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Feedback Stabilizability of MIMO n-D Linear Systems

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Abstract. The problem of output feedback stabilizability of multi-input-multi-output (MIMO) multidimensional (n-D) linear systems is investigated using n-D polynomial matrix theory. A simple necessary and sufficient condition for output feedback stabilizability of a given MIMO n-D linear system is derived in terms of the generating polynomials associated with any matrix fraction descriptions of the system. When a given unstable plant is feedback stabilizable, constructive method is provided for obtaining a stabilizing compensator. Moreover, a strictly causal compensator can always be constructed for a causal (not necessarily strictly causal) plant. A non-trivial example is illustrated.

Key Words: n-D System, Polynomial matrices, Matrix fraction description, Generating polynomials, Output feedback, Stabilizability.

1. Introduction

The problem of feedback stabilization of multi-input-multi-output (MIMO) linear systems has drawn much attention in the past years because of its importance in control and systems (see, e.g., [1]–[8] and the references therein). Consider the feedback system shown in Figure 1, where P represents a plant and C represents a compensator. The relationship between $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{e}_1$, $\mathbf{e}_2$ can be expressed as:

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix}}_{H_{eu}} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \tag{1}$$

A given plant P is said to be (output) feedback stabilizable if and only if there exists a compensator C such that the feedback system H_{eu} is stable, i.e., each entry of H_{eu} has no poles in the unstable region [2], [3]. For linear multidimensional (n-D) system, the feedback system is structurally stable[1] if and only if each entry of H_{eu} has no poles in the closed unit polydisc $\overline{U}^n$ [9], [10].

The problem of feedback stabilizability of MIMO 2-D systems using the matrix fraction description (MFD) approach has been investigated by a number of researchers (see, e.g., [4]–[7] and the references therein). It is now well known that by decomposing a given plant P into an MFD $P = D^{-1}N$, where D and N are minor coprime 2-D polynomial matrices of appropriate dimension, a necessary and sufficient condition for feedback stabilizability of P is that all the maximal order minors of the matrix $[D\ \ N]$ have no common zeros in $\overline{U}^2$

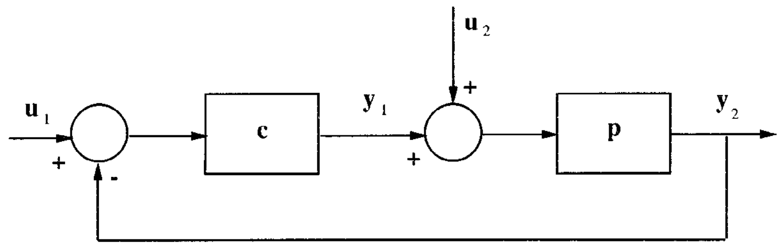

Figure 1. Feedback system.

[4], [5]. Constructive algorithms for the feedback stabilizability and stabilization problem have also been presented for MIMO 2-D systems [4]–[7].

However, generalization of results on MIMO 2-D systems to the n-D ($n \geq 3$) case is a non-trivial task because of some fundamental differences between MIMO 2-D systems and their n-D ($n \geq 3$) counterparts [11]–[14]. In particular, since a given n-D ($n \geq 3$) system P may not always admit a minor coprime MFD [11], [13], existing criterion for feedback stabilizability of MIMO 2-D systems is not applicable to an n-D ($n \geq 3$) system P that does not admit a minor coprime MFD.

Recently, Shankar and Sule have solved the problem of feedback stabilizability and stabilization for single-input-single-output (SISO) systems over a general integral domain, which include SISO n-D systems as special cases [15]. Their method has later been extended to the MIMO case by Sule [8]. However, unlike those earlier results on MIMO 2-D systems [4]–[7] which used mainly polynomial matrix theory, the method presented by Sule in [8] relies heavily on the mathematical theory of commutative algebra and topology, with which some control and systems engineers may be unfamiliar.

Although the theory of commutative algebra and topology is necessary for discussing the feedback stabilizability of linear systems over commutative rings as in [8], it may not be so when one is only interested in linear n-D systems. The objective of this paper is to present a solution to the problem of feedback stabilizability of MIMO linear n-D systems using only the polynomial matrix theory, and thus avoiding the sophisticated theory of commutative algebra and topology. Using polynomial matrix manipulations, we are able to develop a computationally more efficient method for constructing a stabilizing n-D compensator when a given n-D plant is stablizable.

After recalling some necessary definitions and related known results in the next section, a tractable criterion for feedback stabilizability of MIMO n-D systems is presented and proved in Section 3. This section also shows how to construct a strictly causal stabilizing n-D compensator when a given causal (not necessarily strictly causal) n-D plant is stabilizable. Comparison of the main results of this paper with Sule's results [8] is given at the end of Section 3. A non-trivial example is illustrated in Section 4 and conclusion is in Section 5.

