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A generalization of Serre's conjecture and some related issues

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Abstract

Several topics concerned with multivariate polynomial matrices like unimodular matrix completion, matrix determinantal or primitive factorization, matrix greatest common factor existence and subsequent extraction along with relevant primeness and coprimeness issues are related to a conjecture which may be viewed as a type of generalization of the original Serre problem (conjecture) solved nonconstructively in 1976 and constructively, more recently. This generalized Serre conjecture is proved to be equivalent to several other unsettled conjectures and, therfore, all these conjectures constitute a complete set in the sense that solution to any one also solves all the remaining. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

In recent years there has been an increasing interest in multidimensional linear systems theory, due to the wide range of its applications in circuits, systems, con-

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trols, signal processing and other areas (see, e.g., [1–5,11,12,18–24]). This is a consequence of the array of impressive developments in one as well as multivariate multiport network synthesis, analysis and design of multi-input multi-output (multivariable) multidimensional feedback control systems, and multivariable multidimensional digital filtering, made possible by an in-depth exploitation of the theory of polynomial matrices. Since many multidimensional systems and signal processing problems can be formulated as finitely generated projective modules over a polynomial ring [22], or as multivariate polynomial matrices [2–4], it is of importance to adapt, transfer, and apply recent results from the field of algebraic geometry and computer algebra into multidimensional linear systems theory.

Constructive algorithms are now available for finding a free basis of any finitely generated projective module over a polynomial ring. Some of these algorithms are now conveniently implementable by computer algebra system packages for algebraic geometry, commutative algebra, and the theory of resolution of singularities. The evolution of Serre's problem from 1955 to 1976 led to two nonconstructive proofs by D. Quillen and A.A. Suslin of what is referred to as Serre's Conjecture [8], namely that projective modules over polynomial rings are free. This important result was proved constructively by Logar and Sturmfels [9] in 1992 and by Park and Woodburn [10] in 1995.

Let $F \in D^{m\times l}$, $l \leqslant m$, be a full rank matrix whose entries are in the unique factorization domain D. Let M be the set of determinants of all $l \times l$ submatrices (maximal order minors) of F and since M is a subset of D, denote the ideal in D generated by the elements of M by $I = \langle M\rangle$. It is well known that the matrix F is (a) zero prime if $I = D$, (b) minor prime if the elements of M have no nontrivial common divisor in D (i.e., excluding units in D) and (c) (right) factor prime if whenever F is factorable as $F = F_1U$, $U \in D^{l\times l}$, $F_1 \in D^{m\times l}$, then U is unimodular, i.e., its determinant is a unit in D. Considerable research has been conducted in the case when $D = K[z_1, \dots, z_n]$ [11,12] is the n-variate polynomial ring where K is an arbitrary but fixed field of coefficients. When $D = K[z_1]$, which is a principal ideal ring, the three notions of primeness are all equivalent; in fact $K[z_1]$ has been the setting for the well-understood theory of the polynomial approach to multivariable (multi-input multi-output) system theory initiated by Rosenbrock [13]. The case when $D = K[z_1, z_2]$ has been completely tackled by Guiver and Bose [14], who showed that the matrix factorization could be constructed via computations in the ground field K without the need of any algebraically closed extension field. Factor and minor primeness are equivalent in this case but zero primeness is not implied by either, even though the cardinality of the set of common zeros of the maximal minors is always finite for this bivariate case. In fact, the constructive results in [14] hold even when the matrix elements belong to $D = E[z_1]$, where E is any Euclidean domain. It has been pointed out by Oberst [15] that the reason why system theory in two dimensions is more complete and manageable than in higher dimensions is because every second syzygy module is free over the polynomial ring $K[z_1, z_2]$ since every module has projective dimension at most two. This implies that over $K[z_1, z_2]$, every finitely generated reflexive module is free [16]. The framework of D-modules adopted by several researchers like Oberst [15] uses a framework of algebraic analysis based on homological algebra and differential operators. The ring D of differential operators bears resemblance to a polynomial ring.

