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Notes on *n*-D Polynomial Matrix Factorizations

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Abstract. This paper discusses a relationship between the prime factorizability of a normal full rank n-D ($n > 2$) polynomial matrix and its reduced minors. Two conjectures regarding the n-D polynomial matrix prime factorization problem are posed, and a partial solution to one of the conjectures is provided. Another related open problem of factorizing an n-D polynomial matrix that is not of normal full rank as a product of two n-D polynomial matrices of smaller size is also considered, and a partial solution to this problem is presented. An illustrative example is worked out in details.

Key Words: n-D systems, n-D polynomial matrices, matrix factorizations, reduced minors, Gröbner bases, Quillen–Suslin theorem

1. Introduction

The problems of multivariate (n-D) polynomial matrix factorizations have attracted much attention over the past decades because of their wide applications in circuits, systems, controls, signal processing and other areas (see, e.g., [1]–[13]). One of such a factorization problem is to decompose a normal full rank n-D polynomial matrix into a product of two n-D polynomial matrices, with one of them being prime in some sense [3], [6], [11]. This prime factorization problem has long been solved for 1-D and 2-D polynomial matrices [3], [4], [14]. However, it is a challenging open problem for n-D ($n > 2$)[1] polynomial matrices [6], [11], because of some fundamental differences between n-D polynomial matrices and their 1-D and 2-D counterparts [6], [7], [11]. Although some recent efforts have been made towards solving this and other related factorization problems [8]–[13], the prime factorization problem remains largely unresolved.

In this paper, we attempt to establish a relationship between the prime factorizability of a normal full rank n-D polynomial matrix and its reduced minors by posing two conjectures. As a partial solution to one of the conjectures, we present a simple sufficient condition for the factorizability of a class of n-D polynomial matrices. When a matrix in this class is factorizable, a constructive method is provided to carry out the actual factorization. As a by-product, we also show how to factorize some special n-D polynomial matrix that is not of normal full rank as a product of two n-D polynomial matrices of smaller size. The new results are derived by exploiting the celebrated Quillen–Suslin theorem [15]–[17] that can now be implemented using the efficient Gröbner basis approach [18]–[21], and some properties of reduced minors [7], [12], [22].

The organization of the paper is as follows. In the next section, we recall some definitions, and then raise two conjectures regarding zero and minor prime factorizations for n-D polynomial matrices. A partial solution to one of the conjectures posed is presented in Section 3, along with new results on factorizations of a class of n-D polynomial matrices that are not of full normal rank. An example is illustrated in Section 4 and conclusions are given in Section 5.

2. Preliminaries and Problem Formulation

In the following, we shall denote $\mathbf{C}(\mathbf{z}) = \mathbf{C}(z_1, \ldots, z_n)$ the set of rational functions in complex variables $z_1, \cdots, z_n$ with coefficients in the field of complex numbers $\mathbf{C}$; $\mathbf{C}[\mathbf{z}]$ the set of polynomials in complex variables $z_1, \cdots, z_n$ with coefficients in $\mathbf{C}$; $\mathbf{C}^{m\times l}[\mathbf{z}]$ the set of $m \times l$ matrices with entries in $\mathbf{C}[\mathbf{z}]$, *etc.* Throughout this paper, the argument $(\mathbf{z})$ is omitted whenever its omission does not cause confusion.

DEFINITION 1: ([6]) *Let $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$, with $m \geq l$. Then F is said to be:*

(i) zero right prime (ZRP) if there exists no n-tuple $\mathbf{z}^0 \in \mathbf{C}^n$ which is a common zero of the $l \times l$ minors of F;

(ii) minor right prime (MRP) if the $l \times l$ minors of F are factor coprime;

(iii) factor right prime (FRP) if in any polynomial decomposition $F = F_1 F_2$, the $l \times l$ matrix F_2 is a unimodular matrix, i.e., $\det F_2 = k_0 \in \mathbf{C}^*$.[2]

Zero left prime (ZLP) and minor left prime (MLP) *etc*. can be similarly defined.

DEFINITION 2: ([7], [22]) *Let $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank,*[3] *with $m > l$, and let $a_1, \ldots, a_\beta$ denote the $l \times l$ minors of the matrix F, where $\beta = \binom{m}{l} = \frac{m!}{(m-l)!l!}$. Extracting the greatest common divisor (g.c.d.) d of $a_1, \ldots, a_\beta$ gives:*

$$a_i = d b_i, \qquad i = 1, \ldots, \beta. \tag{1}$$

Then, $b_1, \ldots, b_\beta$ are called the "reduced minors" (or equivalently, the "generating polynomials") of F.

Reduced minors of a normal full rank matrix $\tilde{F} \in \mathbf{C}^{m\times l}[\mathbf{z}]$, with $m < l$, can be defined by replacing F with F^T in Definition 2, where $(\cdot)^T$ denotes transposition. We do not define reduced minors for a square matrix.

