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Feedback Stabilization of MIMO nD Linear Systems

Zhiping Lin

***Abstract*—The open problem of the existence of double coprime factorizations (DCFs) for multi-input/multi-output (MIMO) multidimensional (nD) linear systems is considered in this paper. It is shown that DCFs exist for a class of MIMO nD linear systems. A simple and efficient method is proposed for the construction of DCFs. The main result of the paper is to show how to construct a coprime (over the ring of stable rational functions) matrix fraction description of a given MIMO nD system with a stable reduced minor. A parameterization of all stabilizing compensators is given for a MIMO nD system in this class. An example is illustrated.**

***Index Terms*—Feedback stabilization, multidimensional systems, parameterization, reduced minors.**

I. INTRODUCTION

The problem of (output) feedback stabilization of multi-input/multi-output (MIMO) linear systems has drawn much attention in the past years because of its importance in control and systems (see [1]–[10] and the references therein). Consider a standard feedback system with P representing a plant and C a compensator. Let

$$H_{eu} = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix}. \tag{1}$$

P is said to be feedback stabilizable if and only if there exists a compensator C such that the feedback system H_{eu} is stable, i.e., each entry of H_{eu} has no poles in the unstable region [2]. For linear discrete multidimensional (nD) systems, the feedback system is structurally stable[1] if and only if each entry of H_{eu} has no poles in the closed unit polydisc $\overline{U}^n = \{(z_1, \ldots, z_n)\colon |z_1| \le 1, \ldots, |z_n| \le 1\}$ [11], [12].

The problem of feedback stabilization of MIMO two-dimensional (2-D) systems using the matrix fraction description (MFD) approach has been investigated by a number of researchers (see, e.g., [3]–[6] and the references therein). Constructive algorithms for the feedback stabilizability and stabilization problem have been presented for MIMO 2-D systems [3]–[6]. Furthermore, the parameterization of *all* stabilizing compensators for a given stabilizable 2-D system has been given in [3], which is a generalization of the celebrated result on the parameterization of *all* stabilizing compensators for a given one-dimensional (1-D) system [1], [2].

Due to some fundamental differences between 2-D and nD ($n \ge 3$)[2] polynomial matrices [13]–[16], results on stabilization of MIMO 2-D systems cannot be directly generalized to their nD counterparts. Although a necessary and sufficient condition for the feedback stabilizability of MIMO nD systems has recently been derived using the concept of reduced minors [7], [8], it is still unknown whether or not there exists a double coprime factorization (DCF) for a general stabilizable MIMO nD linear system. In fact, this problem is a special case of a more general problem on the existence of DCFs for linear systems over rings posed by Vidyasagar *et al.* in 1982 [2]. For MIMO nD linear systems,

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[1]In this paper, stability means structural stability rather than BIBO stability [11].

[2]In what follows, the term "nD" implies ($n \ge 3$) unless otherwise specified.

Lin conjectured in [8] that a stabilizable system also has a DCF. Very recently, a new result is presented in [10] on the existence of DCFs for a class of MIMO three-dimensional (3-D) systems, which leads to the parameterization of *all* stabilizing compensators. See [10] for a more detailed discussion on feedback stabilization of MIMO nD systems.

However, the method proposed in [10] has several limitations. Firstly, the method is valid for 3-D systems only. Secondly, the method may not work if the content associated with a left or a right MFDs of a given MIMO 3-D system has a multiple zero. Thirdly, the resultant DCFs may have complex coefficients even when the original system transfer matrix has only real coefficients. Fourthly, it may be computationally rather involved in solving a Bézout identity.

In this paper, we show that for a class of MIMO nD linear systems, it is always possible to construct DCFs, thus it is one further step toward proving the conjecture raised in [8] and solving the open problem posed in [2]. The proposed new method does not have any limitations mentioned above for the class of nD systems under discussion. The main result in the paper is to show how to construct a coprime (over the ring of stable rational functions) MFD of a given MIMO nD system with a stable reduced minor.

The organization of the paper is as follows. In the next section, after reviewing some necessary notation and definition, the main results will be presented. An illustrative example will then be given in Section III. Section IV ends this paper with a conclusion. To save space, we refer the reader to the cited references for some definitions and properties which require rather lengthy descriptions, such as reduced minors and causality.

II. Main Results

In the following, we shall denote $\mathbf{R}$ the field of real numbers; $\mathbf{R}(\mathbf{z}) = \mathbf{R}(z_1, \ldots, z_n)$ the set of rational functions in complex variables $z_1, \ldots, z_n$ with coefficients in $\mathbf{R}$; $\mathbf{R}[\mathbf{z}]$ the set of nD polynomials over $\mathbf{R}$; $\mathbf{R_s}(\mathbf{z})$ the subset of rational functions in $\mathbf{R}(\mathbf{z})$ having no poles in $\overline{U}^n$; $\mathbf{R}^{m\times l}[\mathbf{z}]$ the set of $m\times l$ matrices with entries in $\mathbf{R}[\mathbf{z}]$, $\mathbf{R_s}^{m\times l}(\mathbf{z})$ the set of $m\times l$ matrices with entries in $\mathbf{R_s}(\mathbf{z})$, etc. Throughout this paper, an nD polynomial is called a stable polynomial if it has no zeros in $\overline{U}^n$, otherwise it is called an unstable polynomial. The argument $(\mathbf{z})$ is omitted whenever its omission does not cause confusion.