A flurry of research activity has been witnessed in the last few years when $D = K[z_1, \ldots, z_n]$, $n \geqslant 3$, and the problem is still open, though much better understood than was the case a decade back. The three notions of primeness cited above are all different in this case. The notions of zero primeness, unimodular matrix completion and polynomial matrix inverse are linked by the Quillen–Suslin proof of the freeness of projective modules over polynomial rings (referred to in the literature as Serre's conjecture) [8]. However, unlike zero and minor primeness, the notion of factor primeness cannot be characterized exclusively by the variety of the ideal generated by the maximal order minors in the ring $D = K[z_1, \ldots, z_n]$, $n \geqslant 3$, and also the exclusive consideration of the matrix full rank condition is not sufficient. This necessitated the generalization of the factor primeness concept to that of factor primeness in the generalized sense with both being equivalent in the full-rank case [17]. Multivariate polynomial matrix primeness, coprimeness, and greatest common divisor (if it exists) extraction results have been advanced by several researchers in special cases [18,19]. Oberst [15] established the duality between multidimensional linear shift-invariant systems and finitely generated modules over $D = K[z_1, \ldots, z_n]$. He showed that for $n \geqslant 3$, a transfer function might have two minimal matrix fraction descriptions or realizations that are not comparable. Unlike in the $n = 1$ and also $n = 2$ cases, it is not possible to speak of greatest common polynomial matrix divisors as these may not be unique up to unimodularity when $n \geqslant 3$. However, under the hypothesis of existence and uniqueness up to unimodularity, constructive algorithmic procedures like Gröbner bases have been used to obtain the factor provided specific constraints are met as in [18,20].

In linear algebra as well as in multidimensional systems, the case where F is not a full rank matrix is important and deserves some attention [11,19]. Let $F \in K^{m\times r}[z_1, \ldots, z_n]$ be of rank l with $l < m$ and $l < r$, where K is an arbitrary but fixed field of coefficients. One main question concerning matrices not of full rank is whether or not F admits a factorization $F = A_1A_2$ such that $A_1 \in K^{m\times l}[z_1, \ldots, z_n]$ and $A_2 \in K^{l\times r}[z_1, \ldots, z_n]$ [11]. This is the so-called rank degeneracy elimination problem and is of considerable importance in the general multidimensional linear systems theory as well as in linear algebra. If the above matrix factorization can be carried out, we can then study the structural properties of the full rank matrix A_1 or A_2 [5,11,19]. Such a factorization is always possible when $n \leqslant 2$, but not so when $n > 2$ [11]. This problem will also be considered in the paper after the discussion of full rank matrices.

2. Notations and objectives

Let $K[\mathbf{z}] = K[z_1, \dots, z_n]$ denote the set of polynomials in n variables $z_1, \dots, z_n$ with coefficients in a specified field K; $K^{m\times l}[\mathbf{z}]$ the set of $m \times l$ matrices with entries in $K[\mathbf{z}]$; K^* the set of all nonzero elements in K, i.e., $K^* = K\backslash\{0\}$; $0_{l,m}$ the $l \times m$ zero matrix and I_m the $m \times m$ identity matrix.

Henceforth, the argument ($\mathbf{z}$) is omitted whenever that does not cause confusion. Here we consider the case where F is a full rank matrix and postpone discussion of the complementary case to a subsection devoted exclusively to it in the next section.

Definition 1 [21,22]. Let $F \in K^{m\times l}[\mathbf{z}]$ with $m \geqslant l$, and let $a_1, \dots, a_\beta$ denote the $l \times l$ minors of the matrix F, where $\beta = \binom{m}{l} = m!/(m-l)!l!$. Extracting the greatest common divisor (g.c.d.) d of $a_1, \dots, a_\beta$ gives

$$a_i = d\, b_i, \quad i = 1, \dots, \beta. \tag{1}$$

Then, $b_1, \dots, b_\beta$ are called the generating set [21] or reduced minors [22] of F.

Definition 2 [11]. Let F be given in Definition 1. Then F is said to be:

(i) zero right prime (ZRP) if $a_1, \dots, a_\beta$ are zero coprime, i.e., there exist $h_1, \dots, h_\beta \in K[\mathbf{z}]$ such that $\sum_{i=1}^{\beta} h_i\, a_i = k_0 \in K^*$;

(ii) minor right prime (MRP) if $a_1, \dots, a_\beta$ are factor coprime, i.e., $d = k_1 \in K^*$.

$G \in K^{l\times m}[\mathbf{z}]$ with $m \geqslant l$ is said to be zero left prime (ZLP) or minor left prime (MLP) if G^{T} is ZRP or MRP, where G^{T} is the transpose of G.

In 1955, J.P. Serre raised the question as to whether finitely generated projective modules over polynomial rings (algebraic vector bundles over an affine space of n-variate polynomials whose coefficients are in K) are free (trivial bundles). This question, referred to as Serre's conjecture, has been shown to be equivalent to the unimodular matrix completion question: can a ZRP matrix $F \in K^{m\times l}[\mathbf{z}]$ ($m > l$) be completed to a square matrix $U = [F\ E] \in K^{m\times m}[\mathbf{z}]$ such that $E \in K^{m\times(m-l)}[\mathbf{z}]$ and $\det U = k_0 \in K^*$? As mentioned in the previous section, both nonconstructive and constructive methods have been advocated to solve Serre's conjecture [8]. Notice, from Definition 2, that F is ZRP implies that the $l \times l$ minors $a_1, \dots, a_\beta$ of F are zero coprime.

