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An Algebraic Approach to Strong Stabilizability of Linear nD MIMO Systems

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***Abstract*—Although some necessary conditions for the strong stabilizability of linear multidimensional (nD) multiple-input–multiple-output (MIMO) systems have been available recently, very little is known about sufficient conditions for the same problem. This note presents two sufficient conditions for strong stabilizability of some classes of linear nD MIMO systems obtained using an algebraic approach. A simple necessary and sufficient condition is also given for the strong stabilizability of a special class of linear nD MIMO systems. An advantage of the proposed algebraic approach is that a stable stabilizing compensator can be constructed for an nD plant satisfying the sufficient conditions for the strong stabilizability presented in this note. Illustrative examples are given.**

***Index Terms*—Algebraic approach, feedback stabilization, multidimensional systems, strong stabilizability.**

I. INTRODUCTION

Consider a linear multidimensional ($nD, n \geq 2$) multiple-input–multiple-output (MIMO) system represented by an nD rational matrix $P(\mathbf{z})$ where $\mathbf{z} \triangleq (z_1, \ldots, z_n)$. P is said to be (output) feedback stabilizable if there exists a compensator $C(\mathbf{z})$ such that each entry of the following closed-loop transfer matrix is stable, i.e., it has no poles in the closed unit polydisc $\overline{U}^n \triangleq \{\mathbf{z} : |z_1| \leq 1, \ldots, |z_n| \leq 1\}$ (see, e.g., [1] for more details)

$$\begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix}. \tag{1}$$

If P is stabilizable by means of a stable compensator C, we say that P is strongly stabilizable [2]–[4]. There are several advantages in introducing a stable compensator for a stabilizable system. One of the advantages is that it does not introduce extra unstable zeros in the closed-loop system [2]. Another advantage is that two unstable plants can be simultaneously stabilized by a single compensator if a certain other plant is strongly stabilizable [3].

Although the problems of strong stabilizability and stabilization of linear one-dimensional systems have long been solved [5], their nD counterparts have attracted some attention only recently, due to their complexity [3], [4], [6]–[8]. Strong stabilizability of linear nD single-input–single-output (SISO) systems was investigated in [3]–[6]. Several necessary conditions for the strong stabilizability of linear nD MIMO systems were given in [7]. For the special classes of multiple-input–single-output (MISO) and single-input–multiple-output (SIMO) systems, necessary and sufficient conditions for the strong stabilizability have also been derived and presented in [7]. Investigation of strong stabilizability of time-delay linear systems was considered in [8]. However, although a sufficient condition for the strong stabilizability for a class of linear nD MIMO systems was presented in [7], the approach was not constructive. Attacking the problem of strong stabilizability and stabilization of a general linear nD MIMO system by menas of an algebraic approach remains an open problem [7].

In this paper, we present two sufficient conditions for the strong stabilizability of some classes of linear nD MIMO systems obtained using an algebraic approach. The new sufficient conditions are given in the next section. Furthermore, a necessary and sufficient condition for a special class of linear nD MIMO systems is presented in Section III. Three illustrative examples are given in Section IV. Section V then concludes this note.

Throughout this note, denote by $\mathbf{R}(\mathbf{z})$ the set of nD rational functions with coefficients in the field of real numbers $\mathbf{R}$; by $\mathbf{R}[\mathbf{z}]$ the set of nD polynomials over $\mathbf{R}$; by $\mathbf{R_s}(\mathbf{z})$ the set of stable nD rational functions, i.e., nD rational functions whose denominators have no zeros in $\overline{U}^n$; by $\mathbf{R}^{m\times l}[\mathbf{z}]$ the set of $m \times l$ matrices with entries in $\mathbf{R}[\mathbf{z}]$, etc. Similarly, denote by $\mathbf{C}(\mathbf{z})$ the set of nD rational functions over the field of complex numbers, *etc.* I_l denotes the $l \times l$ identity matrix and $(\cdot)^T$ the matrix transposition. For simplicity, arguments such as $\mathbf{z}$ are omitted when their omission does not cause confusion.

II. SUFFICIENT CONDITIONS

Under the assumption that $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ admits a minor right coprime (MRC) [9] matrix fractional description, i.e., $P = ND^{-1}$ with $D \in \mathbf{R}^{l\times l}[\mathbf{z}]$ and $N \in \mathbf{R}^{m\times l}[\mathbf{z}]$ MRC, it has been shown [7] that P is strongly stabilizable if and only if there exists a $C \in \mathbf{R_s}^{l\times m}(\mathbf{z})$ such that

$$\det(D + CN) \neq 0 \text{ in } \overline{U}^n. \tag{2}$$

For convenience of comparison, we recall a known necessary condition for the strong stabilizability of nD systems.