2. Preliminaries

For convenience, in this section we reproduce some definitions and results which are required for the derivation of new results in the next section. In the following, we shall denote $\mathbf{R}(\mathbf{z}) = \mathbf{R}(z_1, \ldots, z_n)$ the set of rational functions in complex variables $z_1, \ldots, z_n$ with coefficients in the field of real numbers $\mathbf{R}$; $\mathbf{R}[\mathbf{z}]$ the set of polynomials in complex variables $z_1, \cdots, z_n$ with coefficients in the field of real numbers $\mathbf{R}$; $\mathbf{R}^{m\times l}[\mathbf{z}]$ the set of $m \times l$ matrices with entries in $\mathbf{R}[\mathbf{z}]$, *etc*. Throughout this paper, the argument $(\mathbf{z})$ is omitted whenever its omission does not cause confusion.

Next, as in [13], we require some preliminaries regarding the ordering of the submatrices and minors of a matrix. Let

$$F = [\mathbf{f}_1, \cdots, \mathbf{f}_{m+l}] \in \mathbf{R}^{(m+l)\times l}[\mathbf{z}], \tag{2}$$

and consider all the $l \times l$ submatrices of F. The number of these submatrices is $\beta = \binom{m+l}{l}$. If a submatrix F_i $(1 \le i \le \beta)$ is formed by selecting rows $1 \le i_1 < \cdots < i_l \le m + l$, we associate F_i with an l-tuple $(i_1, \ldots, i_l)$. It is easy to see that there exists a one to one correspondence between all the $l \times l$ submatrices of F and the collection of all strictly increasing l-tuple $(i_1, \ldots, i_l)$, where $1 \le i_1 < \cdots < i_l \le m + l$. Now by enumerating the above l-tuple $(i_1, \ldots, i_l)$ in the lexicographic order, the $l \times l$ submatrices of F are ordered accordingly. This ordering of the $l \times l$ submatrices of F will be assumed throughout the paper. The $l \times l$ minors of the matrix F, denoted by $a_1, \ldots, a_\beta$, will always be ordered in the same way as $F_1, \ldots, F_\beta$, i.e., $a_i = \det F_i$, $i = 1, \ldots, \beta$.

Definition 1 [2], [3], [9], [10]. Consider the feedback system in Figure 1. Let $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ represents an n-D plant, $C \in \mathbf{R}^{l\times m}(\mathbf{z})$ represents an n-D compensator. The feedback system is stable if and only if each entry of the n-D rational matrix H_{eu} as defined in (1) has no poles in $\overline{U}^n$. An unstable plant P is said to be output feedback stabilizable if and only if there exists a compensator C (called stabilizing compensator) such that the feedback system is stable.

Definition 2 [11], [13]. Let $D \in \mathbf{R}^{l\times l}[\mathbf{z}]$, $N \in \mathbf{R}^{m\times l}[\mathbf{z}]$, and $F = [D^T \ N^T]^T$, where D^T denotes the transposed matrix of D. Then D and N are said to be:

(i) minor right coprime (MRC) if the $l \times l$ minors of $[D^T \ N^T]^T$ are factor coprime.

(ii) factor right coprime (FRC) if in any polynomial decomposition $F = F_1 F_2$, the $l \times l$ matrix F_2 is a unimodular matrix, i.e., $\det F_2 = k \in \mathbf{R}^*$.[2]

In a dual manner, $\tilde{D} \in \mathbf{R}^{m\times m}[\mathbf{z}]$, and $\tilde{N} \in \mathbf{R}^{m\times l}[\mathbf{z}]$, are said to be minor left coprime (MLC) if $\tilde{D}^T$ and $\tilde{N}^T$ are MRC, *etc*.

Definition 3 [13]. Let $F = [D^T \ N^T]^T \in \mathbf{R}^{(m+l)\times l}[\mathbf{z}]$ be of normal full rank,[3] and let $a_1, \ldots, a_\beta$ denote the $l \times l$ minors of the matrix F, with $a_1 = \det D$, where $\beta = \binom{m+l}{l}$.

Extracting a greatest common divisor (g.c.d.) d of $a_1, \ldots, a_\beta$ gives:

$$a_i = db_i, \qquad i = 1, \ldots, \beta. \tag{3}$$

Then, $b_1, \ldots, b_\beta$ are called the "generating polynomials" of F.

The generating polynomials of $\tilde{F} = [\tilde{D}\ \ \tilde{N}]$ can be similarly defined [13]. The term "generating polynomials" is justified by the following tow propositions [13], which show that the generating polynomials are essentially unique for all left and right MFDs of a given n-D rational matrix.