Consider a generalization of Serre's conjecture: let d be the g.c.d. of all the maximal order minors of F, and $b_1, \dots, b_\beta$ be the reduced minors of F. If $b_1, \dots, b_\beta$ are zero coprime, can F be completed to a square matrix $U = [F\ E] \in K^{m\times m}[\mathbf{z}]$ such that $E \in K^{m\times(m-l)}[\mathbf{z}]$ and $\det U = d$? When $d = k_0 \in K^*$, the generalized Serre's conjecture reduces to the original Serre's conjecture. In the remainder of the paper, we assume that d is not a unit in $K[\mathbf{z}]$, and the field of coefficients K is an algebraically closed field, such as the field of complex number $\mathbb{C}$. The generalized Serre's conjecture has not been proved for the ring $K[\mathbf{z}]$ when $n > 2$.

3. Main results

In this section, we show that the generalized Serre's conjecture, enunciated in the previous section, is equivalent to several other, possibly more tractable, conjectures. This set of conjectures is complete in the sense that solution of any would automatically solve the remaining ones. Another related but not equivalent conjecture on polynomial matrix factorization will also be discussed in this section. We shall then show how to apply these results to multivariate polynomial matrices not of full rank.

3.1. Full rank case

We first require a lemma.

Lemma 1. *Let F be given in Definition* 1. *If* $E \in K^{m\times(m-l)}[\mathbf{z}]$ *such that* $U = [F\ \ E] \in K^{m\times m}[\mathbf{z}]$ *with* $\det U = d$, *then E must be ZRP.*

Proof. By Laplace's expansion, we have

$$d = \det U = \sum_{i=1}^{\beta} a_i e_i = \sum_{i=1}^{\beta} d b_i e_i = d\left(\sum_{i=1}^{\beta} b_i e_i\right), \tag{2}$$

or

$$\left(\sum_{i=1}^{\beta} b_i e_i\right) = 1, \tag{3}$$

where $e_1, \ldots, e_\beta$ are the maximal order minors (called maximal minors for simplicity) of E. From (3), it is obvious that $e_1, \ldots, e_\beta$ are zero coprime. Therefore, E is ZRP. □

In fact, (3) also implies that a necessary condition for F to be completed into $U \in K^{m\times m}[\mathbf{z}]$ with $\det U = d$ is that $b_1, \ldots, b_\beta$ are zero coprime. We conjecture that this condition is also sufficient and relate it to other distinct conjectures stated below.

Conjectures 1–4. *Let F be given in Definition* 1. *If* $b_1, \ldots, b_\beta$ *are zero coprime*, *then we have the following conjectures*:

1. *There exists* $E \in K^{m\times(m-l)}[\mathbf{z}]$ *such that* $U = [F\ \ E] \in K^{m\times m}[\mathbf{z}]$ *with* $\det U = d$.
2. *F can be factored as* $F = F_0 G_0$ *for some* $F_0 \in K^{m\times l}[\mathbf{z}]$, $G_0 \in K^{l\times l}[\mathbf{z}]$ *with* $\det G_0 = d$.
3. *There exists* $H \in K^{l\times m}[\mathbf{z}]$ *such that* $HF = G_0$ *for some* $G_0 \in K^{l\times l}[\mathbf{z}]$ *with* $\det G_0 = d$.
4. *There exists* $B \in K^{(m-l)\times m}[\mathbf{z}]$ *with B being ZLP such that* $BF = 0_{m-l,l}$.

Conjecture 1 is the generalized Serre's conjecture discussed earlier. Conjecture 2 was raised in [19] and a partial solution to it was also given there. Although Conjecture 3 was not explicitly stated in [18], issues related to it were discussed there. However, the relationship among these conjectures has not yet been discussed in the literature.

Proposition 1. *Conjectures* 1–4 *are equivalent.*

Proof. Suppose first that Conjecture 2 is true, i.e., F can be factored as $F = F_0G_0$ for some $F_0 \in K^{m\times l}[\mathbf{z}]$ with F_0, and $G_0 \in K^{l\times l}[\mathbf{z}]$ with $\det G_0 = d$. It follows that the maximal minors of F_0 are $b_1, \dots, b_\beta$. Thus F_0 is ZRP. By the Quillen–Suslin theorem [8,23], there exists $E_0 \in K^{m\times(m-l)}[\mathbf{z}]$ such that $U_0 = [F_0\ E_0] \in K^{m\times m}[\mathbf{z}]$ with $\det U_0 = 1$. Let $V_0 = U_0^{-1}$ and partition V_0 as $V_0 = \left[\begin{smallmatrix} H \\ B \end{smallmatrix}\right]$ such that $H \in K^{l\times m}[\mathbf{z}]$ and $B \in K^{(m-l)\times m}[\mathbf{z}]$.

