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Further Results on *n*-D Polynomial Matrix Factorizations

ZHIPING LIN ezplin@ntu.edu.sg
School of Electrical and Electronic Engineering, Nanyang Technological University, Block S2, Nanyang Avenue, Singapore 639798

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Abstract. In this paper, some new results on zero prime factorization for a normal full rank n-D ($n > 2$) polynomial matrix are presented. Assume that d is the greatest common divisor (g.c.d.) of the maximal order minors of a given n-D polynomial matrix F_1. It is shown that if there exists a submatrix F of F_1, such that the reduced minors of F have no common zeros, and the g.c.d. of the maximal order minors of F equals d, then F_1 admits a zero right prime (ZRP) factorization if and only if F admits a ZRP factorization. A simple ZRP factorizability of a class of n-D polynomial matrices based on reduced minors is given. An advantage is that the ZRP factorizability can be tested before carrying out the actual matrix factorization. An example is illustrated.

Keywords: n-D polynomial matrices, matrix factorizations, zero primeness, reduced minors, Quillen–Suslin theorem

1. Introduction

The long-standing open problem of multivariate (n-D, $n > 2$)[1] polynomial matrix prime factorization was first posed by Youla and Gnavi 20 years ago [1], and has attracted some attention over the past decade (see [2]-[4] for more details). Consider[2] a normal full rank n-D polynomial matrix $F \in \mathbf{C}^{m \times l}[\mathbf{z}]$ with $m > l$. Let $a_1, \ldots, a_\beta$ denote the $l \times l$ minors of F, $b_1, \ldots, b_\beta$ the reduced minors of F, and d the greatest common divisor (g.c.d.) of $a_1, \ldots, a_\beta$. Unlike 2-D polynomial matrices [5], [6], it is, in general, not possible to factorize F as $F = F_0 G_0$ such that both F_0 and G_0 are n-D polynomial matrices, with $\det G_0 = d$ [1], [7], [8]. However, if the reduced minors of F satisfy the zero coprime condition, i.e., $b_1, \ldots, b_\beta$ having no common zeros, it might be possible to carry out the above matrix factorization for F [2]-[4]. Two related but different approaches have recently been developed independently to tackle this special case.

The first approach, advanced by Bose and Charoenlarpnopparut [3], [4], is to consider the module generated by the m rows of F. A Gröbner basis for this module is first computed, and a zero right prime (ZRP) factorization of F may then be obtained from this Gröbner basis. An advantage of this approach is that it is computationally attractive, and it can be applied to n-D polynomial matrices of any dimensions. However, this approach sometimes fails to produce a ZRP factorization even when it exists [4]. Moreover, one does not know the ZRP factorizability of a given n-D polynomial matrix until the actual matrix factorization has been attempted.

1 Throughout the paper, it is assumed that $n > 2$.
2 For related notation and definitions, see [2]. It should also be pointed out that the results presented in this paper can be easily applied to the case when $m < l$ with minor modification.

The second approach to the same problem, adopted by Lin [2], is to build upon existing results on n-D polynomial matrix theory, such as the Quillen–Suslin theorem (see, e.g., [9]), and the properties of reduced minors [10], and then to identify classes of n-D polynomial matrices for which ZRP factorization exists. In particular, the ZRP factorizability is conjectured recently in [2] and recalled in the following.

CONJECTURE 1 *[2] Let $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank with $m > l$, and let d be the g.c.d. of the $l \times l$ minors of F, and $b_1, \ldots, b_\beta$ be the reduced minors of F. If $b_1, \ldots, b_\beta$ have no common zeros, then F can be factorized as*

$$F = F_0 G_0 \tag{1}$$

where $F_0 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_0 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_0 = d$.

It has been proved in [2] that Conjecture 1 is always true if $m = l + 1$, and under some condition, also true for F of arbitrary dimension. An advantage of Lin's approach is that ZRP factorizability can be tested before the actual matrix factorization is carried out for the two classes of n-D polynomial matrices discussed in [2].