PROPOSITION 1 [13] *Let $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ be of normal full rank. If*

$$P = N_1 D_1^{-1} = N_2 D_2^{-1},$$

$b_{11}, \ldots, b_{1\beta}$ are the generating polynomials of $[D_1^T\ \ N_1^T]^T$, $b_{21}, \ldots, b_{2\beta}$ are the generating polynomials of $[D_2^T\ \ N_2^T]^T$, then

$$b_{2i} = b_{1i}, \qquad i = 1, \ldots, \beta. \tag{4}$$

PROPOSITION 2 [13] *Let $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ be of normal full rank. Decompose P into the following MFD's:*

$$P = \tilde{D}^{-1}\tilde{N} = ND^{-1}.$$

Denote by $\tilde{b}_1, \ldots, \tilde{b}_\beta$ the generating polynomials of $[\tilde{D}\ \ \tilde{N}]$, and by $b_1, \ldots, b_\beta$ the generating polynomials of $[D^T\ \ N^T]^T$. Then

$$b_i = \pm\tilde{b}_i', \qquad i = 1, \ldots, \beta, \tag{5}$$

where $\tilde{b}_1', \ldots, \tilde{b}_\beta'$ are obtained by reordering $\tilde{b}_1, \ldots, \tilde{b}_\beta$ appropriately, with $b_1 = \tilde{b}_1$.

Remark 1. The definition of "generating polynomials" given in [13] is equivalent to the definition of "family of reduced minors" in [8]. The results stated in Propositions 1 and 2 were first presented in [13]. They were also stated without proof in [8]. Also notice the original results in [13] are $b_{2i} = kb_{1i}$ for (4) and $b_i = \pm k\tilde{b}_i'$ for (5) for some non-zero constant k. For convenience of exposition, the non-zero constant k is dropped here since it can always be absorbed into a g.c.d. of the maximal order minors of a matrix.

PROPOSITION 3 [13] *An n-D system represented by $P = ND^{-1} \in \mathbf{R}^{m\times l}(\mathbf{z})$ is stable if and only if $b_1 \neq 0$ in $\overline{U}^n$, where $b_1, \ldots, b_\beta$ are the generating polynomials of $[D^T\ \ N^T]^T$.*

PROPOSITION 4 [16] *Let $b_i \in \mathbf{R}[\mathbf{z}]$, for $i = 1, \ldots, \beta$. If $b_1, \ldots, b_\beta$ have no common zeros* *in* $\overline{U}^n$*, then there exist* $\lambda_1, \ldots, \lambda_\beta \in \mathbf{R}[\mathbf{z}]$*, such that*

$$\sum_{i=1}^{\beta} \lambda_i b_i = s \tag{6}$$

for some $s \in \mathbf{R}[\mathbf{z}]$ *with* $s \neq 0$ *in* $\overline{U}^n$.

The following definitions and results are generalization from the 2-D case [4] to the n-D case.

Definition 4. A rational function $n(\mathbf{z})/d(\mathbf{z})$ with $n, d \in \mathbf{R}[\mathbf{z}]$ is called causal if $d(\mathbf{0}) = d(0, \ldots, 0) \neq 0$. It is called strictly causal if in addition $n(\mathbf{0}) = 0$. A rational function matrix $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ is called causal if all its entries are causal. It is called strictly causal if all its entries are strictly causal.

PROPOSITION 5 *If* $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ *is causal (strictly causal), there exists a right MFD* $P = ND^{-1}$ *such that* $\det D(\mathbf{0}) \neq 0$ *(in addition,* $N(\mathbf{0}) = 0_{m,l}$*).*

PROPOSITION 6 *If* $P = ND^{-1} \in \mathbf{R}^{m\times l}(\mathbf{z})$*, and* $\det D(\mathbf{0}) \neq 0$*, then* P *is causal. If in addition* $N(\mathbf{0}) = 0_{m,l}$*, then* P *is strictly causal.*

Similar statements for the above two propositions of course follow for left MFD of P.

3. Main Results

In this section, a tractable criterion for feedback stabilizability of MIMO n-D systems and a construction of a strictly causal stabilizing n-D compensator for a stabilizable causal n-D plant are presented. First, we need the following lemma.

LEMMA 1 *Let* $F_1, F_2 \in \mathbf{R}^{k\times l}[\mathbf{z}]$ *be of normal full rank, with* $k > l$*, and let* d_p *denote a g.c.d. of the* $l \times l$ *minors of* F_p $(p = 1, 2)$*. If*

$$F_1 = UF_2 \tag{7}$$

for some unimodular matrix $U \in \mathbf{R}^{k\times k}[\mathbf{z}]$*, then* $d_1 = r_0 d_2$ *for some* $r_0 \in \mathbf{R}^*$.