We have

$$V_0U_0 = \begin{bmatrix} H \\ B \end{bmatrix} [F_0\ E_0] = \begin{bmatrix} I_l & 0_{l,m-l} \\ 0_{m-l,l} & I_{m-l} \end{bmatrix}, \tag{4}$$

or

$$\begin{bmatrix} H \\ B \end{bmatrix} F_0 = \begin{bmatrix} I_l \\ 0_{m-l,l} \end{bmatrix}. \tag{5}$$

It follows that

$$HF_0 = I_l \tag{6}$$

and

$$BF_0 = 0_{m-l,l}. \tag{7}$$

We then have the following results:

Conjecture 2 $\Rightarrow$ *Conjecture* 3: From (6), $HF = HF_0G_0 = G_0$, with $G_0 \in K^{l\times l}[\mathbf{z}]$, and $\det G_0 = d$.

Conjecture 2 $\Rightarrow$ *Conjecture* 4: From (7), $BF = BF_0G_0 = 0_{m-l,l}G_0 = 0_{m-l,l}$.

Conjecture 2 $\Rightarrow$ *Conjecture* 1:

$$\begin{aligned} \det[F\ E_0] &= \det\left\{[F_0\ E_0]\begin{bmatrix} G_0 & 0_{l,m-l} \\ 0_{m-l,l} & I_{m-l} \end{bmatrix}\right\} \\ &= \det U_0 \det \begin{bmatrix} G_0 & 0_{l,m-l} \\ 0_{m-l,l} & I_{m-l} \end{bmatrix} \\ &= 1 \det G_0 \\ &= d. \end{aligned}$$

It remains to show that Conjectures 1, 3 and 4 imply Conjecture 2. We shall show first that Conjectures 1 and 4 imply Conjecture 3, and then that Conjecture 3 implies Conjecture 2.

Conjecture 1 ⇒ *Conjecture* 3: Suppose that there exists an $E \in K^{m\times(m-l)}[\mathbf{z}]$ such that $U = [F\ E] \in K^{m\times m}[\mathbf{z}]$ with $\det U = d$. By Lemma 1, E is ZRP. By the Quillen–Suslin theorem [8,23], there exists an $F_0 \in K^{m\times l}[\mathbf{z}]$ such that $U_0 = [F_0\ E] \in K^{m\times m}[\mathbf{z}]$ with $\det U_0 = 1$. Let $V_0 = U_0^{-1}$ and partition V_0 as $V_0 = \left[\begin{smallmatrix} H \\ B \end{smallmatrix}\right]$ such that $H \in K^{l\times m}[\mathbf{z}]$ and $B \in K^{(m-l)\times m}[\mathbf{z}]$. We have

$$V_0U_0 = \begin{bmatrix} H \\ B \end{bmatrix}[F_0\ E] = \begin{bmatrix} I_l & 0_{l,m-l} \\ 0_{m-l,l} & I_{m-l} \end{bmatrix}. \tag{8}$$

Replacing F_0 in (8) with F gives

$$\begin{bmatrix} H \\ B \end{bmatrix}[F\ E] = \begin{bmatrix} G_0 & 0_{l,m-l} \\ X & I_{m-l} \end{bmatrix} \tag{9}$$

for some $X \in K^{(m-l)\times l}[\mathbf{z}]$ and $G_0 \in K^{l\times l}[\mathbf{z}]$. From (9),

$$HF = G_0. \tag{10}$$

It remains to show that $\det G_0 = d$. This is immediate if we take the determinants of the matrices at both sides of (9):

$$\det V_0 \det U = \det G_0 \tag{11}$$

or

$$\det G_0 = d. \tag{12}$$

Conjecture 4 ⇒ *Conjecture* 3: Suppose that there exists $B \in K^{(m-l)\times m}[\mathbf{z}]$ with B being ZLP, such that $BF = 0_{m-l,l}$. By the Quillen–Suslin theorem [8,23], there exists $H \in K^{l\times m}[\mathbf{z}]$ such that $V_0 = \left[\begin{smallmatrix} H \\ B \end{smallmatrix}\right]$ with $\det V_0 = 1$. We then have

$$V_0F = \begin{bmatrix} H \\ B \end{bmatrix} F = \begin{bmatrix} G_0 \\ 0_{m-l,l} \end{bmatrix} = F', \tag{13}$$

where the matrix F' is defined, implying that

$$HF = G_0 \tag{14}$$

for some $G_0 \in K^{l\times l}[\mathbf{z}]$. Since the only nonzero $l \times l$ minor of F' is $\det G_0$, the g.c.d. of the $l \times l$ minors of F' is $\det G_0$. Since $\det V_0 = 1$, by Lemma 1 of [24], the g.c.d. of the $l \times l$ minors of F' is the same as the g.c.d. of the $l \times l$ minors of F. It follows that $\det G_0 = d$.