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Proposition 1: [7] Let $P = ND^{-1} \in \mathbf{R}^{m\times l}(\mathbf{z})$ *with* $D \in \mathbf{R}^{l\times l}[\mathbf{z}]$ *and* $N \in \mathbf{R}^{m\times l}[\mathbf{z}]$ MRC. Let $\alpha_1, \ldots, \alpha_M$, with $M \triangleq \binom{m+l}{m}$, be the $l \times l$ minors of $[D^T \quad N^T]$ with $\alpha_1 = \det D$. A necessary condition for P to be strongly stabilizable is that there exist $e_2, \ldots, e_M \in \mathbf{R}_s(\mathbf{z})$ such that

$$\det D + \sum_{i=2}^{M} e_i \alpha_i \neq 0 \text{ in } \overline{U}^n. \tag{3}$$

Although the aforementioned necessary condition has been shown to be also sufficient for the strong stabilizability of linear nD MISO and SIMO systems [7], its sufficiency for the strong stabilizability of a general MIMO nD system is still unknown. In this section, we derive two sufficient conditions for the strong stabilizability of some classes of nD systems. It is assumed in the remainder of this notethat the plant P is of dimension $m \times l$ with $m \geq l \geq 2$.

Proposition 2: Let $P = ND^{-1} \in \mathbf{R}^{m\times l}(\mathbf{z})$ *with* $D \in \mathbf{R}^{l\times l}[\mathbf{z}]$ and $N \in \mathbf{R}^{m\times l}[\mathbf{z}]$. Assume that D is an upper or lower triangle matrix with diagonal elements equal to $d_i = d_{si}d$, $i = 1, \ldots, l$, where $d_{s1}, \ldots, d_{sl}$ are stable nD polynomials and d is an unstable nD polynomial. Let $a_1, \ldots, a_\beta$ denote the $l \times l$ minors of N, where $\beta \triangleq \binom{m}{l}$. A sufficient condition for the strong stabilizability of P is that there exist some $h_1, \ldots, h_\beta \in \mathbf{R}_\mathbf{s}(\mathbf{z})$ such that

$$d + \sum_{i=1}^{\beta} h_i a_i \neq 0 \text{ in } \overline{U}^n. \tag{4}$$

Moreover, when the previous condition holds, a stable stabilizing compensator $C \in \mathbf{R_s}^{l\times m}(\mathbf{z})$ can be constructed.

Proof: Assume that D is an upper triangle matrix having the following form:

$$\begin{bmatrix} d_{s1}d & & \mathbf{U} \\ & \ddots & \\ \mathbf{0} & & d_{sl}d \end{bmatrix} \tag{5}$$

where $\mathbf{U}$ denotes the upper right triangular portion of D with arbitrary nD polynomial entries, while $\mathbf{0}$ denote the lower left portion of D with zero entries. Assume that there exist some $h_1, \ldots, h_\beta \in \mathbf{R}_\mathbf{s}(\mathbf{z})$ such that

$$s_1 \triangleq d + \sum_{i=1}^{\beta} h_i a_i \neq 0 \text{ in } \overline{U}^n. \tag{6}$$

Let $\lambda_1 \triangleq \sum_{i=1}^{\beta} h_i a_i$. We first construct a $Y \in \mathbf{R_s}^{l\times m}(\mathbf{z})$ such that $YN = \lambda_1 I_l$. We adopt a technique similar to the one in [10]. Let $\mathbf{n}_k$ denote the kth row of N ($k = 1, \ldots, m$), $N_1, \ldots, N_\beta$ denote the $l \times l$ submatrices of N, i.e.,

$$N_i = \begin{bmatrix} \mathbf{n}_{i_1} \\ \vdots \\ \mathbf{n}_{i_l} \end{bmatrix} \tag{7}$$

where $1 \leq i_i < \cdots < i_l \leq m$, for $i = 1, \ldots, \beta$. Let $[\mathbf{g}_{i_1} \cdots \mathbf{g} i_l] \triangleq \operatorname{adj} N_i$, for $i = 1, \ldots, \beta$. The β submatrices of N are ordered such that $a_i = \det N_i$, for $i = 1, \ldots, \beta$. An $l \times m$ matrix B_i is now constructed as follows. In columns $i_1, \ldots, i_l$ of B_i, we place $\mathbf{g}_{i_1}, \ldots, \mathbf{g}_{i_l}$. The remaining columns of B_i are filled with zeros. Note that, when $l = m$, we have just $\beta = 1$ and $B_1 = \operatorname{adj} N$. It can be easily verified that