Since DCF is the main concern of this paper, the definition of DCF is recalled.

Definition 1 [2], [10]: Let $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ represent an MIMO nD system. Then P is said to have a DCF if:

1) there exist $\tilde{D}_s \in \mathbf{R_s}^{m\times m}(\mathbf{z}), D_s \in \mathbf{R_s}^{l\times l}(\mathbf{z})$, and $\tilde{N}_s N_s \in \mathbf{R_s}^{m\times l}(\mathbf{z})$;
2) there exist $\tilde{X}_s \in \mathbf{R_s}^{l\times l}(\mathbf{z}), X_s \in \mathbf{R_s}^{m\times m}(\mathbf{z})$, and $\tilde{Y}_s, Y_s \in \mathbf{R_s}^{l\times m}(\mathbf{z})$;
3) $\tilde{D}_s, D_s, \tilde{X}_s, X_s$ are all nonsingular;
4) $P = \tilde{D}_s^{-1}\tilde{N}_s = N_s D_s^{-1}$ and the following Bézout identity holds[3]:

$$\begin{bmatrix} \tilde{X}_s & \tilde{Y}_s \\ -\tilde{N}_s & \tilde{D}_s \end{bmatrix} \begin{bmatrix} D_s & -Y_s \\ N_s & X_s \end{bmatrix} = \begin{bmatrix} I_l & 0_{l,m} \\ 0_{m,l} & I_m \end{bmatrix}. \tag{2}$$

Following conventional definition [2], $P = N_s D_s^{-1}$ is said to be a right coprime (over $\mathbf{R_s}(\mathbf{z})$) MFD of P if there exist $\tilde{X}_s \in \mathbf{R_s}^{l\times l}(\mathbf{z})$ and $\tilde{Y}_s \in \mathbf{R_s}^{l\times m}(\mathbf{z})$ such that $\tilde{X}_s D_s + \tilde{Y}_s N_s = I_l$. Obviously, $N_s D_s^{-1}$ is right coprime if and only if the associated matrix $[D_s^T \;\; N_s^T]$ is of full rank in $\overline{U}^n$. Analogous results hold for left MFDs. Also, for simplicity, the phrase "(over $\mathbf{R_s}(\mathbf{z})$)" is omitted in the rest of the paper.

[3] I_l is the $l\times l$ identity matrix and $0_{m,l}$ denotes the $m\times l$ zero matrix.

In this section, the DCF problem is solved constructively for a class of MIMO nD linear systems. Coprime MFDs, $\tilde{D}_s^{-1}\tilde{N}_s$ and $N_s D_s^{-1}$, are first constructed for P, followed by a solution to the Bézout identity (2).

Theorem 1: Let $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ represent a feedback stabilizable MIMO nD system, and let $b_1, \ldots, b_\beta$ denote the reduced minors of $P(\mathbf{z})$. If there exists some b_J $(1 \le J \le \beta)$ such that $b_J \neq 0$ in $\overline{U}^n$, then P has coprime right and left MFDs.

Proof: Write $P(\mathbf{z})$ as $P = N/d$ where d is the least common multiplier of the denominators of all the entries of P. Decompose P into a left and a right MFDs, $P = (dI_m)^{-1}N = \tilde{D}^{-1}\tilde{N} = ND^{-1} = N(dI_l)^{-1}$. Let $a_1, \ldots, a_\beta$ denote the $l\times l$ minors of $F = [D^T \;\; N^T]^T$, and $\tilde{a}_1, \ldots, \tilde{a}_\beta$ the $m\times m$ minors of $\tilde{F} = [-\tilde{N} \;\; \tilde{D}]$. By a result on reduced minors [14], [17], we have $a_i = d_0 b_i$, and $\tilde{a}_i = \pm\tilde{d}_0 b_i$ $(i = 1, \ldots, \beta)$, where $d_0, \tilde{d}_0 \in \mathbf{R}[\mathbf{z}]$, and the sign depends on the index i. The assumption that P is feedback stabilizable implies that $b_1, \ldots, b_\beta$ have no common zeros in $\overline{U}^n$ [7], [8]. If $d_0 \neq 0$ in $\overline{U}^n$, then $a_1, \ldots, a_\beta$ have no common zeros in $\overline{U}^n$. It follows that F is of full rank in $\overline{U}^n$ and hence ND^{-1} is a desirable coprime right MFD of P. Similarly, if $\tilde{d}_0 \neq 0$ in $\overline{U}^n$, $\tilde{D}^{-1}\tilde{N}$ is a coprime left MFD of P. Therefore, P has coprime MFDs when $d_0 \neq 0$ and $\tilde{d}_0 \neq 0$ in $\overline{U}^n$.