Conjecture 3 ⇒ *Conjecture* 2: Suppose that there exists $H \in K^{l\times m}[\mathbf{z}]$ such that $HF = G_0$ for some $G_0 \in K^{l\times l}[\mathbf{z}]$ with $\det G_0 = d$. We first show that H is ZLP. Let $h_1, \ldots, h_\beta$ denote the $l \times l$ minors of H. By Cauchy–Binet formula, we have

$$\sum_{i=1}^{\beta} h_i a_i = d, \tag{15}$$

or

$$\sum_{i=1}^{\beta} h_i d b_i = d\left(\sum_{i=1}^{\beta} h_i b_i\right) = d. \tag{16}$$

It follows that

$$\sum_{i=1}^{\beta} h_i b_i = 1. \tag{17}$$

Therefore, H is ZLP. By the Quillen–Suslin theorem [8,23], there exists $B \in K^{(m-l)\times m}[\mathbf{z}]$, such that $V_0 = \left[\begin{smallmatrix} H \\ B \end{smallmatrix}\right]$ with $\det V_0 = 1$. We then have

$$V_0 F = \begin{bmatrix} H \\ B \end{bmatrix} F = \begin{bmatrix} G_0 \\ X \end{bmatrix} = F' \tag{18}$$

for some $X \in K^{(m-l)\times l}[\mathbf{z}]$. Since $\det V_0 = 1$, V_0 is ZRP. By Lemma 3 of [19], the reduced minors of F' are zero coprime and the g.c.d. of the $l \times l$ minors of F' is equal to d. Consider now $XG_0^{-1} = \{X \operatorname{adj} G_0\}/\det G_0 = \{X \operatorname{adj} G_0\}/d$. By Cramer's rule, each entry of the matrix $\{X \operatorname{adj} G_0\}$ is just some $l \times l$ minor of F', and hence is divisible by d. It follows immediately that $\{XG_0^{-1}\}$ is a polynomial matrix! We then have

$$F' = \begin{bmatrix} I_l \\ XG_0^{-1} \end{bmatrix} G_0 = F_1 G_0 \tag{19}$$

for some $F_1 \in K^{m\times l}[\mathbf{z}]$. Combining (18) and (19) gives

$$F = V_0^{-1} F_1 G_0 = F_0 G_0, \tag{20}$$

where $F_0 \in K^{m\times l}[\mathbf{z}]$. $\square$

The above result shows that it suffices to consider only one of Conjectures 1–4, say Conjecture 2. It is well known [14] that Conjecture 2 is true for univariate and bivariate polynomial rings over a specified field K. However, for the n-variate ($n > 2$) polynomial ring over K, its validity is still unknown except for some special cases. When $l = 1$, Conjecture 2 is always true since in this case, F is just an $m \times 1$ vector consisting of m polynomials in $K[\mathbf{z}]$, and it is possible [12] to extract the g.c.d. from a finite number of n-variate polynomials. Another special case for which Conjecture 2 is true is when $m = l + 1$, as proved recently by Lin [19]. For other cases where $1 < l < m - 1$, Conjecture 2 remains unresolved. It is our hope that this paper may bring more attention to it as well as to other related issues discussed here.

So far we have restricted our discussion to the case where the reduced minors $b_1, \ldots, b_\beta$ are zero coprime. However, for n-variate ($n > 1$) polynomials, $b_1, \ldots, b_\beta$ may not be zero coprime even if they are factor coprime. In [19], Lin also raised another conjecture for this case.

Conjecture 5 [19]. *Let F be given in Definition* 1. *If d, $b_1, \ldots, b_\beta$ are zero coprime, then F can be factored as*

$$F = F_0 G_0 \tag{21}$$

for some $F_0 \in K^{m\times l}[\mathbf{z}]$, $G_0 \in K^{l\times l}[\mathbf{z}]$ *with* $\det G_0 = d$.

The difference between Conjecture 2 and Conjecture 5 is that $b_1, \ldots, b_\beta$ in Conjecture 5 may not be zero coprime, but $d, b_1, \ldots, b_\beta$ are zero coprime. Surprisingly, these two apparently different conjectures turn out to be equivalent.

Proposition 2. *Conjectures* 2 *and* 5 *are equivalent.*

Proof. *Conjecture* 5 $\Rightarrow$ *Conjecture* 2: It is rather easy to prove this part. Suppose that Conjecture 5 is true. Consider F given in Definition 1. If $b_1, \ldots, b_\beta$ are zero coprime, then $d, b_1, \ldots, b_\beta$ are also zero coprime. Hence, F can be factored as

$$F = F_0 G_0 \tag{22}$$

for some $F_0 \in K^{m\times l}[\mathbf{z}]$, $G_0 \in K^{l\times l}[\mathbf{z}]$ with $\det G_0 = d$.