$$B_i N = a_i I_l. \tag{8}$$

Next, let

$$Y \triangleq \sum_{i=1}^{\beta} h_i B_i \in \mathbf{R_s}^{l\times m}(\mathbf{z}) \tag{9}$$

we then have

$$YN = \left(\sum_{i=1}^{\beta} h_i B_i\right) N = \sum_{i=1}^{\beta} h_i B_i N = \sum_{i=1}^{\beta} h_i a_i I_l = \lambda_1 I_l. \tag{10}$$

Now, let $X = I_l$, $D' = \operatorname{diag}\{d_{s1}, \ldots, d_{sl}\}$, $Y' = D'Y$, and $C = X^{-1}Y' = Y' \in \mathbf{R_s}^{l\times m}(\mathbf{z})$. We have

$$\begin{aligned} D + CN &= D + Y'N \\ &= \begin{bmatrix} d_{s1}d & & \mathbf{U} \\ & \ddots & \\ \mathbf{0} & & d_{sl}d \end{bmatrix} + \begin{bmatrix} d_{s1}\lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & d_{sl}\lambda_1 \end{bmatrix} \\ &= \begin{bmatrix} d_{s1}s_1 & & \mathbf{U} \\ & \ddots & \\ \mathbf{0} & & d_{sl}s_1 \end{bmatrix}. \end{aligned}$$

Therefore, $\det(D+CN) = d_{s1} \ldots d_{sl} s_1^l \neq 0$ in $\overline{U}^n$, and C is a stable stabilizing compensator. The case where D is a lower triangle matrix can be proven similarly. □

Comparing Propositions 1 and 2, it can be seen that the two conditions (3) and (4) are indeed different in the number of terms in the sum. For example, assume that $m = l = 2$, then $M = \binom{4}{2} = 6$, while $\beta = \binom{2}{2} = 1$. Hence, the number of terms in the sum in (3) is $6 - 1 = 5$, while that in (4) is just 1. It is then obvious that (4) is more difficult to satisfy than (3). On the other hand, since both (3) and (4) have the same form of expression, the technique of [7] for testing the necessary condition (3) can be readily adopted here for testing the sufficient condition (4).

The following corollary is an important special case of Proposition 2.

Corollary 1: Let $P = ND^{-1} \in \mathbf{R}^{m\times l}(\mathbf{z})$ with $D \in \mathbf{R}^{l\times l}[\mathbf{z}]$ and $N \in \mathbf{R}^{m\times l}[\mathbf{z}]$. Assume that $D = dI_l$ and let $a_1, \ldots, a_\beta$ denote the $l \times l$ minors of N. A sufficient condition for the strong stabilizability of P is that there exist some $h_1, \ldots, h_\beta \in \mathbf{R}_\mathbf{s}(\mathbf{z})$ such that

$$d + \sum_{i=1}^{\beta} h_i a_i \neq 0 \text{ in } \overline{U}^n. \tag{11}$$

Moreover, when the aforementioned condition holds, a stable stabilizing compensator $C \in \mathbf{R_s}^{l\times m}(\mathbf{z})$ can be constructed.

So far, we have restricted D to be an upper or lower triangle matrix. Now, we consider a more general case without any restriction on D and derive another sufficient condition for the strong stabilizability of $P = ND^{-1}$ as shown in the next proposition.

Proposition 3: Let $P = ND^{-1} \in \mathbf{R}^{m\times l}(\mathbf{z})$ with $D \in \mathbf{R}^{l\times l}[\mathbf{z}]$ *and* $N \in \mathbf{R}^{m\times l}[\mathbf{z}]$. Let $a_1, \ldots, a_\beta$ denote the $l \times l$ minors of N. A sufficient condition for the strong stabilizability of P is that $a_1, \ldots, a_\beta$ have no common zeros in $\overline{U}^n$, or equivalently, there exist $h_1, \ldots, h_\beta \in \mathbf{R}_s(\mathbf{z})$ such that

$$\sum_{k=1}^{\beta} h_k a_k \neq 0 \text{ in } \overline{U}^n. \tag{12}$$

Moreover, when the aforementioned condition holds, a stable stabilizing compensator $C \in \mathbf{R_s}^{l\times m}(\mathbf{z})$ can be constructed.