Assume now that $d_0(\mathbf{z_0}) = 0$ for some $(\mathbf{z_0})$ in $\overline{U}^n$. Then $F(\mathbf{z_0})$ is not of full rank and consequently, ND^{-1} is not a coprime right MFD of P. We show in the following how to obtain a coprime right MFD of P when there exists some b_J $(1 \le J \le \beta)$ such that $b_J \neq 0$ in $\overline{U}^n$.

Since $P = \tilde{D}^{-1}\tilde{N} = ND^{-1}$, we have

$$[-\tilde{N} \;\; \tilde{D}] \begin{bmatrix} D \\ N \end{bmatrix} = 0_{m,l} \tag{3}$$

or

$$\tilde{F}F = 0_{m,l}. \tag{4}$$

After some suitable row permutations on F, we obtain a new polynomial matrix $F_J = [D_J^T \;\; N_J^T]^T$, $N_J \in \mathbf{R}^{m\times l}[\mathbf{z}]$, $D_J \in \mathbf{R}^{l\times l}[\mathbf{z}]$ with $\det D_J = d_0 b_J$. This is equivalent to finding a constant unimodular matrix U_0, which is a product of a finite number of row permutation matrices, such that

$$U_0 F = F_J. \tag{5}$$

Combining (4) and (5) gives

$$\tilde{F}F = \tilde{F}U_0^{-1}U_0 F = \tilde{F}_J F_J = 0_{m,l} \tag{6}$$

where $\tilde{F}_J = \tilde{F}U_0^{-1}$. Partition $\tilde{F}_J$ as $\tilde{F}_J = [-\tilde{N}_J \;\; \tilde{D}_J]$, $\tilde{N}_J \in \mathbf{R}^{m\times l}[\mathbf{z}]$, $\tilde{D}_J \in \mathbf{R}^{m\times m}[\mathbf{z}]$. It can be shown [14], [17] that $\det \tilde{D}_J = \pm\tilde{d}_0 b_J$. Since F_J is obtained by performing row permutations on F, it is clear [8] that the greatest common divisor (g.c.d.) of the $l\times l$ minors of F_J is equal to d_0 and the reduced minors of F_J have no common zeros in $\overline{U}^n$. Similarly, the g.c.d. of the $l\times l$ minors of $\tilde{F}_J$ is equal to $\tilde{d}_0$ and the reduced minors of $\tilde{F}_J$ have no common zeros in $\overline{U}^n$. From (6), we have

$$[-\tilde{N}_J \;\; \tilde{D}_J] \begin{bmatrix} D_J \\ N_J \end{bmatrix} = 0_{m,l} \tag{7}$$

or

$$P_J = \tilde{D}_J^{-1}\tilde{N}_J = N_J D_J^{-1} \tag{8}$$

where P_J is a new rational matrix. From (8), we have

$$\begin{aligned} P_J &= N_J D_J^{-1} \\ &= \frac{N_J \operatorname{adj}(D_J)}{\det D_J} \\ &= \frac{N_J \operatorname{adj}(D_J)}{d_0 b_J}. \end{aligned} \tag{9}$$

By the well-known Cramer's rule [18], every entry of $\{N_J \operatorname{adj}(D_J)\}$ is just some $l \times l$ minor of F_J and is hence divisible by d_0. Therefore, (9) reduces to

$$P_J = \frac{N'_J}{b_J} = N'_J {D'_J}^{0-1} \tag{10}$$

where $N'_J \in \mathbf{R}^{m\times l}[\mathbf{z}]$, $D'_J = b_J I_l \in \mathbf{R}^{l\times l}[\mathbf{z}]$ with $\det\ D'_J = b_J^l \neq 0$ in $\overline{U}^n$. Thus, $F'_J = [D'^T_J \quad N'^T_J]^T$ is of full rank in $\overline{U}^n$. Since $P_J = N'_J {D'_J}^{-1}$, we have

$$\tilde{D}_J^{-1}\tilde{N}_J = N'_J {D'_J}^{-1} \tag{11}$$

or

$$\begin{aligned}[-\tilde{N}_J \quad \tilde{D}_J]\begin{bmatrix} D'_J \\ N'_J \end{bmatrix} &= \tilde{F}_J F'_J = \tilde{F} U_0^{-1} F'_J \\ &= \tilde{F} F_s = 0_{m,l} \end{aligned} \tag{12}$$

where $F_s = U_0^{-1} F'_J \in \mathbf{R}^{(m+l)\times l}[\mathbf{z}]$. Since U_0 is a unimodular matrix, F_s is also of full rank in $\overline{U}^n$ [8]. Partition F_s as $F_s = [D_s^T \quad N_s^T]^T$, $N_s \in \mathbf{R}^{m\times l}[\mathbf{z}]$, $D_s \in \mathbf{R}^{l\times l}[\mathbf{z}]$. From (12), we have

$$\tilde{F} F_s = [-\tilde{N} \quad \tilde{D}] \begin{bmatrix} D_s \\ N_s \end{bmatrix} = 0_{m,l}. \tag{13}$$