Consider now a normal full rank n-D polynomial matrix $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$ with $m > l$. Let $a_1, \ldots, a_\beta$ denote the $l \times l$ minors of F, $b_1, \ldots, b_\beta$ denote the reduced minors of F. By Definition 2, a_i and b_i are related by:

$$a_i = d b_i, \qquad i = 1, \ldots, \beta. \tag{2}$$

Throughout the paper, we assume that d is not a non-zero constant. Although we only consider the case when $m > l$ for convenience of exposition, the results presented can be easily applied to the case when $m < l$ with minor modification. The prime factorization problem considered here is to factorize F as:

$$F = F_0 G_0 \tag{3}$$

with $F_0 \in \mathbf{C}^{m\times l}[\mathbf{z}]$, $G_0 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ and $\det G_0 = d$. We feel that the prime factorizability of an n-D polynomial matrix may be related to its reduced minors and pose the following two conjectures:

CONJECTURE 1: *If $b_1, \ldots, b_\beta$ have no common zeros in $\mathbf{C}^n$, then F can be factorized as in (3) with F_0 being ZRP and* $\det G_0 = d$.

CONJECTURE 2: *If $d, b_1, \ldots, b_\beta$ have no common zeros in $\mathbf{C}^n$, then F can be factorized as in (3) with F_0 being MRP and* $\det G_0 = d$.

When F admits factorization (3) with F_0 being ZRP (MRP), we say that F has a ZRP (MRP) factorization. Recently, Bose and Charoenlarpnopparut have also considered the same n-D ZRP factorization problem [13]. By making use of Gröbner bases for modules, they have proposed an algorithm for carrying out the ZRP factorization, with assumptions that $b_1, \ldots, b_\beta$ have no common zeros in $\mathbf{C}^n$, and that a ZRP factorization for F **exists**. However, given an arbitrary n-D polynomial matrix F with a nontrivial d, it is still unknown whether there exists a ZRP factorization for F, even when $b_1, \ldots, b_\beta$ have no common zeros in $\mathbf{C}^n$. In view of the ZRP factorization algorithm proposed in [13], the critical question now is to show the existence of a ZRP factorization for F. In this paper, we prove that ZRP factorizations do exist for a class of n-D polynomial matrices. We also propose alternative methods for carrying out ZRP factorizations for this class of matrices.

We next consider another closely related polynomial matrix factorization problem. Let $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal rank $r < \min\{m, l\}$. we would like to know whether F_1 can be factorized as:

$$F_1 = F_2 G_2 \tag{4}$$

with $F_2 \in \mathbf{C}^{m\times r}[\mathbf{z}]$ and $G_2 \in \mathbf{C}^{r\times l}[\mathbf{z}]$. Youla and Gnavi [6] have shown that such a factorization is always possible for 1-D and 2-D polynomial matrices, but not for their n-D counterparts in general. However, to our best knowledge, there is no algorithm available to determine whether or not F_1 can be factorized as in (4). In this paper, we also solve the factorization problem (4) for a class of n-D polynomial matrices.

3. Main Results

We first require two lemmas.

LEMMA 1: *Let $A \in \mathbf{C}^{k\times m}[\mathbf{z}]$ be ZLP with $k < m$. Then there exists a ZRP matrix $B \in \mathbf{C}^{m\times l}[\mathbf{z}]$, with $l = m - k$, such that*[4]

$$A B = 0_{k,l}. \tag{5}$$

Moreover, if $B_1 \in \mathbf{C}^{m\times r}[\mathbf{z}]$, *where* r *is a positive integer, such that*

$$A\,B_1 = 0_{k,r}, \tag{6}$$

then

$$B_1 = B\,G \tag{7}$$

for some $G \in \mathbf{C}^{l\times r}[\mathbf{z}]$.

Proof: Since A is ZLP, there exists $H \in \mathbf{C}^{l\times m}[\mathbf{z}]$ such that the matrix $U = [H^T \;\; A^T]^T \in \mathbf{C}^{m\times m}[\mathbf{z}]$ is a unimodular matrix, i.e., $\det U = 1$. This is in fact a result of the celebrated Quillen–Suslin theorem [15]–[17], and there are now algorithms for solving such a matrix completion problem [19]–[21]. Clearly, $V = U^{-1} \in \mathbf{C}^{m\times m}[\mathbf{z}]$ is also a unimodular matrix. Partition V as $V = [B \;\; T]$, where $B \in \mathbf{C}^{m\times l}[\mathbf{z}]$, $T \in \mathbf{C}^{m\times k}[\mathbf{z}]$. We have

$$UV = \begin{bmatrix} H \\ A \end{bmatrix} [B \;\; T] = \begin{bmatrix} I_l & 0_{l,k} \\ 0_{k,l} & I_k \end{bmatrix}, \tag{8}$$

or

$$A\,B = 0_{k,l}. \tag{9}$$

Since $HB = I_l$, B is ZRP [6], [11]. Now consider an arbitrary matrix $B_1 \in \mathbf{C}^{m\times r}[\mathbf{z}]$, where r is a positive integer, such that

