On Minor Prime Factorizations for n-D Polynomial Matrices

Zhiping Lin, *Member, IEEE*, Li Xu, *Senior Member, IEEE*, and Huijin Fan

***Abstract*—A tractable criterion is presented for the existence of minor prime factorizations for a class of multidimensional (n-D) ($n > 2$) polynomial matrices whose reduced minors and greatest common divisors have some common zeros. We also present a constructive method for carrying out the minor prime factorizations when they exist. The proposed method is further extended to a larger class of n-D polynomial matrices by an invertible variable transformation. Three illustrative examples are given to show the effectiveness of the proposed method.**

***Index Terms*—Matrix factorizations, minor primeness, multidimensional (n-D) systems, n-D polynomial matrices, reduced minors.**

I. Introduction

The problems of multidimensional (n-D) polynomial matrix factorizations have attracted much attention over the past decades because of their wide applications in n-D circuits, systems, controls, signal processing, and other areas. Although the problem of prime factorizations for one- and two-dimensional polynomial matrices has been completely solved several decades ago, its n-D ($n > 2$) counterpart is more challenging and has not yet been fully resolved (see, e.g., [1] and the references therein).

Let $\mathbf{C}[\mathbf{z}] = \mathbf{C}[z_1, \ldots, z_n]$ denote the set of polynomials in complex variables $z_1, \ldots, z_n$ with coefficients in the field of complex numbers $\mathbf{C}$ and $\mathbf{C}^{m\times l}[\mathbf{z}]$ the set of $m\times l$ matrices with entries in $\mathbf{C}[\mathbf{z}]$, assuming that $m \geq l$ throughout the paper.

Definition 1: [2] Let $F(\mathbf{z}) \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be a normal full-rank matrix throughout the paper. Then $F(\mathbf{z})$ is said to be:

1) zero right prime (ZRP) if the $l\times l$ minors of $F(\mathbf{z})$ are zero coprime, i.e., devoid of any common zeros;
2) minor right prime (MRP) if the $l \times l$ minors of $F(\mathbf{z})$ are factor coprime, i.e., devoid of any common factors.

Definition 2: [3] Let $F(\mathbf{z}) \in \mathbf{C}^{m\times l}[\mathbf{z}]$, and let $a_1(\mathbf{z}), \ldots, a_\beta(\mathbf{z})$ denote the $l \times l$ minors of matrix $F(\mathbf{z})$, where $\beta \triangleq \binom{m}{l} = (m!)/((m-l)!l!)$. Extracting the greatest common divisor (g.c.d.), denoted by $d(\mathbf{z})$, of $a_1(\mathbf{z}), \ldots, a_\beta(\mathbf{z})$ gives

$$a_i(\mathbf{z}) = d(\mathbf{z})b_i(\mathbf{z}), \qquad i = 1, \ldots, \beta. \tag{1}$$

Then, $b_1(\mathbf{z}), \ldots, b_\beta(\mathbf{z})$ are called the reduced minors of $F(\mathbf{z})$.

Manuscript received May 20, 2004; revised December 19, 2004. This paper was recommended by Associate Editor B. C. Levy.

Z. Lin is with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: ezplin@ntu.edu.sg).

L. Xu and H. Fan are with the Department of Electronics and Information Systems, Akita Prefectural University, Akita 015-0055, Japan (e-mail: xuli@akita-pu.ac.jp).

Digital Object Identifier 10.1109/TCSII.2005.850516

Remark 1: If $F(\mathbf{z})$ is a square matrix, i.e., $m = l$, the only $l \times l$ minor of $F(\mathbf{z})$ is its determinant $a_1(\mathbf{z})$. In such a case, we define $b_1(\mathbf{z}) = 1$ and $d(\mathbf{z}) = a_1(\mathbf{z})$.