It can be shown [5], [17] that $\det\ D_s \not\equiv 0$ since $\det\ \tilde{D} \not\equiv 0$. We then have

$$P = \tilde{D}^{-1}\tilde{N} = N_s D_s^{-1}. \tag{14}$$

Thus, $N_s D_s^{-1}$ is a coprime right MFD of P. It can be similarly argued that P also admits a coprime left MFD $P = \tilde{D}_s^{-1}\tilde{N}_s$. □

It is seen from the above proof procedure that for the class of nD systems satisfying the condition stated in Theorem 1, DCFs can be constructed efficiently. In fact, it is not necessary to obtain the matrix U_0. We summarize in the following an algorithm for constructing a coprime right MFD of P. The algorithm can also be applied for obtaining a coprime left MFD of P after minor modification.

Algorithm 1: Let $P(\mathbf{z})$ be given in Theorem 1. A coprime right MFD of P can be constructed in three steps.

Step 1: Decompose P into a left and a right MFDs, $P = \tilde{D}^{-1}\tilde{N} = ND^{-1}$. Let $F = [D^T \quad N^T]^T$, and $\tilde{F} = [-\tilde{N} \quad \tilde{D}]$. Suppose that D_J is formed from rows $J_1, \ldots, J_l$ of F $(1 \leq J_1 < \ldots < J_l \leq m+l)$, with $\det\ D_J = d_0 b_J$, $b_J \neq 0$ in $\overline{U}^n$. For $i = 1, \ldots, l$, swap row i with row J_i of F. If $J_i = i$ for some i, no swapping is required for such i. After the swapping, we obtain a new polynomial matrix $F_J = [D_J^T \quad N_J^T]^T$.

Step 2: Introduce a new rational matrix $P_J = N_J D_J^{-1}$ and obtain a coprime right MFD of P_J as $P_J = N'_J {D'_J}^{-1}$ [see (9) and (10)], where $D'_J = b_J I_l$.

Step 3: Let $F'_J = [D'^T_J \quad {N'_J}^T]^T$. For $i = 1, \ldots, l$, swap row i with row J_i of F'_J, where J_i is identical to J_i in Step 1. Partitioning the resultant polynomial matrix F_s as $F_s = [D_s^T \quad N_s^T]^T$, $D_s \in \mathbf{R}^{l\times l}[\mathbf{z}]$, $N_s \in \mathbf{R}^{m\times l}[\mathbf{z}]$, we finally get a coprime right MFD of P as $P = N_s {D_s}^{-1}$. □

Remark 1: The above algorithm is equivalent to constructing an $F_s \in \mathbf{R}^{(m+l)\times l}[\mathbf{z}]$ such that F_s is of full rank in $\overline{U}^n$ and $\tilde{F}F_s = 0_{m,l}$.

Theorem 1 is now generalized to a larger class of stabilizable MIMO nD systems that have coprime right and left MFDs. Moreover, a very simple and computationally efficient method is also proposed for solving the Bézout identity (2).

Theorem 2: Let $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ represent a causal[4] feedback stabilizable MIMO nD system. Let ND^{-1} be a right MFD of P, and $F = [D^T \quad N^T]^T$. If there exists a unimodular matrix $U \in \mathbf{R}^{(m+l)\times(m+l)}[\mathbf{z}]$ such that a reduced minor of the polynomial matrix $F_1 = UF$ is devoid of any zeros in $\overline{U}^n$, then P has a DCF.

Proof: We first construct coprime right and left MFDs for $P(\mathbf{z})$, and then solve the Bézout identity (2). Decompose P into a left MFD, $P = \tilde{D}^{-1}\tilde{N}$, and let $\tilde{F} = [-\tilde{N} \quad \tilde{D}]$. Since $P = \tilde{D}^{-1}\tilde{N} = ND^{-1}$, we have

$$[-\tilde{N} \quad \tilde{D}] \begin{bmatrix} D \\ N \end{bmatrix} = [-\tilde{N} \quad \tilde{D}] U^{-1} U \begin{bmatrix} D \\ N \end{bmatrix} = 0_{m,l} \tag{15}$$

or

$$\tilde{F}_1 F_1 = 0_{m,l} \tag{16}$$

where $\tilde{F}_1 = \tilde{F}U^{-1} \in \mathbf{R}^{m\times(m+l)}[\mathbf{z}]$. Suppose that a reduced minor of the matrix $F_1 = UF$ is devoid of any zeros in $\overline{U}^n$. Applying Algorithm 1, we can construct $F'_s \in \mathbf{R}^{(m+l)\times l}[\mathbf{z}]$, with F'_s being of full rank in $\overline{U}^n$, such that