Proof: Assume that $a_1, \ldots, a_\beta$ have no common zeros in $\overline{U}^n$, i.e., there exist $h_1, \ldots, h_\beta \in \mathbf{R}_s(\mathbf{z})$ such that (12) holds [11]. Let $s \triangleq \sum_{k=1}^{\beta} h_k a_k$ and $s_0 \triangleq \min_{\mathbf{z}\in\overline{U}^n} |s| > 0$. In the same way as in the proof of Proposition 2, we can construct a $Y \in \mathbf{R_s}^{l\times m}(\mathbf{z})$

such that $YN = sI_l$. Let $b_1, \ldots, b_\beta$ denote the $l \times l$ minors of Y (note that $b_1, \ldots, b_\beta$ are in general not equal to $h_1, \ldots, h_\beta$). By the Cauchy–Binet formula, we have

$$\det(YN) = \sum_{i=1}^{\beta} b_i a_i = \det(sI_l) = s^l. \tag{13}$$

Let $c_1, \ldots, c_M$ denote the $l \times l$ minors of $[I_l \quad Y]$, and arrange the order of $c_1, \ldots, c_M$ in such a way that $c_1 = \det I_l = 1$ and $c_{M-\beta+1} = b_1, \ldots, c_M = b_\beta$. Similarly, let $\alpha_1, \ldots, \alpha_M$ denote the $l \times l$ minors of $[D^T \quad N^T]^T$, and arrange the order of $\alpha_1, \ldots, \alpha_M$ in such a way that $\alpha_1 = \det D$ and $\alpha_{M-\beta+1} = a_1, \ldots, \alpha_M = a_\beta$. Let w be a positive number. We then have

$$\begin{aligned} f(\mathbf{z}; w) &\triangleq \det\left([I_l \quad wY] \begin{bmatrix} D \\ N \end{bmatrix}\right) \\ &= \alpha_1 + w\alpha_2 c_2 + \cdots + w^{l-1}\alpha_{M-\beta} c_{M-\beta} \\ &\quad + w^l \sum_{i=1}^{\beta} b_i a_i. \end{aligned} \tag{14}$$

As $\alpha_1, \alpha_2 c_2, \ldots, \alpha_{M-\beta} c_{M-\beta}$ have bounded absolute values in $\overline{U}^n$, and $|\sum_{i=1}^{\beta} b_i a_i| \geq s_0^l > 0 \; \forall \mathbf{z}$ in $\overline{U}^n$, we have $|f(\mathbf{z}; w)| > 0$, which means $f(\mathbf{z}; w) \neq 0, \; \forall \mathbf{z}$ in $\overline{U}^n$, for a sufficiently large w. Therefore, a stable stabilizing compensator for the given plant P is given by $C = wY \in \mathbf{R_s}^{l \times m}(\mathbf{z})$. □

Remark 1: It is worthwhile comparing Propositions 3 and 2. Although there is no restriction on D in Proposition 3, the requirement of $a_1, \ldots, a_\beta$ having no common zeros in $\overline{U}^n$ is stronger than condition (4). Hence, Proposition 3 is in fact not a generalization of Proposition 2. It is also noted that both Propositions 2 and 3 are valid even when P admits a matrix fractional description that is not MRC.

III. Necessary and Sufficient Condition

The sufficient conditions for the strong stabilizability of nD systems presented in the previous section are not necessary conditions. In this section, we present a simple necessary and sufficient condition for the strong stabilizability of a special class of linear nD MIMO systems over the complex number field. Throughout this section, it is assumed that a system is represented by an nD rational matrix $P \in \mathbf{C}^{m \times l}(\mathbf{z})$ which admits an MRC matrix fractional description $P = ND^{-1}$, i.e., $D \in \mathbf{C}^{l \times l}[\mathbf{z}]$ and $N \in \mathbf{C}^{m \times l}[\mathbf{z}]$ with D and N MRC.

Lemma 1: Let $d, a_1, \ldots, a_\beta \in \mathbf{C}[\mathbf{z}]$. The following two statements are equivalent.

i) There exist $h_1, \ldots, h_\beta \in \mathbf{C_s}(\mathbf{z})$ such that

$$d + \sum_{i=1}^{\beta} h_i a_i \neq 0 \text{ in } \overline{U}^n \tag{15}$$

ii) For a given positive integer l, there exist $h'_1, \ldots, h'_\beta \in \mathbf{C_s}(\mathbf{z})$ such that

$$d^l + \sum_{i=1}^{\beta} h'_i a_i \neq 0 \text{ in } \overline{U}^n. \tag{16}$$

Proof: Clearly, statement i) implies statement ii). This implication follows immediately by expanding $(d + \sum_{i=1}^{\beta} h_i a_i)^l$ and collecting the appropriate terms.