To our best knowledge, the above two approaches are the only ones available in the literature for attacking the zero prime factorization problem for n-D polynomial matrices. The lack of aggressive progress in this research area is probably due to the fact that factor and zero prime factorization for n-D polynomial matrices is a mathematically highly complicated and challenging problem [11, p. 63]. It is expected that it may take some time before the zero prime factorization problem can be resolved completely. Meanwhile, we believe that any incremental progress would be useful in solving this open problem in part. In this paper, we present some new results which improve existing results on the ZRP factorization problem.

2. Main Results

For convenience of exposition and comparison with the new results, we recall two related results from [4], [2].

PROPOSITION 1 *[4] Let $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank with $m > l$, and let d be the g.c.d. of the $l \times l$ minors of F_1, and $b_1, \ldots, b_\beta$ be the reduced minors of F_1. Assume that $b_1, \ldots, b_\beta$ have no common zeros. Compute a Gröbner basis $\mathbf{G}$ for the module generated by rows of F_1 using any ordering. If there exists a set of l linearly independent elements (which are row vectors) of $\mathbf{G}$, such that all rows of F_1 belong to the module generated by these l elements of $\mathbf{G}$, then a ZRP factorization of F_1 has been found,*

$$F_1 = F_0 G_0, \tag{2}$$

where $F_0 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_0 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ is formed from the above mentioned l elements of $\mathbf{G}$ with $\det G_0 = d$.

PROPOSITION 2 *[2] Let $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank with $m = l + 2$, and let d be the g.c.d. of the $l \times l$ minors of F_1. If there exists an $(l + 1) \times l$ submatrix F of F_1, such that* *the reduced minors of F have no common zeros,* **and the g.c.d. of the $l \times l$ minors of F equals** *d, then F_1 can be factorized as*

$$F_1 = F_2 G_2 \tag{3}$$

where $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_2 = d$.

Remark 1. It is easy to see that a necessary condition for an arbitrary n-D polynomial matrix $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ with $m > l$, to admit a ZRP factorization is its reduced minors having no common zeros. This condition is satisfied if there exists an $(l+1) \times l$ submatrix F of F_1, such that the reduced minors of F have no common zeros, **and the g.c.d. of the $l \times l$ minors of F equals** d, where d is the g.c.d. of the $l \times l$ minors of F_1. However, without adding **and the g.c.d. of the $l \times l$ minors of F equals** d in the above Proposition, the reduced minors of F_1 may have some common zeros even when the reduced minors of F have no common zeros. Hence, Proposition 2 of [2] was in fact incorrect. For example, let $F_1' = [z_1 z_2 \quad z_1 z_2^2 \quad z_3]^T \in \mathbf{C}^{3\times 1}[\mathbf{z}]$ where $(\cdot)^T$ denotes transpose. Obviously, the g.c.d. of the minors of F_1' equals 1. Since the reduced minors (1 and z_2) of the submatrix F_0' formed from the first two rows of F_1' have no common zeros, by Proposition 2 of [2], F_1' should admit a ZRP factorization $F_1' = F_2 G_2$ with $F_2 \in \mathbf{C}^{3\times 1}[\mathbf{z}]$ being ZRP and $G_2 = d = 1$. However, such a factorization is impossible since the reduced minors of F_1' have a common zero at $(0, 0, 0)$. Hence, F_1' is a counterexample to Proposition 2 of [2], but not to Proposition 2 in this paper, since the g.c.d. of the minors of F_0' does not equal the g.c.d. of the minors of F_1', and therefore, the new assumption made in Proposition 2 is not satisfied. The author is very grateful to an anonymous reviewer for pointing out this error in Proposition 2 of [2].

Remark 2. It should also be pointed out at this point that the proof presented in [2] for Proposition 2 there was not correct either since it was assumed in the proof that the g.c.d. of the $l \times l$ minors of the $(l+1) \times l$ submatrix equaled the g.c.d. of the $l \times l$ minors of the $m \times l$ $(m = l+2)$ matrix. In fact, it can be seen that the proof for Proposition 2 of [2], although wrong for Proposition 2 of [2], is **a correct** proof for Proposition 2 of the present paper. Moreover, Proposition 2 of this paper turns out to be a spcecial case of Corollay 1 to be presented later.

An important feature of Proposition 2 is that we can test the ZRP factorizability of F_1 *before* carrying out the actual matrix factorization for F_1. In the following, we present some new results which not only generalize Proposition 2, but may also improve Proposition 1.