$$\tilde{F}_1 F'_s = 0_{m,l} \tag{17}$$

or

$$\tilde{F} U^{-1} F'_s = \tilde{F} F_s = 0_{m,l} \tag{18}$$

where $F_s = U^{-1}F'_s \in \mathbf{R}^{(m+l)\times l}[\mathbf{z}]$. Since U is a unimodular matrix, F_s is also of full rank in $\overline{U}^n$ [8]. Partition F_s as $F_s = [D_s^T \quad N_s^T]^T$, $N_s \in \mathbf{R}^{m\times l}[\mathbf{z}]$, $D_s \in \mathbf{R}^{l\times l}[\mathbf{z}]$. From (18), we have

$$\tilde{F} F_s = [-\tilde{N} \quad \tilde{D}] \begin{bmatrix} D_s \\ N_s \end{bmatrix} = 0_{m,l} \tag{19}$$

or

$$P = \tilde{D}^{-1}\tilde{N} = N_s D_s^{-1}. \tag{20}$$

Thus, $N_s D_s^{-1}$ is a coprime right MFD of P. It can be similarly argued that P also admits a coprime left MFD $P = \tilde{D}_s^{-1}\tilde{N}_s$.

It remains to solve the Bézout identity (2). As demonstrated in [3] and [10], the most critical part for solving the Bézout identity (2) is to obtain $\tilde{X} \in \mathbf{R}^{l\times l}[\mathbf{z}]$, $X \in \mathbf{R}^{m\times m}[\mathbf{z}]$, $\tilde{Y}, Y \in \mathbf{R}^{l\times m}[\mathbf{z}]$, such that

$$\tilde{X} D_s + \tilde{Y} N_s = S_1 \tag{21}$$

and

$$\tilde{N}_s Y + \tilde{D}_s X = S_2 \tag{22}$$

for some $S_1 \in \mathbf{R}^{l\times l}[\mathbf{z}]$, $S_2 \in \mathbf{R}^{m\times m}[\mathbf{z}]$ with $\det\ S_1 \neq 0$, $\det\ S_2 \neq 0$ in $\overline{U}^n$.

From the way F'_s is constructed by Algorithm 1, we know that $\det\ D_J = b_J^l \neq 0$ in $\overline{U}^n$, where D_J is formed from rows $J_1, \ldots, J_l$ of F'_s $(1 \leq J_1 < \cdots < J_l \leq m+l)$. Now construct a constant matrix H of size $l \times (m+l)$ by placing the ith column of the identity matrix I_l in the J_ith column of H, and zeros at other columns of H. It is then easy to show that

$$HF'_s = D_J \tag{23}$$

or

$$HUU^{-1}F'_s = \tilde{H}F_s = D_J \tag{24}$$

where $\tilde{H} = HU$. Partition $\tilde{H}$ as $\tilde{H} = [\tilde{X} \quad \tilde{Y}]$, $\tilde{X} \in \mathbf{R}^{l\times l}[\mathbf{z}]$, $\tilde{Y} \in \mathbf{R}^{l\times m}[\mathbf{z}]$. Equation (24) becomes

$$\tilde{X} D_s + \tilde{Y} N_s = D_J. \tag{25}$$

[4]See [3] and [8] for the definition of causality of nD systems.

Equation (22) can be solved similarly. Once (21) and (22) have been solved, it is easy to solve the Bézout identity

$$\begin{bmatrix} \tilde{X}_s & \tilde{Y}_s \\ -\tilde{N}_s & \tilde{D}_s \end{bmatrix} \begin{bmatrix} D_s & -Y_s \\ N_s & X_s \end{bmatrix} = \begin{bmatrix} I_l & 0_{l,m} \\ 0_{m,l} & I_m \end{bmatrix} \tag{26}$$

for some $\tilde{X}_s \in \mathbf{R}_{\mathbf{s}}^{l\times l}(\mathbf{z})$, $X_s \in \mathbf{R}_{\mathbf{s}}^{m\times m}(\mathbf{z})$, and $\tilde{Y}_s, Y_s \in \mathbf{R}_{\mathbf{s}}^{l\times m}(\mathbf{z})$. Since P is causal by assumption, using a technique similar to the one in [5], we can find $X_s(\mathbf{z})$ and $\tilde{X}_s(\mathbf{z})$ such that $\det X_s(0,\ldots,0) = \det X_s(\mathbf{0}) \neq 0$, and $\det \tilde{X}_s(\mathbf{0}) \neq 0$. This immediately implies that $X_s(\mathbf{z})$ and $\tilde{X}_s(\mathbf{z})$ are nonsingular. The details can be worked out similarly as in [5], [8], and [10], and are omitted here to save space. $\square$

Remark 2: For the 2-D case, several constructive methods have been proposed for solving the 2-D version of (21) and (22) [3]–[6]. However, for the nD case, the only available method for solving (21) and (22) is from [19], which is computationally rather involved. In fact, the authors of [19] did not tell how to construct a stable nD polynomial that vanishes at the variety of the ideal generated by the maximal order minors of $[D_s^T \quad N_s^T]^T$ or $[-\tilde{N}_s \quad \tilde{D}_s]$. As mentioned in [9], the construction of such a stable nD polynomial is crucial for solving (21) and (22). One of the contributions of this paper is the development of a very simple and computationally efficient method for solving (21) and (22) for the class of nD systems satisfying the condition stated in Theorem 2.