Let $F(\mathbf{z})$ be given as in Definition 2. The prime factorization problem is to investigate whether or not $F(\mathbf{z})$ can be factorized as

$$F(\mathbf{z}) = F_0(\mathbf{z})G_0(\mathbf{z}) \tag{2}$$

such that $F_0(\mathbf{z}) \in \mathbf{C}^{m\times l}[\mathbf{z}], G_0(\mathbf{z}) \in \mathbf{C}^{l\times l}[\mathbf{z}]$, and $\det G_0(\mathbf{z}) = d(\mathbf{z})$, and to carry out the above factorization when it exists.

It was conjectured in [4] and [5] that there always exists a prime factorization (2) if $b_1(\mathbf{z}), \ldots, b_\beta(\mathbf{z})$ are zero coprime, and in such a case (2) is the ZRP factorization. This Lin–Bose conjecture has recently been proved in [6]–[8]. The interesting relationship between zero prime factorization and module theory was also explored and explained in [6]–[8]. Constructive methods for carrying out the ZRP factorization for several special classes of n-D polynomial matrices have been proposed in [4] and [9]. Combining the results of [5] and [6]–[8], it can be readily shown that there also exists a prime factorization for $F(\mathbf{z})$ if $b_1(\mathbf{z}), \ldots, b_\beta(\mathbf{z})$ are not zero coprime, but $b_1(\mathbf{z}), \ldots, b_\beta(\mathbf{z}), d(\mathbf{z})$ are zero coprime, and in such a case (2) is the MRP factorization. However, if $b_1(\mathbf{z}), \ldots, b_\beta(\mathbf{z}), d(\mathbf{z})$ are not zero coprime, the MRP factorization (2) may not always exist. Indeed, the infeasibility of minor prime factorization in general was demonstrated through examples over two decades ago in [2] and [10]. To the best of our knowledge, the MRP factorization problem (2) with $b_1(\mathbf{z}), \ldots, b_\beta(\mathbf{z}), d(\mathbf{z})$ being not zero coprime is still very challenging [1]. It is expected that it may take some time before this minor prime factorization problem can be resolved completely. Meanwhile, we believe that any incremental progress would be useful in solving this open problem and in motivating further research in this direction.

In this paper, we propose a tractable criterion for the existence of minor prime factorizations for a class of n-D polynomial matrices whose reduced minors and g.c.d.s have some common zeros. We also present a constructive method for carrying out the minor prime factorization when it exists. The proposed method is further extended to a larger class of n-D polynomial matrices by an invertible variable transformation. Three illustrative examples are given to show the effectiveness of the proposed method. Related work and possible further research is also briefly discussed at the end of the paper.

II. Main Results

Before presenting the main results on minor prime factorizations, we first require two lemmas.

Lemma 1: Let $F(\boldsymbol{z}) \in \mathbf{C}^{m\times l}[\boldsymbol{z}]; a_1(\boldsymbol{z}),\ldots,a_\beta(\boldsymbol{z})$ be the $l\times l$ minors of $F(\boldsymbol{z})$, let $b_1(\boldsymbol{z}),\ldots,b_\beta(\boldsymbol{z})$ be the reduced minors of $F(\boldsymbol{z})$, and let $d(\boldsymbol{z})$ be the g.c.d. of $a_1(\boldsymbol{z}),\ldots,a_\beta(\boldsymbol{z})$. If $d(\boldsymbol{z}) = z_1 - f(z_2,\ldots,z_n)$, then $F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ is of normal rank $l-1$.

Proof: Since $d(\boldsymbol{z}) = z_1 - f(z_2,\ldots,z_n)$ is the g.c.d. of $a_1(\boldsymbol{z}),\ldots,a_\beta(\boldsymbol{z})$, it is obvious that $d(f(z_2,\ldots,z_n),z_2,\ldots,z_n) \equiv 0$ and, hence, the normal rank of $F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ is less than or equal to $l-1$. It suffices to show that there exist some fixed $z_2=z_{21},\ldots,z_n=z_{n1}$, such that $\operatorname{rank} F(f(z_{21},\ldots,z_{n1}),z_{21},\ldots,z_{n1}) = \operatorname{rank} F(f_0,z_{21},\ldots,z_{n1}) = l-1$, where $f_0 \triangleq f(z_{21},\ldots,z_{n1})$.