Conjecture 2 $\Rightarrow$ *Conjecture* 5: Suppose that Conjecture 2 is true. Let F be given in Definition 1. Consider a new matrix $F_1 = \left[\begin{smallmatrix} F \\ D \end{smallmatrix}\right] \in K^{(m+l)\times l}[\mathbf{z}]$, where $D = \operatorname{diag}\{d, \ldots, d\} \in K^{l\times l}[\mathbf{z}]$. Let d_1 denote the g.c.d. of the $l \times l$ minors of F_1. We first show that $d_1 = k_0 d$ for some $k_0 \in K^*$. From the way F_1 is constructed, any $l \times l$ minor of F_1 is either an $l \times l$ minor of F or is formed from l rows of F_1 with at least one row from D. In either case, the minor is divisible by d. Hence, d_1 is divisible by d. On the other hand, since F is an $m \times l$ submatrix of F_1 and d is the g.c.d. of the $l \times l$ minors of F, d is divisible by d_1. Therefore, $d_1 = k_0 d$. Without loss of generality, let $k_0 = 1$, so we have $d_1 = d$.

We next show that the reduced minors of F_1 are zero coprime. It is easy to see that the reduced minors of F_1 contains the subset $\{b_1, \ldots, b_\beta, d^{l-1}\}$. By assumption $d, b_1, \ldots, b_\beta$ are zero coprime. This implies that $d^{l-1}, b_1, \ldots, b_\beta$ are zero coprime. It follows that the reduced minors of F_1 are also zero coprime. Therefore, F_1 can be factored as

$$F_1 = F_2 G_0 \tag{23}$$

for some $F_2 \in K^{(m+l)\times l}[\mathbf{z}]$, $G_0 \in K^{l\times l}[\mathbf{z}]$ with $\det G_0 = d$. Partition F_2 as $F_2 = \left[\begin{smallmatrix} F_0 \\ X \end{smallmatrix}\right]$, such that $F_0 \in K^{m\times l}[\mathbf{z}]$. From (23) we have

$$\begin{bmatrix} F \\ D \end{bmatrix} = \begin{bmatrix} F_0 \\ X \end{bmatrix} G_0, \tag{24}$$

or

$$F = F_0 G_0. \tag{25}$$

The proof is thus completed. $\square$

We now consider another related matrix factorization problem: given $F \in K^{l\times l}[\mathbf{z}]$ with $\det F = \prod_{j=1}^{J} f_j$ ($f_j \in K[\mathbf{z}]$, $j = 1, \ldots, J$), to factorize F as $F = \prod_{j=1}^{J} F_j$ with $F_j \in K^{l\times l}[\mathbf{z}]$ and $\det F_j = f_j$ ($j = 1, \ldots, J$). This is the so-called determinantal factorization problem. It is well known [14] that univariate and bivariate polynomial square matrices always admit determinantal factorizations. However, it has been pointed out [11] that some n-variate ($n > 2$) polynomial matrix does not have a determinantal factorization. Thus, it is interesting to know whether or not a given n-variate polynomial matrix admits a determinantal factorization. We raise a conjecture for this problem.

Conjecture 6. *Let* $F \in K^{l\times l}[\mathbf{z}]$ *with* $\det F = \prod_{j=1}^{J} f_j$ $(f_j \in K[\mathbf{z}],\ j = 1, \dots, J)$. *If* $f_1, \dots, f_J$ *are pairwise zero coprime, i.e.,* f_i *and* f_k *are zero coprime for* $1 \leqslant i, k \leqslant J,\ i \neq k$, *then* F *can be factored as* $F = \prod_{j=1}^{J} F_j$ *with* $F_j \in K^{l\times l}[\mathbf{z}]$ *and* $\det F_j = f_j$ $(j = 1, \dots, J)$.

We next prove that Conjecture 6 is implied by Conjecture 2.

Proposition 3. *If Conjecture* 2 *is true, so is Conjecture* 6.

Proof. *Conjecture* 2 $\Rightarrow$ *Conjecture* 6: The proof is in fact similar to the proof for Proposition 2 and is sketched here. Let F be given in Conjecture 6, and let $d_1 = f_2, \dots, f_J$. From the way d_1 is constructed and by the assumption on the pairwise zero coprimeness on $f_1, \dots, f_J$, it is clear that f_1 and d_1 are zero coprime. By considering the new matrix $F_0 = \begin{bmatrix} F \\ D_1 \end{bmatrix}$, where $D_1 = \operatorname{diag}\{d_1, \dots, d_1\}$, and arguing similarly as in the proof for Proposition 2, it can be shown that F can be factored as $F = F_1 F'$ such that $F_1, F' \in K^{l\times l}[\mathbf{z}]$ and $\det F_1 = f_1$, $\det F' = d_1$. Continuing the above procedure J times, we finally have $F = \prod_{j=1}^{J} F_j$ with $F_j \in K^{l\times l}[\mathbf{z}]$ and $\det F_j = f_j$ $(j = 1, \dots, J)$. □

Unfortunately, at this stage we are not able to show if Conjecture 6 implies Conjecture 2.