LEMMA 1 *Let $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank with $m > l$, and let d be the g.c.d. of the $l \times l$ minors of F_1. If there exists an $l \times l$ submatrix G_1 of F_1, such that* $\det G_1 = k_0 d$, *for some nonzero constant k_0, then F_1 can be factorized as*

$$F_1 = F_2 G_2 \tag{4}$$

where $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_2 = d$.

Proof: Without loss of generality, assume that $k_0 = 1$, and G_1 is formed from the first l rows of F_1. We have,

$$F_1 = \begin{bmatrix} G_1 \\ F_3 \end{bmatrix}, \tag{5}$$

where $F_3 \in \mathbf{C}^{(m-l)\times l}[\mathbf{z}]$.

Let $F_4 = F_3\,G_1^{-1}$. By Cramer's rule [12], $F_4 = F_3 \cdot \operatorname{adj} G_1 / \det G_1 = (F_3 \cdot \operatorname{adj} G_1)/d$. Notice that any entry of $(F_3 \cdot \operatorname{adj} G_1)$ is just an $l \times l$ minor of F_1 and hence contains d as its divisor. Therefore, F_4 is an n-D polynomial matrix. We then have

$$F_1 = \begin{bmatrix} I_l \\ F_3\,G_1^{-1} \end{bmatrix} G_1 = \begin{bmatrix} I_l \\ F_4 \end{bmatrix} G_1 = F_7\,G_1 \tag{6}$$

where $F_7 \in \mathbf{C}^{m\times l}[\mathbf{z}]$, and $G_1 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_1 = d$. Clearly, F_7 is ZRP. Let $F_2 = F_7$, $G_2 = G_1$. The proof is thus completed. □

We now present the main result of this paper. The objective of Proposition 3 is trying to reduce the ZRP factorization problem for F_1 to the one for F, where F is a submatrix of F_1 satisfying certain condition.

Proposition 3 *Let $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank with $m > l$, and let d be the g.c.d. of the $l \times l$ minors of F_1. If there exists an $s \times l$ ($m > s \geq l$) submatrix F of F_1, such that the reduced minors of F have no common zeros, and the g.c.d. of all the $l \times l$ minors of F equals d, then the following two statements are equivalent:*

(i) F_1 can be factorized as

$$F_1 = F_2\,G_2 \tag{7}$$

where $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_2 = d$.

(ii) F can be factorized as

$$F = F_0\,G_2 \tag{8}$$

where $F_0 \in \mathbf{C}^{s\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_2 = d$.

Moreover, if F admits a ZRP factorization in (8), a ZRP factorization for F_1 can be readily obtained by letting $F_2 = F_1\,G_2^{-1} \in \mathbf{C}^{m\times l}[\mathbf{z}]$, and $F_1 = F_2\,G_2$.

Proof: It is easy to show that (i) implies (ii). In fact, if $F_1 = F_2\,G_2$ with $\det G_2 = d$, we can partition F_1 as $F_1 = \begin{bmatrix} F \\ F_3 \end{bmatrix}$, and F_2 as $F_2 = \begin{bmatrix} F_0 \\ F_3' \end{bmatrix}$, where $F, F_0 \in \mathbf{C}^{s\times l}[\mathbf{z}]$. It follows immediately that $F = F_0\,G_2$. Since by assumption, the g.c.d. of the $l \times l$ minors of F equals d, and $\det G_2 = d$, it follows easily that the $l \times l$ minors of F_0 equal to the reduced minors of F, and thus have no common zeros. Therefore, F_0 is ZRP.