$$A\,B_1 = 0_{k,r}. \tag{10}$$

Combining (8) and (10) leads to

$$\begin{bmatrix} H \\ A \end{bmatrix} [B_1 \;\; B \;\; T] = \begin{bmatrix} G & I_l & 0_{l,k} \\ 0_{k,r} & 0_{k,l} & I_k \end{bmatrix}, \tag{11}$$

where $G = HB_1 \in \mathbf{C}^{l\times r}[\mathbf{z}]$. Simple algebra on (11) gives

$$\begin{bmatrix} H \\ A \end{bmatrix} [B_1 - BG \;\; B \;\; T] = \begin{bmatrix} 0_{l,r} & I_l & 0_{l,k} \\ 0_{k,r} & 0_{k,l} & I_k \end{bmatrix}, \tag{12}$$

or

$$U[B_1 - BG] = 0_{m,r}. \tag{13}$$

Since $\det U = 1$, we must have $[B_1 - BG] = 0_{m,r}$, or $B_1 = BG$. ■

LEMMA 2: *Let* $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$ *be of normal full rank with* $m = l + 1$, $b_1, \ldots, b_\beta$ *be the reduced minors of* F. *If* $b_1, \ldots, b_\beta$ *have no common zeros in* $\mathbf{C}^n$, *then there exists a ZLP row vector* $\tilde{\mathbf{b}}_0 \in \mathbf{C}^{1\times m}[\mathbf{z}]$ *such that*

$$\tilde{\mathbf{b}}_0\,F = 0_{1,l}. \tag{14}$$

Proof: Since F is of normal full rank and of size $m \times l$ with $m = l + 1$, without loss of generality, we can assume that the $l \times l$ submatrix $D \in \mathbf{C}^{l \times l}[\mathbf{z}]$ formed from the first l rows of F is nonsingular, i.e., $\det D \not\equiv 0$ and $F = [D^T \; N^T]^T$, where $N \in \mathbf{C}^{1 \times l}[\mathbf{z}]$. Define an n-D rational matrix $P = ND^{-1} \in \mathbf{C}^{1 \times l}(\mathbf{z})$ and obtain a left matrix fraction description (MFD) of $P = \tilde{D}^{-1}\tilde{N}$ where $\tilde{D} \in \mathbf{C}[\mathbf{z}]$, $\tilde{N} \in \mathbf{C}^{1 \times l}[\mathbf{z}]$. Since $P = \tilde{D}^{-1}\tilde{N} = ND^{-1}$, we have

$$[-\tilde{N} \;\; \tilde{D}] \begin{bmatrix} D \\ N \end{bmatrix} = 0_{1,l} \tag{15}$$

or

$$\tilde{\mathbf{b}}_0 \, F = 0_{1,l} \tag{16}$$

where $\tilde{\mathbf{b}}_0 = [-\tilde{N} \;\; \tilde{D}] = [\tilde{b}_1 \cdots \tilde{b}_m]$. Without loss of generality, we can assume that $\tilde{\mathbf{b}}_0$ is already MLP, for otherwise one can always pull out the g.c.d. of $\tilde{b}_1, \ldots, \tilde{b}_m$ using methods available in the literature (see, e.g., [1]).Hence, the reduced minors of $\tilde{\mathbf{b}}_0$ are just $\tilde{b}_1, \ldots, \tilde{b}_m$. According to a known result on reduced minors associated with MFDs of an n-D rational matrix [7], we have

$$b_i = \pm \tilde{b}'_i, \qquad i = 1, \ldots, m, \tag{17}$$

where $\tilde{b}'_1, \ldots, \tilde{b}'_m$ are obtained by reordering $\tilde{b}_1, \ldots, \tilde{b}_m$ appropriately. Since $b_1, \ldots, b_\beta$ have no common zeros in $\mathbf{C}^n$, it follows from (17) that $\tilde{b}'_1, \ldots, \tilde{b}'_m$ and hence $\tilde{b}_1 \cdots \tilde{b}_m$ have no common zeros in $\mathbf{C}^n$. This implies that $\tilde{\mathbf{b}}_0$ is ZLP. ■

We now present a simple necessary and sufficient condition for the ZRP factorizability of $F \in \mathbf{C}^{m \times l}[\mathbf{z}]$ when $m = l + 1$.

PROPOSITION 1: *Let $F \in \mathbf{C}^{m \times l}[\mathbf{z}]$ be of normal full rank with $m = l + 1$, $a_1, \ldots, a_\beta$ be the $l \times l$ minors of F, $b_1, \ldots, b_\beta$ be the reduced minors of F, i.e., $a_i = db_i$ $(i = 1, \ldots, \beta)$. F can be factorized as*

$$F = F_0 G_0 \tag{18}$$

where $F_0 \in \mathbf{C}^{m \times l}[\mathbf{z}]$ is ZRP, and $G_0 \in \mathbf{C}^{l \times l}[\mathbf{z}]$ with $\det G_0 = d$, if and only if $b_1, \ldots, b_\beta$ have no common zeros in $\mathbf{C}^n$.