Since $b_1(\boldsymbol{z}),\ldots,b_\beta(\boldsymbol{z})$ are factor coprime, there exists $b_h(\boldsymbol{z})$ for some $1\le h\le\beta$ such that $b_h(f(z_2,\ldots,z_n),z_2,\ldots,z_n) \not\equiv 0$. Otherwise, $(z_1 - f(z_2,\ldots,z_n))$ would be a common divisor of $b_1(\boldsymbol{z}),\ldots,b_\beta(\boldsymbol{z})$, leading to a contradiction. It follows that there exist some fixed $z_2=z_{21},\ldots,z_n=z_{n1}$, such that $b_h(f_0,z_{21},\ldots,z_{n1})\neq 0$. With a slight abuse of notation, consider the one-dimensional (1-D) polynomial matrix $F(z_1) \triangleq F(z_1,z_{21},\ldots,z_{n1})$. Clearly, the $l\times l$ minors of $F(z_1)$ are $a_1(z_1),\ldots,a_\beta(z_1)$, where $a_i(z_1) = b_i(z_1,z_{21},\ldots,z_{n1})d(z_1,z_{21},\ldots,z_{n1}) = b_i(z_1,z_{21},\ldots,z_{n1})(z_1-f_0)$, for $i=1,\ldots,\beta$. In particular, $a_h(z_1) = b_h(z_1,z_{21},\ldots,z_{n1})(z_1-f_0)$. Since $b_h(f_0,z_{21},\ldots,z_{n1}) \neq 0, (z_1-f_0)$ is a simple factor of $a_h(z_1)$. By transforming $F(z_1)$ into its Smith form, it follows immediately that $\operatorname{rank} F(f_0) = \operatorname{rank} F(f(z_{21},\ldots,z_{n1}),z_{21},\ldots,z_{n1}) = l-1$. Therefore, $F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ is of normal rank $l-1$. □

Lemma 2: Let $A\in\mathbf{C}^{l\times l}[\boldsymbol{z}]$, and $\det A(\boldsymbol{z}) = d(\boldsymbol{z}) = z_1 - f(z_2,\ldots,z_n)$, then there exists a ZRP vector $\boldsymbol{w}\in\mathbf{C}^{l\times 1}[z_2,\ldots,z_n]$ such that

$$A(f(z_2,\ldots,z_n),z_2,\ldots,z_n)\boldsymbol{w}(z_2,\ldots,z_n) = [0,\ldots,0]^T. \tag{3}$$

Proof: From Remark 1, there is only one reduced minor of $A(\boldsymbol{z})$, which is just $b_1(\boldsymbol{z})=1$. Obviously, $b_1(\boldsymbol{z})$ and $d(\boldsymbol{z})$ have no common zeros. It follows from the proof procedure of [11, Theorem 1] that there exists a ZRP vector $\boldsymbol{w}\in\mathbf{C}^{l\times 1}[z_2,\ldots,z_n]$ such that

$$A(f(z_2,\ldots,z_n),z_2,\ldots,z_n)\boldsymbol{w}(z_2,\ldots,z_n) = [0,\ldots,0]^T. \tag{4}$$

□

We are now ready to present the main result.

