $[z]_E$ is Δ^0_2 if, for all $x, y \in [z]_E$, we have $\lim_s h(x,y,s) = 1$. Of course, if $x \in [z]_E$ and $y \notin [z]_E$, then $\lim_s h(x,y,s) = 0$, so in this case the class $[z]_E$ really is Δ^0_2, uniformly in any single element x in the class. On the other hand, even if $[z]_E$ is not Δ^0_2 under this h, it could still be a Δ^0_2 set, under some other computable approximation. For this reason, our next theorem does not preclude the possibility that cofinitely many E-equivalence classes might be Δ^0_2, but it does say that cofinitely many classes cannot be uniformly limit-computable.

For an example of these notions, let E be the relation $E^{ce}_{\max}$, saying of i and j that W_i and W_j have the same maximum. More formally, $i\ E^{ce}_{\max}\ j$ iff

$$(\forall x \forall s \exists t \exists y, z \geq x)[(x \in W_{i,s} \implies y \in W_{j,t})\ \&\ (x \in W_{j,s} \implies z \in W_{i,t})].$$

We can define h here by letting $h(i,j,s) = 1$ when either $\max(W_{i,s}) = \max(W_{j,s})$ or else $\max(W_{i,s}) > \max(W_{i,t})$ and $\max(W_{j,s}) > \max(W_{j,t})$ (where t is the greatest number $< s$ with $h(i,j,t) = 1$), and taking $h(i,j,s) = 0$ otherwise. Then the $E^{ce}_{\max}$-class **Inf** of those i with W_i infinite is the only class which fails to be Δ^0_2 under this h, and since the set **Inf** is in fact Π^0_2-complete, it cannot be Δ^0_2 under any other h either. Recall that $E^{ce}_{\max}$ is complete among Π^0_2 equivalence relations under $\leq^3_c$, but not under $\leq^4_c$. The following theorem generalizes this result.

Theorem 4.12 *Suppose that E is complete under $\leq^4_c$ among Π^0_2 equivalence relations. Let h be any computable Π^0_2-approximating function for E. Then E must contain infinitely many equivalence classes which are not Δ^0_2 under this h.*

Proof. Suppose that $z_0, \ldots, z_n$ were numbers such that $\langle z_i, z_j \rangle \notin E$ for each $i < j$, and such that every E-class except these $(n+1)$ classes $[z_i]_E$ is Δ^0_2 under h. For each e, we will build four c.e. sets which show that φ_e is not a 4-reduction from the relation $=^{ce}$ to E. (Recall that $i =^{ce} j$ iff $W_i = W_j$, and that this Π^0_2-equivalence relation is complete under finitary reducibility, making it a natural choice to show 4-incompleteness of E.)

Fix any e, and choose four fresh indices a, b, c and d of c.e. sets $A = W_a$, $B = W_b$, $C = W_c$, and $D = W_d$, which we enumerate according to the following instructions. First, we wait until $\varphi_e(i,a,b,c,d)$ has converged for each $i < 4$. (By the Recursion Theorem, these indices may be assumed to know their own values.) Set $\hat{a} = \varphi_e(0,a,b,c,d)$, $\hat{b} = \varphi_e(1,a,b,c,d)$, etc. If φ_e is a 4-reduction, then $A = B$ iff $\hat{a}\ E\ \hat{b}$, and $A = C$ iff $\hat{a}\ E\ \hat{c}$, and so on.

At an odd stage $2s+1$, we first compare $\hat{a}$ and $\hat{b}$, using the computable Π^0_2-approximating function h for E. If $h(\hat{a}, \hat{b}, s) = 1$ and $A_{2s} = B_{2s}$, then we add to A_{2s+1} some even number not in B_{2s}, so $A_{2s+1} \neq B_{2s+1}$. On the other hand, if

$h(\hat{a}, \hat{b}, s) = 0$ and $A_{2s} \neq B_{2s}$, then we make $A_{2s+1} = B_{2s+1} = A_{2s} \cup B_{2s}$. (The purpose of these maneuvers is to ensure that $\lim_s h(\hat{a}, \hat{b}, s)$ diverges, so that $\hat{a}$ and $\hat{b}$ lie in one of the properly Π^0_2 E-classes.)

Next we do exactly the same procedure with $\hat{c}$ and $\hat{d}$ in place of $\hat{a}$ and $\hat{b}$, and using a new odd number if needed, instead of a new even number. This completes stage $2s+1$, ensuring that $\lim_s h(\hat{c}, \hat{d}, s)$ also diverges.

At stage $2s+2$, fix the $i \leq n$ such that $h(\hat{a}, z_i, s') = 1$ for the greatest possible $s' \leq s$, and similarly the $j \leq n$ such that $h(\hat{c}, z_j, s'') = 1$ for the greatest possible $s'' \leq s$. (If there are several such i, choose the least; likewise for j. If there is no such i or no such j, then we do nothing at this stage.) If $i = j$, then add a new even number to both A_{2s+2} and B_{2s+2}, thus ensuring that they are both distinct from C_{2s+2} and D_{2s+2} (and keeping $A_{2s+2} = B_{2s+2}$ iff $A_{2s+1} = B_{2s+1}$). If $i \neq j$, then we add all the even numbers in A_{2s+1} to both C_{2s+2} and D_{2s+2}, and add all the odd numbers in C_{2s+1} to both A_{2s+2} and B_{2s+2}. (This is the only step in which even numbers are enumerated into C or D, or odd numbers into A or B.) This completes stage $2s+2$, and the construction.