Proof: (Sufficiency) Assume that $b_1, \ldots, b_\beta$ have no common zeros in $\mathbf{C}^n$. By Lemma 2, there exists a ZLP row vector $\tilde{\mathbf{b}}_0 \in \mathbf{C}^{1 \times m}[\mathbf{z}]$ such that

$$\tilde{\mathbf{b}}_0 \, F = 0_{1,l}. \tag{19}$$

By Lemma 1, there exists a ZRP matrix $F_0 \in \mathbf{C}^{m \times l}[\mathbf{z}]$, such that

$$\tilde{\mathbf{b}}_0 \, F_0 = 0_{1,l}. \tag{20}$$

Since $\tilde{\mathbf{b}}_0 F = 0_{1,l}$, applying Lemma 1 gives

$$F = F_0 G_0 \tag{21}$$

where $G_0 \in \mathbf{C}^{l\times l}[\mathbf{z}]$. It remains to show that $\det G_0 = d$. Let $\det G_0 = g$ and $f_1, \ldots, f_\beta$ be the $l \times l$ minors of F_0. From (21), we have

$$a_i = f_i g, \qquad i = 1, \ldots, \beta, \tag{22}$$

or

$$db_i = g f_i, \qquad i = 1, \ldots, \beta. \tag{23}$$

Since F_0 is ZRP, $f_1, \ldots, f_\beta$ have no nontrivial common divisors. Hence, d must be a divisor of g. On the other hand, since $b_1, \ldots, b_\beta$ have no nontrivial common divisors by Definition 2, g must be a divisor of d. Hence, we have $g = k_0 d$ for some $k_0 \in \mathbf{C}^*$. We may assume that $k_0 = 1$. Therefore, $g = d$, or $\det G_0 = d$.

Necessity: Assume that $F = F_0 G_0$, with F_0 being ZRP and $\det G_0 = d$. Arguing similarly as in the above proof for sufficiency, we can also arrive at equation (23) with $g = d$, i.e.

$$b_i = f_i, \qquad i = 1, \ldots, \beta. \tag{24}$$

The assumption F_0 being ZRP implies that $f_1, \ldots, f_\beta$ have no common zeros in $\mathbf{C}^n$. It is then clear from (24) that $b_1, \ldots, b_\beta$ cannot have any common zero in $\mathbf{C}^n$. ■

The main reason for requiring the size of the matrix F to be $(l+1) \times l$ in Lemma 2 and Proposition 1 is that it guarantees the existence of a ZLP row vector $\tilde{\mathbf{b}}_0$ such that $\tilde{\mathbf{b}}_0 F = 0_{1,l}$, as algorithms are available for extracting the g.c.d. of $\tilde{b}_1, \ldots, \tilde{b}_m$ [1]. If F is of size $(l+k) \times l$ where $k > 1$, we do not know whether there exists a ZLP matrix $\tilde{B}_0 \in \mathbf{C}^{k\times(l+k)}[\mathbf{z}]$ such that $\tilde{B}_0 F = 0_{k,l}$. In fact, it is even not known whether there exists a ZLP row vector $\tilde{\mathbf{b}}_0 \in \mathbf{C}^{1\times(l+k)}[\mathbf{z}]$ such that $\tilde{\mathbf{b}}_0 F = 0_{1,l}$ when $k > 1$. We shall consider this problem in more details later.

We next apply Proposition 1 to the factorization of an n-D polynomial matrix that is not of normal full rank as a product of two n-D polynomial matrices of smaller size.

COROLLARY 1: *Let $F_1 \in \mathbf{C}^{m\times r}[\mathbf{z}]$ be of normal rank l with $m = l + 1$ and $r > l$. If there exists an $m \times l$ submatrix F of F_1, such that the reduced minors of F have no common zeros in $\mathbf{C}^n$, then F_1 can be factorized as*

$$F_1 = F_2 G_2 \tag{25}$$

where $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times r}[\mathbf{z}]$.

Proof: Without loss of generality, we can assume that the $m \times l$ submatrix F is formed from the first l columns of F_1. That is, $F_1 = [F \quad A]$, where $A \in \mathbf{C}^{m\times(r-l)}[\mathbf{z}]$. Since the reduced minors of F have no common zeros in $\mathbf{C}^n$, by Proposition 1, we have $F = F_2 G_0$, where $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_0 \in \mathbf{C}^{l\times l}[\mathbf{z}]$, or

$$F_1 = [F \quad A] = [F_2G_0 \quad A] = [F_2 \quad A]\begin{bmatrix} G_0 & 0_{l,r-l} \\ 0_{r-l,l} & I_{r-l} \end{bmatrix}. \tag{26}$$