Theorem 1: Let $F\in\mathbf{C}^{m\times l}[\boldsymbol{z}]$, let $a_1(\boldsymbol{z}),\ldots,a_\beta(\boldsymbol{z})$ be the $l\times l$ minors of $F(\boldsymbol{z})$, and let $d(\boldsymbol{z}) = z_1 - f(z_2,\ldots,z_n)$ be the g.c.d. of $a_1(\boldsymbol{z}),\ldots,a_\beta(\boldsymbol{z})$. A necessary and sufficient condition for $F(\boldsymbol{z})$ to admit a minor prime factorization $F(\boldsymbol{z}) = F_0(\boldsymbol{z})G_0(\boldsymbol{z})$ with $F_0\in\mathbf{C}^{m\times l}[\boldsymbol{z}], G_0\in\mathbf{C}^{l\times l}[\boldsymbol{z}]$, and $\det G_0(\boldsymbol{z}) = d(\boldsymbol{z})$ is that there exists a normal full rank $(l-1)\times l$ submatrix $F_1(z_2,\ldots,z_n) \triangleq F_1(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ of $F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ such that the reduced minors of $F_1(z_2,\ldots,z_n)$ are zero coprime.

Proof: (Necessity) Suppose that $F(\boldsymbol{z})$ admits a minor prime factorization

$$F(\boldsymbol{z}) = F_0(\boldsymbol{z})G_0(\boldsymbol{z}) \tag{5}$$

with $F_0(\boldsymbol{z})\in\mathbf{C}^{m\times l}[\boldsymbol{z}], G_0(\boldsymbol{z})\in\mathbf{C}^{l\times l}[\boldsymbol{z}]$, and $\det G_0(\boldsymbol{z}) = d(\boldsymbol{z})$. From (5), we have

$$F(f(z_2,\ldots,z_n),z_2,\ldots,z_n) = F_0(f(z_2,\ldots,z_n),z_2,\ldots,z_n)G_0(f(z_2,\ldots,z_n),z_2,\ldots,z_n). \tag{6}$$

By Lemma 2, there exists a ZRP vector $\boldsymbol{w}\in\mathbf{C}^{l\times 1}[z_2,\ldots,z_n]$ such that

$$G_0(f(z_2,\ldots,z_n),z_2,\ldots,z_n)\boldsymbol{w}(z_2,\ldots,z_n) = [0,\ldots,0]^T. \tag{7}$$

Multiplying (6) from the right-hand side by $\boldsymbol{w}(z_2,\ldots,z_n)$ and recalling (7) gives

$$F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)\boldsymbol{w}(z_2,\ldots,z_n) = [0,\ldots,0]^T. \tag{8}$$

By Lemma 1, the normal rank of $F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ is equal to $l-1$. Hence, there exists a normal full rank $(l-1)\times l$ submatrix $F_1(z_2,\ldots,z_n) \triangleq F_1(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ of $F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ such that

$$F_1(z_2,\ldots,z_n)\boldsymbol{w}(z_2,\ldots,z_n) = [0,\ldots,0]^T. \tag{9}$$

Since $\boldsymbol{w}(z_2,\ldots,z_n)$ is a ZRP vector, the reduced minors of $\boldsymbol{w}(z_2,\ldots,z_n)$ are simply its entries, which are zero coprime. As $F_1(z_2,\ldots,z_n)$ and $\boldsymbol{w}(z_2,\ldots,z_n)$ are of normal rank $l-1$ and 1, respectively, and satisfy (9), their reduced minors are essentially the same (i.e., differ possibly by the ordering and sign of the reduced minors only) [3], it follows immediately that the reduced minors of $F_1(z_2,\ldots,z_n)$ are also zero coprime.