Before ending this section, following [1]–[3], [10], we can give a parameterization of *all* stabilizing compensators for a stabilizable nD system $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ satisfying the condition stated in Theorem 2

$$\begin{aligned} C = \left(\tilde{X}_s - Q\tilde{N}_s\right)^{-1}\left(\tilde{Y}_s + Q\tilde{D}_s\right)&: \\ Q \in \mathbf{R}_{\mathbf{s}}^{l\times m}(\mathbf{z}) \text{ and } \det\left(\tilde{X}_s - Q\tilde{N}_s\right) &\not\equiv 0. \end{aligned} \tag{27}$$

III. Example

Consider a causal unstable 2×2 four-dimensional (4-D) system represented by

$$P(z_1,\ldots,z_4) = \frac{1}{g}\begin{bmatrix} g & 0 \\ z_3z_4 & f \end{bmatrix} \tag{28}$$

where $g = 1 + z_1 - z_2$, $f = 1 - 4z_1z_2$. Decompose P into MFDs, $P = \tilde{D}^{-1}\tilde{N} = ND^{-1}$, where

$$D = \tilde{D} = \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}, \qquad N = \tilde{N} = \begin{bmatrix} g & 0 \\ z_3z_4 & f \end{bmatrix}.$$

Let $F = [D^T \quad N^T]^T$, $\tilde{F} = [-\tilde{N} \quad \tilde{D}]$, and let $a_1,\ldots,a_6$ denote the 2×2 minors of F. We have, $a_i = d_0b_i$ for $i = 1,\ldots,6$, where $d_0 = g = 1 + z_1 - z_2$, and $b_1,\ldots,b_6$ are the reduced minors of F given by

$$g, \quad 0, \quad f, \quad -g, \quad -z_3z_4, \quad f.$$

The set of all the common zeros of $b_1,\ldots,b_6$ can be calculated as

$$\begin{gathered} \left(-\frac{1+\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2}, z_{30}, 0\right), \left(-\frac{1+\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2}, 0, z_{40}\right) \\ \left(\frac{-1+\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}, z_{30}, 0\right), \left(\frac{-1+\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}, 0, z_{40}\right) \end{gathered}$$

where z_{30} and z_{40} are arbitrary complex numbers. It is clear that $b_1,\ldots,b_6$ have no common zeros in $\overline{U}^4$. Hence, $P(z_1,\ldots,z_4)$ is feedback stabilizable [7], [8]. However, since d_0 has some zeros in $\overline{U}^4$, such as (−0.5, 0.5, 0, 0), F is not of full rank in $\overline{U}^4$. Thus, ND^{-1} is not a coprime right MFD of P. Since the general nD polynomial matrix factorization problem is still open [15], [16], we do not know whether or not a right factor with determinant equal to d_0 can be extracted from F directly. The method proposed in [10] cannot be applied here either since d_0 depends on both z_1 and z_2, and F is a 4-D polynomial matrix. As none of the reduced minors of P is a stable polynomial, Theorem 1 cannot be directly applied. However, we observe that there exists a unimodular matrix U such that a reduced minor of UF becomes a stable polynomial. In fact, using a result from [20], we have

$$\begin{aligned} 2\left(1+\sqrt{2}\right)g + f &= 2\left(1+\sqrt{2}\right)(1 + z_1 - z_2) + 1 - 4z_1z_2 \\ &= \underbrace{4\left(\frac{1+\sqrt{2}}{2} + z_1\right)\left(\frac{1+\sqrt{2}}{2} + z_2\right)}_{s} \end{aligned} \tag{29}$$

where s is obviously a stable polynomial. Choose U as

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & e & 0 & 1 \end{bmatrix} \tag{30}$$

where $e = 2(1+\sqrt{2})$. Then

$$\begin{aligned} UF &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & e & 0 & 1 \end{bmatrix} \begin{bmatrix} g & 0 \\ 0 & g \\ g & 0 \\ z_3z_4 & f \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} g & 0 \\ 0 & g \\ g & 0 \\ z_3z_4 & s \end{bmatrix}}_{F'}. \end{aligned} \tag{31}$$

The reduced minors of F' are

$$g, \quad 0, \quad s, \quad -g, \quad -z_3z_4, \quad s.$$

Since s is stable, Algorithm 1 can be applied to F'.