We first show that the normal rank of $[F_2 \quad A]$ is equal to l. Since $[F \quad A]$ and F are both of normal rank l, all the columns of A can be generated by linear combinations of the l columns of F over $C(\mathbf{z})$, i.e., there exists $W \in \mathbf{C}^{l\times(r-l)}(\mathbf{z})$ such that $FW = A$, or $F_2G_0W = A$. Hence $F_2W_0 = A$ for $W_0 = G_0W \in \mathbf{C}^{l\times(r-l)}(\mathbf{z})$. This implies that all the columns of A can be generated by linear combinations of the l columns of F_2 over $C(\mathbf{z})$. Therefore $[F_2 \quad A]$ is of normal rank l. Since F_2 is ZRP, according to a result due to Youla and Gnavi [6], we can factorize $[F_2 \quad A]$ as:

$$[F_2 \quad A] = F_2G_1 \tag{27}$$

for some $G_1 \in \mathbf{C}^{l\times r}[\mathbf{z}]$. Substituting (27) into (26) gives

$$F_1 = F_2G_2 \tag{28}$$

where

$$G_2 = G_1\begin{bmatrix} G_0 & 0_{l,r-l} \\ 0_{r-l,l} & I_{r-l} \end{bmatrix} \in \mathbf{C}^{l\times r}[\mathbf{z}]. \tag{29}$$

The proof is thus completed. ■

We next show that under certain condition, Conjecture 1 is also true for $m = l + 2$. We require first the following lemma.

LEMMA 3: *Let $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$, $F_2 \in \mathbf{C}^{r\times l}[\mathbf{z}]$ and $U \in \mathbf{C}^{m\times r}[\mathbf{z}]$ such that*

$$F_1 = UF_2 \tag{30}$$

with $m \geq r > l$. Let $b_{11}, \ldots, b_{1\beta}$ be the reduced minors of F_1, where $\beta = \binom{m}{l}$, d_1 be the g.c.d. of all the $l \times l$ minors of F_1, $b_{21}, \ldots, b_{2\alpha}$ be the reduced minors of F_2, where $\alpha = \binom{r}{l}$, and d_2 be the g.c.d. of all the $l \times l$ minors of F_2. If U is ZRP, then $d_1 = k_0d_2$ for some $k_0 \in \mathbf{C}^*$ and $b_{11}, \ldots, b_{1\beta}$ and $b_{21}, \ldots, b_{2\alpha}$ share the same set of common zeros.

Proof: Let $a_{11}, \ldots, a_{1\beta}$ denote the $l \times l$ minors of the matrix F_1, and $a_{21}, \ldots, a_{2\alpha}$ denote the $l \times l$ minors of the matrix F_2. By Definition 2, we have

$$a_{1i} = d_1b_{1i}, \qquad i = 1, \ldots, \beta, \tag{31}$$

and

$$a_{2j} = d_2b_{2j}, \qquad j = 1, \ldots, \alpha. \tag{32}$$

Let U_i denote the $l \times r$ matrix formed by selecting the rows $i_1, \ldots, i_l$ $(1 \leq i_1 < \cdots < i_l \leq m)$ from U, and let $q_{i1}, \ldots, q_{i\alpha}$ denote the $l \times l$ minors of U_i. From (30), and by using the Cauchy-Binet formula [24], it follows that

$$\begin{aligned} a_{1i} &= \sum_{j=1}^{\alpha} q_{ij} a_{2j} \\ &= \sum_{j=1}^{\alpha} q_{ij} d_2 b_{2j} \\ &= d_2 \sum_{j=1}^{\alpha} q_{ij} b_{2j} \qquad i = 1, \ldots, \beta. \end{aligned} \tag{33}$$

Thus, d_2 is a common divisor of $a_{11}, \ldots, a_{1\beta}$. Since by assumption, d_1 is the g.c.d. of $a_{11}, \ldots, a_{1\beta}$, d_2 is necessarily a divisor of d_1.

Next, since U is ZRP, there exists $W \in \mathbf{C}^{r \times m}[\mathbf{z}]$ such that $WU = I_r$ [6], [21]. Premultiplying (30) by W leads to:

$$F_2 = W F_1 \tag{34}$$

It can be similarly argued as above that d_1 is a divisor of d_2. Therefore, $d_1 = k_0 d_2$ for some $k_0 \in \mathbf{C}^*$.

Substituting (31) and $d_1 = k_0 d_2$ into (33) and canceling d_2 from both sides gives

$$k_0 b_{1i} = \sum_{j=1}^{\alpha} q_{ij} b_{2j} \qquad i = 1, \ldots, \beta. \tag{35}$$

It follows that a common zero of $b_{21}, \ldots, b_{2\alpha}$ is necessarily a common zero of $b_{11}, \ldots, b_{1\beta}$. Starting from (34), it can be similarly shown that a common zero of $b_{11}, \ldots, b_{1\beta}$ is also a common zero of $b_{21}, \ldots, b_{2\alpha}$. Therefore, $b_{11}, \ldots, b_{1\beta}$ and $b_{21}, \ldots, b_{2\alpha}$ share the same set of common zeros. ■

Remark 1. When $m = r$, the above lemma reduces to Lemma 1 in [12]. It should be pointed out that Lemma 3 does not hold in general for $m < r$. This is because when $m < r$, (30) does not imply (34), since there does not exist any W such that $WU = I_r$. We also notice that although $b_{11}, \ldots, b_{1\beta}$ and $b_{21}, \ldots, b_{2\alpha}$ share the same set of common zeros, the family of $b_{11}, \ldots, b_{1\beta}$ are in general different from the family of $b_{21}, \ldots, b_{2\alpha}$.