Sufficiency: Assume that there exists a normal full rank $(l-1)\times l$ submatrix $F_1(z_2,\ldots,z_n) \triangleq F_1(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ of $F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ such that the reduced minors of $F_1(z_2,\ldots,z_n)$ are zero coprime. We can then construct a ZRP vector $\boldsymbol{w}\in\mathbf{C}^{l\times 1}[z_2,\ldots,z_n]$ such that [11]

$$F_1(z_2,\ldots,z_n)\boldsymbol{w}(z_2,\ldots,z_n) = [0,\ldots,0]^T. \tag{10}$$

Since $F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$ is of normal rank $l-1$ by Lemma 1, from (10) and the fact that $F_1(z_2,\ldots,z_n)$ is a normal full rank submatrix of $F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)$, we have

$$F(f(z_2,\ldots,z_n),z_2,\ldots,z_n)\boldsymbol{w}(z_2,\ldots,z_n) = [0,\ldots,0]^T. \tag{11}$$

Using the Quillen–Suslin theorem, we can construct a unimodular matrix $U\in\mathbf{C}^{l\times l}[z_2,\ldots,z_n]$, where $\boldsymbol{w}(z_2,\ldots,z_n)$ is the first column of $U(z_2,\ldots,z_n)$, such that [11]

$$F(\boldsymbol{z})U(z_2,\ldots,z_n) = F_0(\boldsymbol{z})\underbrace{\operatorname{diag}\{(z_1-f(z_2,\ldots,z_n)),1,\ldots,1\}}_{D(\boldsymbol{z})}$$

for some $F_0 \in \mathbf{C}^{m\times l}[\boldsymbol{z}]$. It follows that

$$F(\boldsymbol{z}) = F_0(\boldsymbol{z})G_0(\boldsymbol{z})$$

where $G_0(\boldsymbol{z}) = D(\boldsymbol{z})U^{-1}(z_2,\ldots,z_n) \in \mathbf{C}^{l\times l}[\boldsymbol{z}]$ with $\det G(\boldsymbol{z}) = (z_1 - f(z_2,\ldots,z_n)) = d(\boldsymbol{z})$ □.

Remark 2: With minor modification, the previous theorem is clearly valid for the case where

$$d(\boldsymbol{z}) = z_j - f(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_n) \tag{12}$$

for any $1 \le j \le n$. Furthermore, it also applies to the case where $d(\boldsymbol{z})$ is not of the form in (12), but, after the following invertible variable transformation (if a nonsingular constant matrix H of size $n\times n$ can be found)

$$[z_1,\ldots,z_n]^T = H[t_1,\ldots,t_n]^T \tag{13}$$

then $d(t_1,\ldots,t_n)$ is of the form in (12).

III. Examples

In this section we present three illustrative examples.

Example 1: This example was used by Charoenlarpnopparut and Bose in [9] to indicate that their algorithm for the zero prime factorization problem could not be applied to the minor prime factorization problem when the reduced minors have some common zeros. Note that the matrix in [9] has been transposed here. Let

$$F(z_1,z_2,z_3) = \begin{bmatrix} z_1(z_1-1) & z_1-1 \\ z_1 - z_2z_3 - z_3 & z_2^2+2z_2+2 \\ z_1z_3 - 2z_1 - z_3 & z_2+z_3-1 \end{bmatrix}. \tag{14}$$

It is easy to check that the g.c.d. of the 2×2 minors of $F(z_1,z_2,z_3)$ is $d(z_1,z_2,z_3) = z_3 + z_1z_2 + z_1$, and the reduced minors are $b_1 = (z_1-1)(z_2+1), b_2 = -z_2z_3 - z_3 + 2z_2 + 3$, and $b_3 = z_1 - 1$. Since $b_1(z_1,z_2,z_3), b_2(z_1,z_2,z_3), b_3(z_1,z_2,z_3)$, and $d(z_1,z_2,z_3)$ have a common zero at $(1,-2,1)$, neither the prime factorization algorithm of [9] nor that of [11] can be applied to this example. However, as $d(z_1,z_2,z_3)$ is of the form $z_3 - f(z_1,z_2)$, we can apply Theorem 1 to test whether $F(z_1,z_2,z_3)$ admits a minor prime factorization.