To show that (ii) implies (i), we first notice that the case $s = l$ reduces to Lemma 1. In the following, we assume that $m > s > l$. Without loss of generality, we can assume that F is formed from the first s rows of F_1. Thus,

$$F_1 = \begin{bmatrix} F \\ F_3 \end{bmatrix}, \tag{9}$$

where $F_3 \in \mathbf{C}^{(m-s)\times l}[\mathbf{z}]$. By assumption, F admits a ZRP factorization

$$F = F_0\, G_2 \tag{10}$$

where $F_0 \in \mathbf{C}^{s\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_2 = d$. By the Quillen–Suslin theorem (see, e.g., [9]), there exists $B \in \mathbf{C}^{s\times(s-l)}[\mathbf{z}]$ such that $V_0 = [F_0\ B] \in \mathbf{C}^{s\times s}[\mathbf{z}]$ and $\det V_0 = 1$. Let $U_0 = V_0^{-1}$. Clearly, $U_0 \in \mathbf{C}^{s\times s}[\mathbf{z}]$, $\det U_0 = 1$ and

$$U_0\, V_0 = I_s, \tag{11}$$

or

$$U_0\, F_0 = \begin{bmatrix} I_l \\ 0_{s-l,l} \end{bmatrix}, \tag{12}$$

or

$$U_0\, F = \begin{bmatrix} G_2 \\ 0_{s-l,l} \end{bmatrix}. \tag{13}$$

Let

$$U = \begin{bmatrix} U_0 & 0_{s,m-s} \\ 0_{m-s,s} & I_{m-s} \end{bmatrix}. \tag{14}$$

Clearly, $U \in \mathbf{C}^{m\times m}[\mathbf{z}]$, $\det U = 1$, and

$$U\, F_1 = U \begin{bmatrix} F \\ F_3 \end{bmatrix} = \begin{bmatrix} U_0\, F \\ F_3 \end{bmatrix} = \begin{bmatrix} G_2 \\ 0_{s-l,l} \\ F_3 \end{bmatrix}. \tag{15}$$

Let

$$F_5 = \begin{bmatrix} G_2 \\ 0_{s-l,l} \\ F_3 \end{bmatrix}, \tag{16}$$

and let $a'_1, \ldots, a'_\beta$ denote the $l \times l$ minors of F_5. Since $F_5 = U\,F_1$ and U is a unimodular matrix, by Lemma 3 of [2], d is the g.c.d. of $a'_1, \ldots, a'_\beta$. Since G_2 is an $l \times l$ submatrix of F_5 with $\det G_2 = d$, by Lemma 1, we have

$$F_5 = F_7\,G_2 \tag{17}$$

where $F_7 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP.

From (15)-(17), we have

$$F_1 = U^{-1}F_5 = U^{-1}F_7\,G_2 = F_2\,G_2 \tag{18}$$

where $F_2 = U^{-1}F_7 \in \mathbf{C}^{m\times l}[\mathbf{z}]$, and $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_2 = d$. Since both U^{-1} and F_7 are ZRP, by Corollary 2 of [2], F_2 is ZRP. Finally, notice that G_2 in (18) is the same G_2 in (10). Therefore, if F admits a ZRP factorization in (10), a ZRP factorization for F_1 can be readily obtained by just letting $F_2 = F_1\,G_2^{-1} \in \mathbf{C}^{m\times l}[\mathbf{z}]$, and $F_1 = F_2\,G_2$. □

Remark 3. It should be emphasized that in practice, it is not necessary to construct U_0 and U in order to factorize F_1. Once a ZRP factorization $F = F_0\,G_2$ is available, We can simply compute $F_2 = F_1\,G_2^{-1} \in \mathbf{C}^{m\times l}[\mathbf{z}]$. Then $F_1 = F_2\,G_2$ is the desired ZRP factorization for F_1.

Remark 5. The above proposition can also be combined with Proposition 1 (originally from [3], [4]) to improve the computational efficiency. Let $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank with $m > l$, and let d be the g.c.d. of the $l \times l$ minors of F_1. If there exists an $s \times l$ ($m > s \geq l$) submatrix F of F_1, such that the reduced minors of F have no common zeros, and the g.c.d. of the $l \times l$ minors of F equals d, then, instead of computing a Gröbner basis for the module generated by all the rows of F_1, as suggested by Bose and Charoenlarpnopparut in [3], [4] (see also Proposition 1 here), we can simply calculate a Gröbner basis for the module generated by all the rows of F. If F admits a ZRP factorization given in (8), then F_1 will also admit a ZRP factorization given in (7). This will be illustrated by an example shortly. Notice that the improvement on computational efficiency is more significant when $m \gg s$.