A special case of Lemma 3 is when F_2 is ZRP, i.e., $d_2 = 1$ and $b_{21}, \ldots, b_{2\alpha}$ have no common zeros in $\mathbf{C}^n$. For this case, Lemma 3 reduces to the following corollary.

COROLLARY 2: *If both $U \in \mathbf{C}^{m \times r}[\mathbf{z}]$ and $F_2 \in \mathbf{C}^{r \times l}[\mathbf{z}]$ are ZRP, then the product $F_1 = U F_2$ is also ZRP.*

We now show that Conjecture 1 is also true for matrices of dimension $(l + 2) \times l$ under certain condition.

PROPOSITION 2: *Let $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank with $m = l + 2$, and let d be the g.c.d. of the $l \times l$ minors of F_1. If there exists an $(l + 1) \times l$ submatrix F of F_1, such that the reduced minors of F, denoted by $b_1, \ldots, b_{l+1}$, have no common zeros in $\mathbf{C}^n$, then F_1 can be factorized as*

$$F_1 = F_2 G_2 \tag{36}$$

where $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_2 = d$.

Proof: Without loss of generality, we can assume that F is formed from the first $l+1$ rows of F_1. That is, $F_1 = [F^T \;\; \tilde{\mathbf{f}}_1^T]^T$, where $\tilde{\mathbf{f}}_1 \in \mathbf{C}^{1\times l}[\mathbf{z}]$. Since F is of dimension $(l + 1) \times l$ and its reduced minors $b_1, \ldots, b_{l+1}$ have no common zeros in $\mathbf{C}^n$, by Lemma 2, there exists a ZLP row vector $\tilde{\mathbf{f}}_0 \in \mathbf{C}^{1\times(l+1)}[\mathbf{z}]$ such that

$$\tilde{\mathbf{f}}_0 F = 0_{1,l}. \tag{37}$$

Let $\tilde{\mathbf{f}}'_0 = [\tilde{\mathbf{f}}_0 \; 0] \in \mathbf{C}^{1\times(l+2)}[\mathbf{z}]$. It is clear that $\tilde{\mathbf{f}}'_0$ is also ZLP, and satisfies

$$\tilde{\mathbf{f}}'_0 F_1 = 0_{1,l}. \tag{38}$$

By Lemma 1, there exists ZRP $F_0 \in \mathbf{C}^{(l+2)\times(l+1)}[\mathbf{z}]$ and $G_1 \in \mathbf{C}^{(l+1)\times l}[\mathbf{z}]$, such that

$$\tilde{\mathbf{f}}'_0 F_0 = 0_{1,l}, \tag{39}$$

and

$$F_1 = F_0 G_1. \tag{40}$$

Let $b_1, \ldots, b_\beta$ denote the reduced minors of F_1. Since F_0 is ZRP and $(l+2) > (l+1) > l$, by Lemma 3, the reduced minors of G_1 have the same set of common zeros with that of $b_1, \ldots, b_\beta$. Since F is an $(l + 1) \times l$ submatrix formed from the first $l + 1$ rows of F_1, it is clear that $b_1, \ldots, b_{l+1}$ is a proper subset of $b_1, \ldots, b_\beta$. The assumption that $b_1, \ldots, b_{l+1}$ have no common zeros in $\mathbf{C}^n$ implies that $b_1, \ldots, b_\beta$ have no common zeros in $\mathbf{C}^n$. It follows immediately that the reduced minors of G_1 also have no common zeros in $\mathbf{C}^n$. Furthermore, by Lemma 3, the g.c.d. of the $l \times l$ minors of G_1 is equal to d (we assume that $k_0 = 1$). Since G_1 is of dimension $(l + 1) \times l$, by Proposition 1, G_1 can be factorized as

$$G_1 = G_3 G_2 \tag{41}$$

where $G_3 \in \mathbf{C}^{(l+1)\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_2 = d$. Substituting (41) into (40) gives

$$F_1 = F_0 G_3 G_2 = F_2 G_2 \tag{42}$$

where $F_2 = F_0 G_3 \in \mathbf{C}^{m\times l}[\mathbf{z}]$. Since both F_0 and G_3 are ZRP, by Corollary 2, F_2 is also ZRP. ■

Remark 2. Unlike Proposition 1, in Proposition 2 the condition that there exists an $(l + 1) \times l$ submatrix whose reduced minors have no common zeros in $\mathbf{C}^n$ is a sufficient but not necessary one for ZRP factorizability of F_1. As we pointed out earlier, for F_1 of size $(l+k) \times l$ with $k > 1$, it is still unknown whether there exists a ZLP row vector $\tilde{\mathbf{b}}_0 \in \mathbf{C}^{1\times(l+k)}[\mathbf{z}]$ such that $\tilde{\mathbf{b}}_0 F_1 = 0_{1,l}$. Imposing the condition that the reduced minors of an $(l+1) \times l$ submatrix have no common zeros in $\mathbf{C}^n$ is to ensure the existence of such a ZLP row vector.