Step 1: Consider the 2×2 matrix D_J formed from rows 1 and 4 of F', with $\det D_J = d_0b_J = gs$. Swap row 2 with row 4 of F'. We obtain a new polynomial matrix

$$F_J = \begin{bmatrix} g & 0 \\ z_3z_4 & s \\ g & 0 \\ 0 & g \end{bmatrix} = \begin{bmatrix} D_J \\ N_J \end{bmatrix}. \tag{32}$$

Step 2: Introduce a new rational matrix P_J and obtain a coprime right MFD of P_J as follows:

$$\begin{aligned} P_J &= N_JD_J^{-1} \\ &= \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}\begin{bmatrix} g & 0 \\ z_3z_4 & s \end{bmatrix}^{-1} \\ &= \frac{1}{s}\begin{bmatrix} s & 0 \\ -z_3z_4 & g \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} s & 0 \\ -z_3z_4 & g \end{bmatrix}}_{N_J'}\underbrace{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}}_{D_J'}^{-1}. \end{aligned} \tag{33}$$

Hence, $N_J' D_J'^{-1}$ is a coprime right MFD of P_J.

Step 3: Let $F'_J = [D'^T_J \quad N'^T_J]^T$. Obviously, F'_J is of full rank in $\overline{U}^4$. Swap row 2 with row 4 of F'_J. We obtain a new polynomial matrix

$$F'_s = \begin{bmatrix} s & 0 \\ -z_3z_4 & g \\ s & 0 \\ 0 & s \end{bmatrix} \tag{34}$$

which is also of full rank in $\overline{U}^4$. It is easy to check that $\tilde{F}'F'_s = 0_{m,l}$, where $\tilde{F}' = \tilde{F}U^{-1}$.

To obtain a coprime right MFD of P, we have to premultiply F'_s by U^{-1}, i.e.,

$$U^{-1}F'_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -e & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ -z_3z_4 & g \\ s & 0 \\ 0 & s \end{bmatrix} = \underbrace{\begin{bmatrix} s & 0 \\ -z_3z_4 & g \\ s & 0 \\ ez_3z_4 & f \end{bmatrix}}_{F_s}. \tag{35}$$

It follows that $\tilde{F}F_s = 0_{m,l}$. Partitioning F_s as $F_s = [D_s^T \quad N_s^T]^T$, we have arrived at a coprime right MFD of P as

$$P(z_1,\ldots,z_4) = N_sD_s^{-1} = \begin{bmatrix} s & 0 \\ ez_3z_4 & f \end{bmatrix} \begin{bmatrix} s & 0 \\ -z_3z_4 & g \end{bmatrix}^{-1}. \tag{36}$$

Next, a coprime left MFD of P can be easily obtained for this example as

$$P(z_1,\ldots,z_4) = \tilde{D}_s^{-1}\tilde{N}_s = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ z_3z_4 & f \end{bmatrix}. \tag{37}$$

It remains to solve the Bézout identity (2), or equivalently, (21) and (22). We observe from (34) that $\det\, D'_J = s^2 \neq 0$ in $\overline{U}^n$, where D'_J is formed from rows 1 and 4 of the matrix F'_s. Construct a 2×4 constant matrix H as

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is then easy to verify that

$$HF'_s = sI_2 \tag{38}$$

or

$$\tilde{H}F_s = sI_2 \tag{39}$$

where

$$\tilde{H} = HU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e & 0 & 1 \end{bmatrix}.$$

Obviously, $\det(sI_2) = s^2 \neq 0$ in $\overline{U}^4$. Partition $\tilde{H}$ as $\tilde{H} = [\tilde{X} \quad \tilde{Y}]$, where

$$\tilde{X} = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} \quad \text{and} \quad \tilde{Y} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equation (39) becomes

$$\tilde{X}D_s + \tilde{Y}N_s = sI_2. \tag{40}$$

Similarly, we can construct

$$X = \begin{bmatrix} s^2 & 0 \\ -ez_3z_4(1-s) & e \end{bmatrix}, \quad Y = \begin{bmatrix} s(1-s) & 0 \\ -z_3z_4(1-s) & 1 \end{bmatrix}$$

such that

$$\tilde{N}_sY + \tilde{D}_sX = S_2 \tag{41}$$

where

$$S_2 = \begin{bmatrix} 1 & 0 \\ z_3z_4 & s \end{bmatrix}.$$

Obviously, $\det\, S_2 = s \neq 0$ in $\overline{U}^4$.