Next, assume that statement ii) is true, i.e., there exist $h'_1, \ldots, h'_\beta \in \mathbf{C_s}(\mathbf{z})$ such that $d^l + \sum_{i=1}^{\beta} h'_i a_i \neq 0$ in $\overline{U}^n$. By [7, Th. 2.1], d^l has a single-valued logarithm function $\log d^l$ on $V(I) \cap \overline{U}^n$, where I is the ideal generated by $a_1, \ldots, a_\beta$ and $V(I)$ is the variety of I. Since $\log d^l = l \log d$, it follows immediately that d has a single-valued logarithm function $\log d$ on $V(I) \cap \overline{U}^n$. By [7, Th. 2.1], there exist $h_1, \ldots, h_\beta \in \mathbf{C_s}(\mathbf{z})$ such that $d + \sum_{i=1}^{\beta} h_i a_i \neq 0$ in $\overline{U}^n$. Therefore, statement ii) implies i), and the proof is completed. □

However, if we require the coefficients to be in $\mathbf{R}$, then statement ii) does not imply i) in general, as it can be seen from [6, Ex. 3].

Proposition 4: Let $P = ND^{-1} \in \mathbf{C}^{m \times l}(\mathbf{z})$ with $D \in \mathbf{C}^{l \times l}[\mathbf{z}]$ and $N \in \mathbf{C}^{m \times l}[\mathbf{z}]$ being MRC. Assume that $D = dI_l$, and that the ideal generated by the $l \times l$ minors of $[D^T \quad N^T]$, with the exception of $\det D$, is equal to the ideal generated by the $l \times l$ minors of N, denoted by $a_1, \ldots, a_\beta$. Then, a necessary and sufficient condition for the strong stabilizability of P is that there exist some $h_1, \ldots, h_\beta \in \mathbf{C_s}(\mathbf{z})$ such that

$$d + \sum_{i=1}^{\beta} h_i a_i \neq 0 \text{ in } \overline{U}^n. \tag{17}$$

Proof: (Sufficiency) It follows from Corollary 1.

(Necessity) Let $\alpha_1, \ldots, \alpha_M$ be the $l \times l$ minors of $[D^T \quad N^T]$ with $\alpha_1 = \det D$. Suppose that P is strongly stabilizable. By Proposition 1 (and also [7, Th. 3.1]), there exist $e_2, \ldots, e_M \in \mathbf{C}_s(\mathbf{z})$ such that

$$\det D + \sum_{i=2}^{M} e_i \alpha_i \neq 0 \text{ in } \overline{U}^n. \tag{18}$$

By assumption, $D = dI_l$, and also the ideal generated by the $l \times l$ minors of $[D^T \quad N^T]$, with the exception of $\det D$, is equal to the ideal generated by the $l \times l$ minors of N. Thus, (18) implies that there exist $h'_1, \ldots, h'_\beta \in \mathbf{C_s}(\mathbf{z})$ such that

$$d^l + \sum_{i=1}^{\beta} h'_i a_i \neq 0 \text{ in } \overline{U}^n. \tag{19}$$

By Lemma 1, (19) implies that there exist some $h_1, \ldots, h_\beta \in \mathbf{C_s}(\mathbf{z})$ such that

$$d + \sum_{i=1}^{\beta} h_i a_i \neq 0 \text{ in } \overline{U}^n. \tag{20}$$

□

Remark 2: Admittedly, the class of linear nD MIMO systems discussed in Proposition 4 is rather special. However, to our best knowledge, this is the only available necessary and sufficient condition for the strong stabilizability of linear nD MIMO systems other than MISO and SIMO systems. We can check whether two ideals are equal using Gröbner bases [12]. It is also noted that Proposition 4 may not be applied to the case where P does not admit an MRC matrix fractional description. Nevertheless, a two-dimensional (2-D) system always admits an MRC matrix fractional description [9].

IV. Examples

Three examples are given in this section to illustrate the new results presented in this note.