3.2. Degenerate rank case

Systems that admit a full column or row rank factorization are those whose *projective dimension* (the minimum of the lengths of projective resolutions) is at most one [6]. Also, a minimal left (right) annihilator of a bivariate polynomial matrix belonging to $K[\mathbf{z}] = K[z_1, z_2]$ is either zero or can be chosen to be a matrix with full row (column) rank. The full rank condition, however, becomes restrictive in the case of three or more indeterminates. The rank deficient case and other notions of primeness that do not require the imposition of the restriction of full rank was studied, recently, in [7]. For a discursive documentation, see the monograph by Zerz [27]. Therefore, the counterparts of the results, obtained so far in this paper, need to be investigated when F is not of full rank, as is done next.

Assumption 1. *Let* $F \in K^{m\times r}[z_1, \dots, z_n]$ *be of rank* l *with* $l < m$ *and* $l < r$. *Assume that* $F_1, \dots, F_k$ *are all the* $m \times l$ *full rank submatrices of* F, *and* d_i *is the* *g.c.d. of all the* $l \times l$ *minors of* F_i, $i = 1, \ldots, k$. *From* [5, 21], $F_1, \ldots, F_k$ *have the same reduced minors, denoted by* $b_1, \ldots, b_\beta$. *For simplicity, we call* $b_1, \ldots, b_\beta$ *the column reduced minors of* F.

Note that for a matrix not of full rank, its column reduced minors may not be the same as its row reduced minors. With the above notation and assumption, we can raise the following conjectures concerning multivariate polynomial matrices not of full rank.

Conjectures 7–9. *Let* F *be given in Assumption* 1. *If* $b_1, \ldots, b_\beta$ *are zero coprime, then we have the following conjectures*:

7. F *can be factored as* $F = A_1 A_2$ *for some* $A_1 \in K^{m\times l}[\mathbf{z}]$, $A_2 \in K^{l\times r}[\mathbf{z}]$ *with* A_1 *being ZRP*, $\det G_i = d_i$, *where* G_i *is the* ith $l \times l$ *submatrix of* A_2 *corresponding to* F_i, $i = 1, \ldots, k$.
8. *There exists* $H \in K^{l\times m}[\mathbf{z}]$ *such that* $HF = A_2$ *for some* $A_2 \in K^{l\times r}[\mathbf{z}]$ *such that* $\det G_i = d_i$, *where* G_i *is the* ith $l \times l$ *submatrix of* A_2 *corresponding to* F_i, $i = 1, \ldots, k$.
9. *There exists* $B \in K^{(m-l)\times m}[\mathbf{z}]$ *with* B *being ZLP*, *such that* $BF = 0_{m-l,r}$.

Conjectures 7–9 may be considered as generalizations of Conjectures 2–4. It should be pointed out that when F is not a full rank matrix, it cannot be completed into a square matrix whose determinant is nonzero. In the following proposition, we show that Conjectures 7–9 are equivalent to Conjectures 2–4. For convenience of exposition, we re-state Conjecture 2 in the following.

Conjecture 2′. *Let* F *be given in Assumption* 1. *If* $b_1, \ldots, b_\beta$ *are zero coprime, then* F_1 *can be factored as* $F_1 = A_1 G_1$ *for some* $A_1 \in K^{m\times l}[\mathbf{z}]$ *with* A_1 *being ZRP*, $G_1 \in K^{l\times l}[\mathbf{z}]$ *with* $\det G_1 = d_1$.

Proposition 4. *Conjectures* 2′, 7–9 *are equivalent.*

Proof. We only show the equivalence of Conjecture 2′ and Conjecture 7. The other equivalences can be shown similarly as in the proof of Proposition 1.

Without loss of generality, assume that F_1 is formed from the first l columns of F, i.e., $F = [F_1\ C]$, where $C \in K^{m\times(r-l)}[\mathbf{z}]$.

Conjecture 7 ⇒ *Conjecture* 2′: Assume that F can be factored as $F = A_1 A_2$ for some $A_1 \in K^{m\times l}[\mathbf{z}]$, $A_2 \in K^{l\times r}[\mathbf{z}]$ with A_1 being ZRP, $\det G_i = d_i$, where G_i is the ith $l \times l$ submatrix of A_2 corresponding to F_i, $i = 1, \ldots, k$.