Substituting $z_3 = -(z_1z_2+z_1)$ into $F(z_1,z_2,z_3)$ and after simplification, we have

$$F(z_1,z_2,-(z_1z_2+z_1)) = \begin{bmatrix} z_1(z_1-1) & z_1-1 \\ z_1(z_2^2+2z_2+2) & z_2^2+2z_2+2 \\ z_1(z_2 - z_1z_2 - z_1 - 1) & z_2 - z_1z_2 - z_1 - 1 \end{bmatrix}.$$

Obviously, $F(z_1,z_2,-(z_1z_2+z_1))$ is of normal rank 1. The reduced minors of its first row (one of its 1×2 submatrices) are z_1 and 1, which are zero coprime. By Theorem 1, there exists a minor prime factorization for $F(z_1,z_2,z_3)$. To construct such a factorization, let $U = \begin{bmatrix} 1 & 0 \\ -z_1 & 1 \end{bmatrix}$, with $\det U = 1$. We have

$$F(z_1,z_2,-(z_1z_2+z_1))U = \begin{bmatrix} 0 & z_1-1 \\ 0 & z_2^2+2z_2+2 \\ 0 & z_2 - z_1z_2 - z_1 - 1 \end{bmatrix}$$

which gives

$$F(z_1,z_2,z_3)U = \begin{bmatrix} 0 & z_1-1 \\ -(z_2+1) & z_2^2+2z_2+2 \\ -1 & z_2+z_3-1 \end{bmatrix} \underbrace{\begin{bmatrix} z_3+z_1z_2+z_1 & 0 \\ 0 & 1 \end{bmatrix}}_{D}.$$

Thus

$$F(z_1,z_2,z_3) = \underbrace{\begin{bmatrix} 0 & z_1-1 \\ -(z_2+1) & z_2^2+2z_2+2 \\ -1 & z_2+z_3-1 \end{bmatrix}}_{F_0(z_1,z_2,z_3)} \underbrace{\begin{bmatrix} z_3+z_1z_2+z_1 & 0 \\ z_1 & 1 \end{bmatrix}}_{G_0(z_1,z_2,z_3)}..$$

Clearly, $\det G_0(z_1,z_2,z_3) = z_3 + z_1z_2 + z_1 = d(z_1,z_2,z_3)$, and we have obtained the desired MRP factorization for $F(z_1,z_2,z_3)$.

Example 2: Consider the following example from [2]. Note that the matrix in [2] has been transposed here. Let

$$F(z_1,z_2,z_3) = \begin{bmatrix} z_2z_3 - z_1^3 & z_3^2 - z_1^2z_2 \\ z_2^2 - z_1z_3 & z_2z_3 - z_1^3 \\ z_1 & 0 \\ 0 & z_1 \end{bmatrix}. \tag{15}$$

It is easy to check that the g.c.d. of the 2×2 minors of $F(z_1,z_2,z_3)$ is $d(z_1,z_2,z_3) = z_1$, and the reduced minors are $b_1 = b_2 = z_2z_3 - z_1^3, b_3 = z_3^2 - z_1^2z_2, b_4 = z_2^2 - z_1z_3, b_5 = z_1^5 - 3z_1^2z_2z_3 + z_1z_2^3 + z_3^3$, and $b_6 = z_1$. Since $F(z_1,z_2,z_3)$ is not MRP, it is interesting to know whether $F(z_1,z_2,z_3)$ admits an MRP factorization. It was proved in [2], using a rather sophisticated technique, that $F(z_1,z_2,z_3)$ does not have an MRP factorization. As $d(z_1,z_2,z_3)$ is of the form $z_1 - f(z_2,z_3)$ (with $f(z_2,z_3)\equiv 0$), we can apply Theorem 1 to test whether or not $F(z_1,z_2,z_3)$ admits an MRP factorization.