Once (40) and (41) have been solved, it is routine to solve the Bézout identity. Choose

$$\tilde{X}_s = \frac{1}{s}\begin{bmatrix} s & 0 \\ 0 & e \end{bmatrix}, \quad \tilde{Y}_s = \frac{1}{s}\begin{bmatrix} 1-s & 0 \\ 0 & 1 \end{bmatrix}$$
$$X_s = \frac{1}{s}\begin{bmatrix} s^2 & 0 \\ -ez_3z_4(1-s) & e \end{bmatrix}$$

and

$$Y_s = \frac{1}{s}\begin{bmatrix} s(1-s) & 0 \\ -z_3z_4(1-s) & 1 \end{bmatrix}.$$

It can be verified easily that

$$\begin{bmatrix} \tilde{X}_s & \tilde{Y}_s \\ -\tilde{N}_s & \tilde{D}_s \end{bmatrix} \begin{bmatrix} D_s & -Y_s \\ N_s & X_s \end{bmatrix} = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & I_2 \end{bmatrix}. \tag{42}$$

The derivation is just routine calculation and is omitted here to save space. Obviously, $\tilde{X}_s, X_s, \tilde{Y}_s, Y_s \in \mathbf{R}_s^{2\times 2}(z_1,\ldots,z_4)$, and X_s, $\tilde{X}_s$ are both nonsingular. Therefore, a DCF of $P(z_1,\ldots,z_4)$ has been obtained. Finally, *all* stabilizing compensators for the given unstable 4-D system P are parameterized by

$$C = \left(\tilde{X}_s - Q\tilde{N}_s\right)^{-1}\left(\tilde{Y}_s + Q\tilde{D}_s\right): \quad Q \in \mathbf{R}_s^{2\times 2}(z_1,\ldots,z_4) \text{ and } \det\left(\tilde{X}_s - Q\tilde{N}_s\right) \not\equiv 0. \tag{43}$$

IV. Conclusion

In this paper, we have solved the open problem of the existence of DCFs for a class of MIMO nD linear systems. A simple and efficient method has been proposed for the construction of DCFs when they exist. The main result of the paper is to show how to construct a coprime (over the ring of stable rational functions) matrix fraction description of a given MIMO nD system with a stable reduced minor. A parameterization of all stabilizing compensators has also been given for an MIMO nD system in this class.

For the class of MIMO nD systems under discussion, the proposed method has several advantages compared with the recent result of [10]. Firstly, the new method is valid not only for 3-D systems, but also for nD ($n > 3$) systems. Secondly, it works even when an unstable g.c.d. associated with an MFD of a given MIMO nD system has multiple zeros. Thirdly, the resultant DCFs always have real coefficients when the original system transfer matrix has only real coefficients. Fourthly, the proposed method is very simple and computationally efficient in solving the Bézout identity (2). An illustrative example has been worked out in details to support the new results presented in the paper.

However, the new results presented in this paper are applicable only to the class of MIMO nD linear systems whose reduced minors satisfying the condition stated in Theorem 2. Thus, the problem of the existence of a DCF for a general stabilizable MIMO nD linear system is still open. Further investigation is required.

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The Routh–Hurwitz Array and Realization of Characteristic Polynomials

Michael Margaliot and Gideon Langholz

***Abstract*—In this paper we show that the Routh–Hurwitz array of a given characteristic polynomial provides all the information required to realize the polynomial using an electrical circuit. This new interpretation also leads to an intuitive proof of the Routh–Hurwitz stability criterion.**

***Index Terms*—Realization theory, Routh–Hurwitz criterion, stability.**

I. Introduction

Stability of dynamical systems is of fundamental importance in control theory. Because of that, the Routh–Hurwitz (RH) stability criterion, which is very-well known and widely used, is often revisited in the literature [1], [3], and [5]. To quote the most recent paper on this subject [1]: "The importance of the topic and the complexity of the mathematical tools required in most of the proofs motivate the continued interest in the subject."

In this paper we provide a new perspective of the RH array, which also leads to a very intuitive proof of the RH stability criterion, by relating it to physical entities rather than using mathematical abstractions. We do so by considering the following problem [4]: Given a characteristic polynomial $D(s)$ of order n, realize it as an electrical circuit. We construct the circuit recursively for any n and show that the values of its components are just the entries in the first column of the RH array of $D(s)$.

If the components' values are all positive, they are passive and the energy stored in the circuit can never increase. On the other hand, if there is a sign change among the components, then at least one component will gain energy so that the total energy of the circuit will increase. Hence, to determine stability we must examine sign changes among the components' values which, as we just noted, are exactly the entries in the first column of the RH array.

The remainder of the paper is organized as follows. In Section II we construct a linear electrical circuit recursively, analyze its properties, and show how it can be used to realize $D(s)$. In Section III we use the energy stored in the circuit to derive a simple proof of the RH stability test. Section IV discusses singular cases and the final section summarizes.

II. The Linear Circuit

We begin by defining recursively a linear electrical circuit C^n, $n = 1, 2, \ldots$. The circuit C^1 [see Fig. 1(a)] consists of the parallel connection of a capacitor and a resistor with resistance $R = 1\Omega$. The circuit C^2 [see Fig. 1(b)] is obtained by replacing the resistor in C^1 with the series connection of an inductor and a 1 Ω resistor. For any $n > 2$, the circuit C^n is defined recursively.

- If n is odd, then C^n is the circuit obtained by replacing the resistor in C^{n-1} with the parallel connection of a capacitor $1/K_n$ and a resistor $R = 1\ \Omega$, as shown in Fig. 1(c).

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