Now combining Proposition 3 in this paper with Proposition 1 of [2], we have the following corollary which can be used to test the ZRP factorizability of F_1 before carrying out the actual matrix factorization.

Corollary 1 *Let $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank with $m = l + k$, $k \geq 2$, and let d be the g.c.d. of the $l \times l$ minors of F_1. If there exists an $(l+1) \times l$ submatrix F of F_1, such that the reduced minors of F have no common zeros, and the g.c.d. of all the $l \times l$ minors of F equals d, then F_1 can be factorized as*

$$F_1 = F_2\,G_2 \tag{19}$$

where $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is ZRP, and $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_2 = d$.

Remark 4. Corollary 1 in fact includes Lemma 1 as a special case, since if there exists an $l \times l$ submatrix G_1 of F_1, such that $\det G_1 = k_0 d$, for some nonzero constant k_0, there will also exist an $(l+1) \times l$ submatrix F of F_1, such that the reduced minors of F have no common zeros, and the g.c.d. of all the $l \times l$ minors of F equals d. Notice also that when $k = 2$, the above corollary reduces to Proposition 2. However, the proofs are quite different even for this special case. In fact, comparing the proof for Proposition 3 in this paper with the proof for Proposition 2 in [2], it is easy to see that Proposition 3 does provide a much more efficient way for obtaining a ZRP factorization for F_1, as it will also be illustrated by an example shortly. It may be worthwhile at this point to point out that an attempt was also made in [2] to generalize Proposition 2 to the case where $k > 2$ without much success. Another criterion for the existence of ZRP factorization for F_1 ($k > 2$) was derived in [2] under a stronger condition. In particular, it *was not* possible to test the ZRP factorizability of F_1 *before* carrying out the actual matrix factorization for F_1. The reader is referred to [2] for more details on this result.

Consider now the example from [2]. Let

$$F_1 = \begin{bmatrix} 2z_1^2 z_2 z_3 - z_1^2 z_2 + 3z_2 + z_3 + 2 & 2z_1^2 z_3 + z_1 z_2 + z_1 z_3 - z_1^2 + 2z_1 + 2 \\ 2z_2 z_3 - z_2 & 2z_3 - 1 \\ 1 & z_1 \\ 2z_2 z_3 - z_2 + z_3 & 2z_3 + z_1 z_3 - 1 \end{bmatrix}.$$

The g.c.d. of the 2×2 minors of F_1 is $d = (1 - z_1 z_2)$. Let F denote the 3×2 submatrix formed from the first 3 rows of F_1. It can be checked [2] that the reduced minors of F have no common zeros, and the g.c.d. of the 2×2 minors of F equals to d. By Corollary 1, F_1 admits a ZRP factorization. By Proposition 3, to obtain a ZRP factorization for F_1, it suffices to obtain a ZRP factorization for F. Indeed, by Proposition 1 of [2], F admits a ZRP factorization given by

$$\begin{aligned} F &= F_0 G_2 \\ &= \begin{bmatrix} z_2 + z_3 + 2 & 2z_1^2 z_3 - z_1^2 + 2 \\ 0 & 2z_3 - 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ z_2 & 1 \end{bmatrix}, \end{aligned}$$

where F_0 is ZRP and $\det G_2 = d = (1 - z_1 z_2)$. The details are omitted here since it is similar to that in [2]. By Proposition 3, a ZRP factorization for F_1 can be readily obtained:

$$\begin{aligned} F_1 &= (F_1 G_2^{-1}) G_2 \\ &= F_2 G_2 \\ &= \begin{bmatrix} z_2 + z_3 + 2 & 2z_1^2 z_3 - z_1^2 + 2 \\ 0 & 2z_3 - 1 \\ 1 & 0 \\ z_3 & 2z_3 - 1 \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ z_2 & 1 \end{bmatrix}. \end{aligned} \tag{20}$$

As can be seen, the above procedure for arriving at the same ZRP factorization is much simpler than that in [2].