Substituting $z_1 = 0$ into $F(z_1,z_2,z_3)$, we have

$$F(0,z_2,z_3) = \begin{bmatrix} z_2z_3 & z_3^2 \\ z_2^2 & z_2z_3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

There are only two normal full rank 1×2 submatrices of $F(0,z_2,z_3)$, which are the first and second rows. Obviously, the reduced minors of each of these two rows are z_2 and z_3. Since z_2 and z_3 are not zero coprime, by Theorem 1, $F(z_1,z_2,z_3)$ does not have an MRP factorization. It can be seen that our approach is more elementary and simpler than the one given in [2].

Example 3: In this example, let

$$F'(z_1,z_2,z_3) = \begin{bmatrix} z_1(z_1-1) & z_1-1 \\ z_1 - z_2z_3 + z_3^2 - z_3 & z_2^2 - 2z_2z_3 + z_3^2 + 2z_2 - 2z_3 + 2 \\ z_1z_3 - 2z_1 - z_3 & z_2 - 1 \end{bmatrix}.$$

The g.c.d. of the 2×2 minors of $F'(z_1,z_2,z_3)$ is $d'(z_1,z_2,z_3) = z_3 + z_1z_2 - z_1z_3 + z_1$, and it could be easily verified that the reduced minors $b_1'(z_1,z_2,z_3), b_2'(z_1,z_2,z_3), b_3'(z_1,z_2,z_3)$, and $d'(z_1,z_2,z_3)$

have a common zero at $(1,-1,1)$. As $d'(z_1,z_2,z_3)$ is not of the form in (12), we cannot apply Theorem 1 directly to test the minor prime factorizability of $F'(z_1,z_2,z_3)$. Observe that $d'(z_1,z_2,z_3) = z_3 + z_1(z_2 - z_3) + z_1$ and, if we set $t_1 = z_1, t_2 = z_2 - z_3$, and $t_3 = z_3$ or equivalently

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{16}$$

then $d'(z_1,z_2,z_3)$ can be transformed into $d'(t_1,t_2,t_3) = t_3 + t_1t_2 + t_1$, which is now of the form in (12). Correspondingly, the invertible variable transformation of (13) follows from (16) as

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{H} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \tag{17}$$

and $F'(z_1,z_2,z_3)$ becomes

$$F'(t_1,t_2,t_3) = \begin{bmatrix} t_1(t_1-1) & t_1 - 1 \\ t_1 - t_2t_3 - t_3 & t_2^2 + 2t_2 + 2 \\ t_1t_3 - 2t_1 - t_3 & t_2 + t_3 - 1 \end{bmatrix}. \tag{18}$$

It is ready to verify that the g.c.d. of the 2×2 minors of $F'(t_1,t_2,t_3)$ is just $d'(t_1,t_2,t_3)$. Using the results of Example 1 and applying the variable transformation of (16), we can obtain the desired MRP factorization for $F'(z_1,z_2,z_3)$ as follows:

$$\begin{aligned} F'(z_1,z_2,z_3) &= \underbrace{\begin{bmatrix} 0 & z_1 - 1 \\ -(z_2 - z_3 + 1) & z_2^2 - 2z_2z_3 + z_3^2 + 2z_2 - 2z_3 + 2 \\ -1 & z_2 - 1 \end{bmatrix}}_{F_0'(z_1,z_2,z_3)} \\ &\quad \cdot \underbrace{\begin{bmatrix} z_3 + z_1z_2 - z_1z_3 + z_1 & 0 \\ z_1 & 1 \end{bmatrix}}_{G_0'(z_1,z_2,z_3)}, \end{aligned}$$

where $\det G_0'(z_1,z_2,z_3) = z_3 + z_1z_2 - z_1z_3 + z_1 = d'(z_1,z_2,z_3)$.