Example 1: Consider an unstable 2-D 2×2 plant represented by

$$P(z_1, z_2) = \frac{1}{1 + z_1 - z_2} \begin{bmatrix} 1 & 2z_1 \\ 2z_2 & 1 \end{bmatrix}.$$

Let $P = ND^{-1}$, where

$$D = \begin{bmatrix} 1 + z_1 - z_2 & 0 \\ 0 & 1 + z_1 - z_2 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 2z_1 \\ 2z_2 & 1 \end{bmatrix}.$$

Since $D = dI_2$ with $d = 1 + z_1 - z_2$, we can apply Corollary 1 or Proposition 2 to test the strong stabilizability of the given system P. As N is a square matrix, it has only one 2×2 minor, which is equal to $a_1 = 1 - 4z_1 z_2$. A sufficient condition for the strong stabilizability of P is the existence of some $h \in \mathbf{R_s}(z_1, z_2)$ such that $d + ha_1 \neq 0$ in $\overline{U}^2$. This condition turns out to be equivalent to the strong stabilizability of a 2-D SISO system represented by $p = a_1/d$, which was discussed in [4]. In fact, from [4], we can find $h = 1/(2(1+\sqrt{2})) \in \mathbf{R_s}(z_1, z_2)$ such that

$$\begin{aligned} s_1 \triangleq d + ha_1 &= (1 + z_1 - z_2) + \frac{1}{2(1+\sqrt{2})}(1 - 4z_1z_2) \\ &= \frac{4}{2(1+\sqrt{2})}\left(\frac{1+\sqrt{2}}{2} + z_1\right)\left(\frac{1+\sqrt{2}}{2} - z_2\right) \\ &\neq 0 \text{ in } \overline{U}^2. \end{aligned}$$

From the proof procedure of Proposition 2, we can also obtain a stabilizing compensator as follows. Let

$$Y = h\,(\text{adj } N) = \frac{1}{2(1+\sqrt{2})}\begin{bmatrix} 1 & -2z_1 \\ -2z_2 & 1 \end{bmatrix}$$

$X = I_2$, and $C = X^{-1}Y = Y \in \mathbf{R_s}^{2\times 2}(z_1, z_2)$. It is easy to verify that $\det(D + CN) = s_1^2 \neq 0$ in $\overline{U}^2$. Hence, C is the required stable stabilizing compensator.

Example 2: Consider an unstable 2-D 3×2 plant represented by

$$P(z_1, z_2) = \frac{1}{z_2}\begin{bmatrix} z_1 & 0 \\ 0 & 1 \\ 1 - z_1z_2 & 0 \end{bmatrix}.$$

Let $P = ND^{-1}$, where

$$D = \begin{bmatrix} z_2 & 0 \\ 0 & z_2 \end{bmatrix} \quad N = \begin{bmatrix} z_1 & 0 \\ 0 & 1 \\ 1 - z_1z_2 & 0 \end{bmatrix}.$$

The 2×2 minors of $[D^T \quad N^T]$ are

$$\det D = z_2^2,\ 0,\ z_2,\ 0, -z_1z_2,\ 0,\ z_2(z_1z_2 - 1),\ z_1,\ 0,\ z_1z_2 - 1$$

and the 2×2 minors of N are

$$z_1,\ 0,\ z_1z_2 - 1.$$

Because the ideal generated by the 2×2 minors of N contains the unit 1 (z_1 and $z_1z_2 - 1$ have no common zero in $\mathbf{C}^2$), it is obvious that the ideal generated by the 2×2 minors of $[D^T \quad N^T]$, with the exception of $\det D$, is equal to the ideal generated by the 2×2 minors of N. Moreover, $D = z_2I_2$. Hence, we can apply Proposition 4 to this example to see whether the necessary and sufficient condition for the strong stabilizability (17) is satisfied. Set $a_1 \triangleq z_1$, $a_2 \triangleq 0$, and $a_3 \triangleq z_1z_2 - 1$, and let $h_1 = 2z_2$, $h_2 = 0$, and $h_3 = -2$. It is then easy to verify that

$$\begin{aligned} s_1 &\triangleq d + \sum_{i=1}^{3} h_ia_i = d + h_1a_1 + h_3a_3 \\ &= z_2 + 2z_2(z_1) + (-2)(z_1z_2 - 1) \\ &= z_2 + 2 \neq 0 \text{ in } \overline{U}^n. \end{aligned} \tag{21}$$

Therefore, by Proposition 4, the given 2-D plant P is strongly stabilizable. To construct a stable stabilizing compensator, we can follow the procedure presented in the proof of Proposition 2, just like we did in Example 1. The details are omitted here but the required compensator is given by $C = X^{-1}Y \in \mathbf{R_s}^{2\times 3}(z_1, z_2)$ where $X = I_2$ and

$$Y = \begin{bmatrix} 2z_2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \tag{22}$$

satisfies $YN = 2I_2$. It is now trivial to verify that $\det(D + CN) = s_1^2 \neq 0$ in $\overline{U}^2$, i.e., the closed-loop system is stable.