COROLLARY 3: *Let $F_1 \in \mathbf{C}^{m\times r}[\mathbf{z}]$ be of normal rank l with $m = l+2$ and $r > l$. If there exists an $(l+1) \times l$ submatrix F of F_1, such that the reduced minors of F have no common zeros in $\mathbf{C}^n$, then F_1 can be factorized as*

$$F_1 = F_2 G_2 \tag{43}$$

where $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times r}[\mathbf{z}]$.

A proof is similar to that for Corollary 1 (with Proposition 1 replaced by Proposition 2) and is hence omitted here.

We now refine Proposition 2 and Corollary 3 to the general case for an n-D polynomial matrix of arbitrary size. In the following proposition, we present an algorithm for testing the ZRP factorizability of an arbitrary n-D polynomial matrix F, and for carrying out the ZRP factorization of F when exists.

PROPOSITION 3: *Let $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank with $m = l+k$, $k \geq 1$, and let d denote the g.c.d. of the $l \times l$ minors of F. If the following algorithm can be executed to the statement* **Exit** *instead of the statement* **Stop and exit**, *then F admits ZRP factorization $F = AF_0$ with $A \in \mathbf{C}^{m\times l}[\mathbf{z}]$ being ZRP, $F_0 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ and $\det F_0 = d$.*

INITIALIZATION: *Let $J = k$ and $F_J = F$*

WHILE *($J \neq 0$)* **DO**

IF *(there exists an $(l+1) \times l$ submatrix of F_J, such that its reduced minors have no common zeros in $\mathbf{C}^n$)*

Factorize F_J as $F_J = A_J F_{J-1}$, where $A_J \in \mathbf{C}^{(l+J)\times(l+J-1)}[\mathbf{z}]$ is ZRP and $F_{J-1} \in \mathbf{C}^{(l+J-1)\times l}[\mathbf{z}]$

ELSE

Stop and exit.

END IF

$J = J - 1$

IF *($J = 0$)*

Let $A = A_k A_{k-1} \cdots A_1$

Exit.

END IF

END WHILE

A proof is omitted here as it would be similar to the one for Proposition 2 (with repetition of k times). When $k = 1$, Proposition 3 specializes to Proposition 1, and when $k = 2$ to Proposition 2. However, it should be pointed out that while the ZRP factorizability of F can be determined by its reduced minors before carrying out the actual factorization for $k = 1, 2$, it is not so when $k > 2$, as it can be seen from the above algorithm. More investigation is still required for the case when $k > 2$. It should also be noted that as in Proposition 2, the condition for ZRP factorizability stated in Proposition 3 is only a sufficient one for $k > 2$.

COROLLARY 4: *Let $F_1 \in \mathbf{C}^{m\times r}[\mathbf{z}]$ be of normal rank l with $m = l + k$, $k \geq 1$ and $r > l$. If there exists an $m \times l$ submatrix F of F_1, such that F admits a ZRP factorization, then F_1 can be factorized as*

$$F_1 = F_2 G_2 \tag{44}$$

where $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times r}[\mathbf{z}]$.

4. Example

In this section, we present an example to illustrate Proposition 2, which covers Proposition 1 as a special case and can be generalized easily to Proposition 3. Most of the computations are implemented using the program SINGULAR [23].

Example: Let

$$F_3 = \begin{bmatrix} 2z_1^2z_2z_3 - z_1^2z_2 + 3z_2 + z_3 + 2 & 2z_1^2z_3 + z_1z_2 + z_1z_3 - z_1^2 + 2z_1 + 2 \\ 2z_2z_3 - z_2 & 2z_3 - 1 \\ 1 & z_1 \\ 2z_2z_3 - z_2 + z_3 & 2z_3 + z_1z_3 - 1 \end{bmatrix}$$

The g.c.d. of the 2×2 minors of F_3 is $d_3 = (1 - z_1z_2)$, and the reduced minors are:

$$\begin{aligned}
b_{31} &= (2z_3 - 1)(z_2 + z_3 + 2),\\
b_{32} &= -(2z_1^2z_3 - z_1^2 + 2),\\
b_{33} &= (2z_3 - 1)(-z_1^2z_3 + z_2 + z_3 + 2) - 2z_3,\\
b_{34} &= -(2z_3 - 1),\\
b_{35} &= -z_3(2z_3 - 1),\\
b_{36} &= 2z_3 - 1.
\end{aligned}$$