IV. Discussion and Conclusion

In this paper, we have addressed the open problem of minor prime factorizations for n-D polynomial matrices whose reduced minors and g.c.d.s. have some common zeros. A tractable criterion has been presented for the existence of minor prime factorizations for a class of n-D $(n > 2)$ polynomial matrices. We have also presented a constructive method for carrying out the minor prime factorizations when they exist. Three examples have been worked out to illustrate the effectiveness of the proposed method. Although the new method is applicable only when the g.c.d. has a special form, we hope the new result will motivate further work toward this challenging open problem. Although only normal full rank matrices are considered in this paper, factorizations for matrices not of normal full rank could be similarly discussed (see [4] and [8]).

Before concluding this paper, we point out the similarity and difference between the ideas presented in this paper and those in the recent paper [8], where Wang and Feng gave a simple proof of the Lin–Bose conjecture based on Fitting ideals, localization, and the local freedom of projective modules. Specifically, they made good use of the properties of the ideal generated by the $l \times l$ minors of $F(\boldsymbol{z})$, i.e., $a_1(\boldsymbol{z}),\ldots,a_\beta(\boldsymbol{z})$, where $a_i(\boldsymbol{z}) = d(\boldsymbol{z})b_i(\boldsymbol{z}), i = 1,\ldots,\beta$, and the fact that $b_1(\boldsymbol{z}),\ldots,b_\beta(\boldsymbol{z})$ are zero coprime. In this paper, we explore properties of the reduced minors of an $(l-1) \times l$ submatrix associated with $F(\boldsymbol{z})$. Although the method of [8] may not be directly applicable to the minor prime factorization problem discussed in this paper as $b_1(\boldsymbol{z}),\ldots,b_\beta(\boldsymbol{z})$ are not zero coprime here, there is a similarity in both papers in that localization is exploited. It is therefore hoped that the method of this paper could be combined with that of [8]. This is a difficult task in itself and requires further investigation.

Acknowledgment

The authors would like to thank Dr. M. Wang from the Chinese Academy Sciences for helpful discussions and the anonymous reviewers for many comments which have improved the presentation of the paper considerably.

References

[1] N. K. Bose, B. Buchberger, and J. P. Guiver, *Multidimensional Systems Theory and Applications*. Dordrecht, The Netherlands: Kluwer, 2003.

[2] D. C. Youla and G. Gnavi, "Notes on n-dimensional system theory," *IEEE Trans. Circuits Syst.*, vol. CAS-26, no. 2, pp. 105–111, Feb. 1979.

[3] Z. Lin, "On matrix fraction descriptions of multivariable linear n-D systems," *IEEE Trans. Circuits Syst.*, vol. 35, no. 10, pp. 1317–1322, Oct. 1988.

[4] ——, "Notes on n-D polynomial matrix factorizations," *Multidimensional Syst. Signal Process.*, vol. 10, pp. 379–393, 1999.

[5] Z. Lin and N. K. Bose, "A generalization of Serre's conjecture and some related issues," *Linear Algebra Appl.*, vol. 338, pp. 128–138, 2001.

[6] J. F. Pommaret, "Solving Bose conjecture on linear multidimensional systems," in *Proc. Eur. Control Conf.*, 2001, pp. 1853–1855.

[7] V. Srinivas, "A generalized Serre problem," *J. Algebra*, vol. 278, no. 2, pp. 621–627, Aug. 2004.

[8] M. Wang and D. Feng, "On Lin–Bose problem," *Linear Algebra Appl.*, vol. 390, pp. 279–285, Oct. 2004.

[9] C. Charoenlarpnopparut and N. K. Bose, "Multidimensional FIR filter bank design using Gröbner bases," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 46, no. 12, pp. 1475–1486, Dec. 1999.

[10] B. C. Lévy, "2-D polynomial and rational matrices, and their applications for the modeling of 2-D dynamical systems," Ph.D. dissertation, Info. Syst. Lab., Stanford Univ., Stanford, CA, Jun. 1981.

[11] Z. Lin, J. Q. Ying, and L. Xu, "Factorizations for n-D polynomial matrices," *Circuits, Syst., Signal Process.*, vol. 20, pp. 601–618, 2001.