Example 3: Consider an unstable 2-D 3×2 plant represented by $P = ND^{-1}$, where $D = \begin{bmatrix} 2z_1 & 1 \\ 1 & 2z_2 \end{bmatrix}$ and N is the same as in Example 2. It is known from Example 2 that the 2×2 minors of N have no common zero in $\mathbf{C}^2$. By Proposition 3, the system P is strongly stabilizable. To obtain a stable stabilizing compensator, we first contruct the same Y as in (22) in Example 2, such that $YN = 2I_2$. Since $\det(D + YN) = 4z_1z_2 + 4(z_1 + z_2) + 3$ has zeros in $\overline{U}^2$, e.g., $(z_1, z_2) = (0, -3/4)$, the compensator $C = Y$ chosen in Example 2 cannot stabilize the system considered here. To construct a stable compensator, we can follow the procedure presented in the proof of Proposition 3. Let $f(z_1, z_2; w) \triangleq \det\left([I_2 \quad wY]\begin{bmatrix} D \\ N \end{bmatrix}\right)$, where w is a positive number. Expanding $f(z_1, z_2; w)$ and collecting terms gives

$$f(z_1, z_2; w) = (4z_1z_2 - 1) + 4(z_1 + z_2)w + 4w^2.$$

To ensure that $f(z_1, z_2; w) \neq 0$ in $\overline{U}^2$, it suffices to choose a large enough w. In particular, for any positive w and $\forall (z_1, z_2)$ in $\overline{U}^2$, we have $|(4z_1z_2 - 1) + 4(z_1 + z_2)w| \leq |4z_1z_2 - 1| + 4w|(z_1 + z_2)| < 5 + 8w$. Therefore, any w which satisfies $w > 0$ and $4w^2 \geq 5 + 8w$, or equivalently, $w \geq 5/2$ will guarantee $|f(z_1, z_2; w)| > 0$, which implies $f(z_1, z_2; w) \neq 0$, in $\overline{U}^2$. For example, $w = 3$ yields a stable stabilizing compensator $C = X^{-1}3Y = 3Y \in \mathbf{R_s}^{2\times 3}(z_1, z_2)$ where $X = I_2$.

V. Conclusion

In this note, we have presented two sufficient conditions for the strong stabilizability of some classes of linear nD MIMO systems, as well as a simple necessary and sufficient condition for the strong stabilizability of a special class of linear nD MIMO systems. The proposed method is algebraic and a stable stabilizing compensator can always be constructed when a given nD MIMO plant satisfies one of the sufficient conditions presented in the paper.

Finally, it is worth pointing out that the proof of the necessity part of the strong stabilizability of a general nD MIMO system given in [7] involves nonconstructive advanced mathematics. What still remain as open problems are, first, to find a connection of the problem with constructive mathematical tools and to prove or disprove the sufficiency, and second, to establish an algebraic constructive sufficient and necessary condition for a general nD MIMO system.

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Constantly Scaled H_∞ Control Problems for Pseudofull Information Problems

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***Abstract*—This note proposes a class of the constantly scaled H_∞ control problem, where the class is characterized by an assumption also proposed in this note. In general, the scaled H_∞ control problem leads to nonconvex solvability conditions. On the other hand, for the problems in the proposed class, we show that a convex but sufficient solvability condition can be given. Moreover, the convexity of the state feedback and the full information problems can be regarded as the extreme cases of the sufficient condition. This fact motivates us to call the assumption the pseudofull information condition.**

***Index Terms*—Constantly scaled H_∞ control problems, integrated quadratic constraints (IQCs), linear matrix inequalities (LMIs), robust control.**

I. Introduction

For the past two decades, much effort has been devoted to developing analysis and synthesis methods for robust control systems. One of the most important issues in robust control is conservatism of analysis and synthesis methods. In order to reduce the conservatism, scaled small-gain conditions or passivity conditions with multipliers are often employed [1]–[5]. Although those measures provide only sufficient conditions, they provide efficient robustness analysis. In fact, many analysis results can be examined efficiently by casting them as linear matrix inequalities (LMIs) [6]–[8].

While analysis problems aim to find scalings, the corresponding synthesis problems aim to find both scalings and controllers. Unfortunately, the synthesis problems are nonconvex in general, even if scalings are restricted to being constant, i.e., frequency independent. Concerning the constantly scaled H_∞ control problem, several methods have been proposed to find global solutions [9]–[11]. However, those methods demand a large amount of computational effort.