It is easy to test that $b_{31}, \ldots, b_{36}$ have no common zeros in $\mathbf{C}^3$. Hence, F_3 may admit a ZRP factorization. Let F_1 denote the 3×2 submatrix formed from the first 3 rows of F_3. it can be checked that the reduced minors of F_1 (they are b_{31}, b_{32} and b_{34}) also have no common zeros in $\mathbf{C}^3$. By Proposition 2, F_3 is ZRP factorizable. Therefore, we can determine the ZRP factorizability of F_3 without carrying out the actual matrix factorization. To illustrate that F_3 indeed admits a ZRP factorization, we first construct a ZLP row vector

$$\tilde{\mathbf{b}}_3 = [2z_3 - 1 \quad -2z_1^2 z_3 + z_1^2 - 2 \quad -2z_2z_3 - 2z_3^2 + z_2 - 3z_3 + 2 \quad 0]$$

such that

$$\tilde{\mathbf{b}}_3 F_3 = 0_{1,2}.$$

By Lemma 1, we can construct F_5 and G_7,

$$F_5 = \begin{bmatrix} z_2 + z_3 + 2 & 2z_1^2 z_3 - z_1^2 + 2 & 0 \\ 0 & 2z_3 - 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$G_7 = \begin{bmatrix} 1 & z_1 \\ z_2 & 1 \\ 2z_2z_3 - z_2 + z_3 & 2z_3 + z_1z_3 - 1 \end{bmatrix}$$

such that

$$\tilde{\mathbf{b}}_3 F_5 = 0_{1,2}$$

and

$$F_3 = F_5 G_7, \tag{45}$$

where F_5 is ZRP. Let d_7 denote the g.c.d. of the 2×2 minors of G_7 and b_{71}, b_{72} and b_{73} denote the reduced minors of G_7. By Lemma 3, we should have $d_7 = k_0 d_3$ for some $k_0 \in \mathbf{C}^*$, and that b_{71}, b_{72} and b_{73} are free from any common zeros since $b_{31}, \ldots, b_{36}$ have no common zeros in $\mathbf{C}^3$. This is indeed the case, as direct computation gives $d_7 = (1 - z_1z_2) = d_3$, and

$$b_{71} = 1,$$
$$b_{72} = 2z_3 - 1,$$
$$b_{73} = -z_3.$$

Notice that the family of b_{71}, b_{72} and b_{73} are different from the family of $b_{31}, \ldots, b_{36}$. Applying Proposition 1 to G_7 gives

$$G_7 = G_8 G_9, \tag{46}$$

where

$$G_8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ z_3 & 2z_3 - 1 \end{bmatrix}$$

and

$$G_9 = \begin{bmatrix} 1 & z_1 \\ z_2 & 1 \end{bmatrix}.$$

Clearly, G_8 is ZRP, and $\det G_9 = (1 - z_1 z_2) = d_3$. Combining (45) and (46) leads to

$$F_3 = F_6 G_9,$$

where

$$F_6 = F_5 G_8 = \begin{bmatrix} z_2 + z_3 + 2 & 2z_1^2 z_3 - z_1^2 + 2 \\ 0 & 2z_3 - 1 \\ 1 & 0 \\ z_3 & 2z_3 - 1 \end{bmatrix}.$$

Since F_5 and G_8 are both ZRP, by Corollary 2, F_6 must be ZRP. This is indeed the case by checking F_6 directly. ■

5. Conclusions

In this paper, we have made an attempt to establish a relationship between the prime factorizability of an n-D ($n > 2$) polynomial matrix and its reduced minors by raising two conjectures on zero and minor prime factorizability of n-D polynomial matrices. We have proved that Conjecture 1 (zero right prime factorizability) is always true for an n-D polynomial matrix F of dimension $(l + 1) \times l$, and under some condition also true when F is of arbitrary dimension. In particular, ZRP factorizability for an n-D polynomial matrix of dimension $(l + k) \times l$ ($k = 1, 2$) can be easily tested from its reduced minors without carrying out the actual matrix factorization. An illustrative example has been worked out in details.

We have also shown how to factorize some special n-D polynomial matrix that is not of normal full rank as a product of two n-D polynomial matrices of smaller size.

We hope that the conjectures posed and the new results presented in this paper will motivate further research in the area of n-D polynomial matrix factorizations.

Finally, although for simplicity, the ground field is assumed to be the field of complex numbers, all the derived results are still valid with minor modification for an arbitrary coefficient field.

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Notes

1. In what follows, the term "n-D" implies ($n > 2$) unless otherwise specified.
2. $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$, the set of non-zero complex numbers.
3. An $m \times l$ matrix $A(\mathbf{z})$ is of normal full rank if there exists an $r \times r$ minor of $A(\mathbf{z})$ that is not identically zero, where $r = \min\{m, l\}$.
4. Denote $0_{l,m}$ the $l \times m$ zero matrix and I_m the $m \times m$ identity matrix.

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