Since G_1 is the $l \times l$ submatrix of A_2 corresponding to F_1, it is formed from the first l columns of A_2. Hence, we can re-write A_2 as $A_2 = [G_1\ D]$, where $D \in K^{l\times(r-l)}[\mathbf{z}]$. We then have

$$[F_1\ C] = A_1[G_1\ D]. \tag{26}$$

It is then obvious that

$$F_1 = A_1 G_1 \tag{27}$$

with $G_1 \in K^{l\times l}[\mathbf{z}]$ and $\det G_1 = d_1$.

Conjecture 2′ ⇒ *Conjecture* 7: Assume that F_1 can be factored as $F_1 = A_1 G_1$ for some $A_1 \in K^{m\times l}[\mathbf{z}]$ with A_1 being ZRP, $G_1 \in K^{l\times l}[\mathbf{z}]$ with $\det G_1 = d_1$. Since F_1 is a full rank $m \times l$ submatrix of F and A_1 is ZRP, by a known result in [11] (see, also [5]), it can be asserted that F admits a polynomial factorization:

$$F = A_1 A_2 \tag{28}$$

for some $A_2 \in K^{l\times r}[\mathbf{z}]$. Let G_i be the ith $l \times l$ submatrix of A_2 corresponding to F_i, $i = 1, \ldots, k$. It is then straightforward to show that $\det G_i = d_i$, $i = 1, \ldots, k$. □

Finally, we generalize Conjecture 5 to the case where F is not of full rank. Similarly to the proof of Proposition 4, it can also be shown that Conjecture 10 is equivalent to Conjecture 5. It is omitted here to save space.

Conjecture 10. *Let F be given in Assumption* 1. *If* $d_i, b_1, \ldots, b_\beta$ *are zero coprime for some* $i \in \{i, \ldots, k\}$, *then F can be factored as* $F = A_1 A_2$ *for some* $A_1 \in K^{m\times l}[\mathbf{z}]$, $A_2 \in K^{l\times r}[\mathbf{z}]$ *with* A_1 *being MRP*, $\det G_i = d_i$, *where* G_i *is the* ith $l \times l$ *submatrix of* A_2 *corresponding to* F_i, $i = 1, \ldots, k$.

4. Conclusions

The various issues in multivariate polynomial matrix factorization for the case when the maximal minors are zero coprime are linked by a set of equivalent conjectures (these equivalences have been proved), which are complete in the sense that a proof of any one will settle all. For example, in the special case when $l = m - 1$ in Definition 1, it has been proved that zero coprimeness of the m reduced minors is sufficient for an n-variate polynomial matrix F to admit a right factorization $F = F_0 G_0$ with $\det G_0 = d$ [19] and, therefore, from the results in this paper each of Conjectures 1–5 is true. The truth of each of Conjectures 1–5 for the case when $l = 1$ and m is arbitrary, follows dirctly from the truth of Conjecture 2 (that may easily be inferred from Proposition 1 in [28]), and the equivalence results proved in Propositions 1 and 2 of this paper. In fact, the algorithmic approach delineated in [9,10] for unimodular completion in Serre's problem coupled with the computational device used in the proof of Proposition 1 in [28], provides a constructive scheme for proving the truth of Conjecture 2 in the special case when $l = 1$ and m is arbitrary.

It is important to emphasize that when the zero coprimeness condition is not satisfied by the reduced minors, Conjectures 1–4 cannot hold in general. For example, consider the 1×2 matrix $F = [z_1 z_2 \;\; z_1 z_3]^{\mathrm{T}} \in C^{1\times 2}[z_1, z_2, z_3]$. Clearly, it is not possible for F to be completed into a square matrix $U \in C^{2\times 2}[z_1, z_2, z_3]$, with F being the first column of U such that $\det U = z_1$. This is because although the reduced minors z_2 and z_3 are factor coprime, they are not zero coprime.

The matrices in the multivariate matrix polynomial factorization problems and related issues considered in this paper belong to the n-variate polynomial ring over the complex field (which is an algebraically closed field of characteristic zero). One of the most important invariants of a ring, which is related to the cancellation problem as well as other problems of stabilization [25] is its *stability rank* (s.r.), which is a kind of dimensionality of the ring. The precise value of the *stability rank* (known for almost all fields in the bivariate case i.e., when $n = 2$) or its least upper bound is known to be field-dependent. For example s.r. $(R[z_1, \ldots, z_n]) = n + 1$, while s.r. $(C[z_1, \ldots, z_n]) > 1 + n/2$. The counterparts of the results in this paper when the field is not algebraically closed remain to be investigated. The constructive aspects of the results in this paper are implementable by the use of algorithmic algebra like Gröbner bases, generalized Gröbner bases and their variants [26].

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