On the other hand, applying Bose-Charoenlarpnopparut's algorithm [3], [4], we can also obtain a ZRP factorization for F_1 as follows. Using the software package SINGULAR [13], a Gröbner basis, consisting of two row vectors $\mathbf{r}_1$, $\mathbf{r}_2$, for the module generated by all the rows of F_1 is readily obtained:

$$\mathbf{r}_1 = [0 \quad z_1z_2 - 1]; \quad \mathbf{r}_2 = [1 \quad z_1].$$

Let

$$G_0 = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = \begin{bmatrix} 0 & z_1z_2 - 1 \\ 1 & z_1 \end{bmatrix}. \tag{21}$$

Simple algebra shows that F_1 has the following ZRP factorization:

$$\begin{aligned} F_1 &= F_0G_0 = (F_1\,G_0^{-1})G_0 \\ &= \begin{bmatrix} -2z_1^2z_3 + z_1^2 - 2 & 2z_1^2z_2z_3 - z_1^2z_2 + 3z_2 + z_3 + 2 \\ 1 - 2z_3 & 2z_2z_3 - z_2 \\ 0 & 1 \\ 1 - 2z_3 & 2z_2z_3 - z_2 + z_3 \end{bmatrix} \begin{bmatrix} 0 & z_1z_2 - 1 \\ 1 & z_1 \end{bmatrix}. \end{aligned} \tag{22}$$

It can be easily checked that F_0 is ZRP, and $\det G_0 = d = (1 - z_1z_2)$. Notice that F_0 and G_0 in (22) are different from F_2 and G_2 in (20). However, G_0 and G_2 are connected by a unimodular matrix $U = \begin{bmatrix} 0 & 1 \\ -1 & z_2 \end{bmatrix}$, such that $G_2 = UG_0$ (see Remark 3 of [4]).

Since the submatrix F satisfies the condition given in Proposition 3, by Remark 4, instead of calculating a Gröbner basis for the module generated by all the four rows of F_1, we only need to compute a Gröbner basis for the module generated by all the three rows of F. It turns out that the Gröbner basis for F is the same as that for F_1, i.e., $F = F_0'\,G_0$ where G_0 is the same as in (21). By Proposition 3, F_1 admits a ZRP factorization $F_1 = (F_1\,G_0^{-1})G_0 = F_2\,G_0$, the same as in (22). An advantage is that it would be computationally more efficient to calculate a Gröbner basis for F than for F_1, particularly when the size of F is much smaller than that of F_1. □

3. Conclusion

The new results presented in this paper can be summarized in the following:

Let $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of normal full rank with $m > l$, and let d be the g.c.d. of the $l \times l$ minors of F_1.

1. If there exists an $s \times l$ $(m > s \geq l)$ submatrix F of F_1, such that the reduced minors of F have no common zeros, and the g.c.d. of the $l \times l$ minors of F equals d, then F_1 admits a ZRP factorization if and only if F admits a ZRP factorization. Moreover,

once we have $F = F_0 G_2$ with $\det G_2 = d$, a ZRP factorization of F_1 is given by $F_1 = (F_1 G_2^{-1})G_2 = F_2 G_2$ (Proposition 3).

2. If there exists an $s \times l$ ($m > s \geq l$) submatrix F of F_1, such that the reduced minors of F have no common zeros, and the g.c.d. of the $l \times l$ minors of F equals d, then it is only necessary to compute a Gröbner basis for the module generated by all the rows of F instead of F_1 (Remark 4).
3. If there exists an $(l + 1) \times l$ submatrix F of F_1, such that the reduced minors of F have no common zeros, and the g.c.d. of the $l \times l$ minors of F equals d, then F_1 is ZRP factorizable, and its ZRP factorization can be computed constructively (Corollary 1).
4. If there exists an $l \times l$ submatrix G_2 of F_1, such that $\det G_1 = k_0 d$, for some nonzero constant k_0, then F_1 is ZRP factorizable, and its ZRP factorization can be computed easily. Although this result is a special case of 3, it is of some interest in its own right in view of its simplicity, i.e., a ZRP factorization for F_1 can be calculated by simple matrix manipulations (Lemma 1).
5. An error in Proposition 2 of [5] has been corrected (Proposition 2). This was pointed out by an anonymous reviewer.

We believe that the contributions made in this paper are one further step towards completely resolving the open problem of zero prime factorization for n-D polynomial matrices, which is presently a challenge to both mathematicians [11, p. 63] and multidimensional system theorists [2]-[4].

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