We claim first that the odd stages succeeded in their purpose of making $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ all belong to properly Π^0_2 E-classes. At each stage $2s+1$ such that $h(\hat{a}, \hat{b}, s) = 1$, we made A_{2s+1} contain a new even number, which only subsequently entered B if $A_{2s'} = B_{2s'}$ at some stage $s' > s$. Therefore, if $\lim_s h(\hat{a}, \hat{b}, s) = 1$, this even number would show $A \neq B$, yet $\hat{a}\ E\ \hat{b}$, so that φ_e would not be a 4-reduction. So there are infinitely many s with $h(\hat{a}, \hat{b}, s) = 0$, and at all corresponding stages $2s+1$ we made $A_{2s+1} = B_{2s+1}$, which implies $A = B$. If φ_e is a 4-reduction, then we must have $\hat{a}\ E\ \hat{b}$, so there were infinitely (but also coinfinitely) many s with $h(\hat{a}, \hat{b}, s) = 1$. Therefore $\lim_s h(\hat{a}, \hat{b}, s)$ diverged, and so the E-class of $\hat{a}$ must be one of the $[z_i]_E$ with $i \leq n$, with $\hat{b}$ lying in the same class. We now fix this i. A similar analysis on $\hat{c}$ and $\hat{d}$ shows that they both lie in one particular E-class $[z_j]_E$ with $j \leq n$, and that $C = D$.

Recall that $z_0, \ldots, z_n$ were chosen as representatives of distinct E-classes. Therefore, there must exist some stage s_0 such that, at all stages $s > s_0$, we had $h(\hat{a}, z_k, s) = 0 = h(\hat{b}, z_k, s)$ for every $k \neq i$, and also $h(\hat{c}, z_k, s) = 0 = h(\hat{d}, z_k, s)$ for every $k \neq j$. Moreover, we know that $i = j$ iff $z_i\ E\ z_j$. If indeed $i = j$, then at every even stage $> 2s_0$ we were in the $i = j$ situation, and we added a new even number to A and B at each such stage, while no even numbers were added to either C or D at any stage $> 2s_0$. Therefore, if $i = j$, we would have $A \neq C$, yet $\hat{a}\ E\ z_i\ E\ \hat{c}$, which would show that φ_e is not a 4-reduction. On the other hand, if $i \neq j$, then at every even stage $> 2s_0$ we were in the $i \neq j$ situation, and so all even numbers ever added to A were subsequently added to both C and D, and all odd numbers in C were subsequently added to both A and B. However, no odd numbers were ever added to A or B except numbers already in C, and no even numbers were ever added to C or D except numbers already in A. So we must have $A = B = C = D$, yet $\hat{a}\ E\ z_i$ and $\hat{c}\ E\ z_j$, which lie in distinct E-classes. So once again φ_e cannot have been a 4-reduction from $=^{ce}$ to E. This same argument works for every e (by a separate argument for each; there is no need to combine them), and so $=^{ce}\ \not\leq^4_c\ E$. ■

It remains open whether an equivalence relation E which is Π^0_2-complete under $\leq^4_c$ might have cofinitely many (or possibly all) of its classes be Δ^0_2 in some nonuniform way.

5 Questions

Computable reducibility has been independently invented several times, but many of its inventions were inspired by the analogy to Borel reducibility on 2^ω. Therefore, when a new notion appears in computable reducibility, it is natural to ask whether one can repay some of this debt by introducing the analogous notion in the Borel context. We have not attempted to do so here, but we encourage researchers in Borel reducibility to consider this idea. First, do the obvious analogues of n-ary and finitary reducibility bring anything new to the study of Borel reductions? And second, in the context of 2^ω, could one not also ask about ω-reducibility? A Borel ω-reduction from E to F would take an arbitrary countable subset $\{x_0, x_1, \ldots\}$ of 2^ω, indexed by naturals, and would produce corresponding reals $y_0, y_1, \ldots$ with $x_i \ E \ x_j$ iff $y_i \ F \ y_j$. Obviously, a Borel reduction from E to F immediately gives a Borel ω-reduction, and when the study of Borel reducibility is restricted to Borel relations on 2^ω, such ω-reductions always exist. The interesting situation would involve E and F which are not Borel and for which $E \not\leq_B F$: could Borel ω-reductions (or finitary reductions) be of use in such situations? And finally, if the Continuum Hypothesis fails, could the same hold true of κ reductions, or $< \kappa$-reductions, for other $\kappa < 2^\omega$?

There are plenty of specific questions to be asked about computable finitary reducibility. Computable reductions have become a basic tool in computable model theory, being used to compare classes of computable structures under the notion of Turing-computable embeddings (as in [3, 4], for example). In situations where no computable reduction exists, finitary reducibility could aid in investigating the reasons why: is there not even any binary reduction? Or is there a computable finitary reduction, but no computable reduction overall? Or possibly the truth lies somewhere in between? Finitary reducibility has served to answer such questions in several contexts already, as seen in Subsection 4.2, and one hopes for it to be used to sharpen other results as well.

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