Since the synthesis problems are nonconvex in general, much computational effort is inherently required to find solutions. Furthermore, there is no guarantee to obtain the global solution. On the other hand, if a given problem is proven convex, we can find the global solution without spending much time and effort. Hence, it is useful to know what kind of problems can be reduced to convex problems. Motivated by these facts, several classes of problems have been revealed convex so far. The most well-known classes are the state feedback (SF) and the full information (FI) problems [12]. Furthermore, it has been clarified that another class of problems such that the rank of uncertainty is one can be reduced to convex problems [13].

Recently, we have reported that a class of problems which is larger than the SF and the FI problems can be reduced to convex problems [14]. In other words, there exist assumptions which are more relaxed than the SF and the FI assumptions so as to make the problems convex. However, even in the enlarged class, the problems are still restricted to the special cases such that $D_{21} = 0$ or $D_{21}^T D_{21} > 0$ holds, and it is still open whether the problems can be reduced to convex problems for general D_{21}. In addition, no formulas of controllers have been shown in [14].

This note proposes a new class of problems such that neither $D_{21} = 0$ nor $D_{21}^T D_{21} > 0$ holds, and shows that a convex *sufficient* condition can be derived for that class of problems. The new class is characterized by an assumption similar to [14]. Moreover, we show that the convex solvability conditions for the SF and the FI problems can be regarded as the extreme cases of the derived sufficient condition. It follows that the sufficient condition gives necessity in the extreme cases. The sufficient condition will be given based on two solvability conditions which also are derived in this note. Moreover, an explicit formula of a possible controller will be given.

This note is organized as follows. In Section II, we offer a robust stability criterion based on an integral quadratic constraint (IQC). Based on this robust stability criterion, the corresponding synthesis problem is formulated, and the assumption called a pseudofull information condition is proposed in Section III. The main results of this note will be shown in Section IV. The effectiveness of the proposed method will be examined in Section V by using an example.

Notation in this note is fairly standard. M^T and $M^\dagger$ are the transpose and the pseudoinverse of matrix M, respectively. $M^\perp$ is a full-row rank matrix whose rows span the orthogonal complement of the range of M.

II. Robust Stability Analysis

This note deals with the constantly scaled H_∞ control problem primarily based on robust stabilization problems. In order to formulate the robust stabilization problem, we first introduce a robust stability analysis result in this section. The analysis problem is dealt with for a feedback system depicted in Fig. 1, where $\boldsymbol{\Delta} \subseteq \mathbf{R}^{m_1 \times p_1}$ represents uncertainty. We assume $0 \in \boldsymbol{\Delta}$ without loss of generality. $\mathcal{P}(s)$ corresponds to a closed-loop system constructed by a generalized plant and a controller. We assume the following realization for $\mathcal{P}(s)$:

$$\mathcal{P}(s) = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] =: \mathcal{D} + \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}. \tag{1}$$

Then, the robust stability analysis result is given by the following proposition [1], [4], [15].

Proposition 1: Let $\boldsymbol{\Delta} \subseteq \mathbf{R}^{m_1 \times p_1}$ be a given set, and $\mathcal{A} \in \mathbf{R}^{n \times n}$, $\mathcal{B} \in \mathbf{R}^{n \times m_1}$, $\mathcal{C} \in \mathbf{R}^{p_1 \times n}$ and $\mathcal{D} \in \mathbf{R}^{p_1 \times m_1}$ be given matrices. Suppose that there exist $\mathcal{X} > 0$ ($\mathcal{X} \in \mathbf{R}^{n \times n}$), $Q = Q^T \in \mathbf{R}^{m_1 \times m_1}$, $R = R^T \in \mathbf{R}^{p_1 \times p_1}$ and $S \in \mathbf{R}^{p_1 \times m_1}$ such that the following inequalities hold:

$$\begin{bmatrix} I \\ \Delta^T \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} I \\ \Delta^T \end{bmatrix} \geq 0 \qquad \text{for all } \Delta \in \boldsymbol{\Delta} \tag{2}$$

$$\begin{bmatrix} \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T & \mathcal{X}\mathcal{C}^T \\ \mathcal{C}\mathcal{X} & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{B} & 0 \\ \mathcal{D} & I \end{bmatrix} \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} \mathcal{B} & 0 \\ \mathcal{D} & I \end{bmatrix}^T < 0. \tag{3}$$

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