Proof: Let $a_{p1}, \ldots, a_{p\beta}$ denote the $l \times l$ minors of F_p $(p = 1, 2)$ where $\beta = \binom{k}{l}$. Since d_p is a g.c.d. of the $l \times l$ minors of F_p $(p = 1, 2)$, we have

$$a_{pi} = d_p b_{pi} \qquad i = 1, \ldots, \beta;\ p = 1, 2 \tag{8}$$

where $b_{pi} \in \mathbf{R}[\mathbf{z}]$. Let U_i denote the $l \times k$ matrix formed by selecting the rows $i_1, \ldots, i_l$ from U, and let $q_{i1}, \ldots, q_{i\beta}$ denote the $l \times l$ minors of U_i. From (7), and by using the

Cauchy-Binet formula [17], it follows that

$$
\begin{aligned}
a_{1i} &= \sum_{j=1}^{\beta} q_{ij} a_{2j} \\
&= \sum_{j=1}^{\beta} q_{ij} d_2 b_{2j} \\
&= d_2 \sum_{j=1}^{\beta} q_{ij} b_{2j} \qquad i = 1, \ldots, \beta.
\end{aligned}
$$

Thus, d_2 is a common divisor of $a_{11}, \ldots, a_{1\beta}$. Since by assumption, d_1 is a g.c.d. of $a_{11}, \ldots, a_{1\beta}$, d_2 is necessarily a divisor of d_1.

Next, from (7), we have $F_2 = U^{-1} F_1$, where $U^{-1} \in \mathbf{R}^{k \times k}[\mathbf{z}]$ is a unimodular matrix. It can be similarly argued as above that d_1 is a divisor of d_2. Therefore, $d_1 = r_0 d_2$ for some $r_0 \in \mathbf{R}^*$. ∎

The main results of this paper are stated in the following two theorems. Theorem 1 presents a constructive solution to an n-D polynomial matrix equation, while Theorem 2 gives a criterion on the output feedback stabilizability of n-D systems.

THEOREM 1 *Let a causal n-D plant* $P = ND^{-1} \in \mathbf{R}^{m \times l}(\mathbf{z})$ *with* $D \in \mathbf{R}^{l \times l}[\mathbf{z}]$, $N \in \mathbf{R}^{m \times l}[\mathbf{z}]$ *and* $\det D(\mathbf{0}) \neq 0$. *Denote by* $b_1, \ldots, b_\beta$ *the generating polynomials of* $[D^T \;\; N^T]^T$, *where* $\beta = \binom{m+l}{l}$.

If $b_1, \ldots, b_\beta$ *have no common zeros in* $\overline{U}^n$, *then there exists a strictly causal compensator* $C = X^{-1} Y \in \mathbf{R}^{l \times m}(\mathbf{z})$ *with* $X \in \mathbf{R}^{l \times l}[\mathbf{z}]$, $Y \in \mathbf{R}^{l \times m}[\mathbf{z}]$, $\det X(\mathbf{0}) \neq 0$ *and*[4] $Y(\mathbf{0}) = 0_{l,m}$, *such that the generating polynomials of* $[X \;\; Y]$, *denoted by* $e_1, \ldots, e_\beta$, *satisfy*

$$
\sum_{i=1}^{\beta} e_i b_i = s_1 \tag{9}
$$

for some $s_1 \in \mathbf{R}[\mathbf{z}]$ *with* $s_1 \neq 0$ *in* $\overline{U}^n$.

Proof: A proof consists of the following four steps:

Step 1: Since $b_1, \ldots, b_\beta$ have no common zeros in $\overline{U}^n$, by Proposition 4, there exist $\lambda_1, \ldots, \lambda_\beta \in \mathbf{R}[\mathbf{z}]$, such that

$$
\sum_{i=1}^{\beta} \lambda_i b_i = s \tag{10}
$$

for some $s \in \mathbf{R}[\mathbf{z}]$ with $s \neq 0$ in $\overline{U}^n$. Let

$$
F = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{m+l} \end{bmatrix} = \begin{bmatrix} D \\ N \end{bmatrix} \in \mathbf{R}^{(m+l) \times l}[\mathbf{z}], \tag{11}
$$

where $\mathbf{f}_i \in \mathbf{R}^{1\times l}[\mathbf{z}]$ $(i = 1, \ldots, m+l)$. Let $F_1, \ldots, F_\beta$ denote the $l \times l$ submatrices of F, i.e.

$$F_i = \begin{bmatrix} \mathbf{f}_{i_1} \\ \vdots \\ \mathbf{f}_{i_l} \end{bmatrix} \tag{12}$$

where $1 \le i_i < \cdots < i_l \le m+l$, for $i = 1, \ldots, \beta$. Let $a_i = \det F_i$, $G_i = [\mathbf{g}_{i_1} \cdots \mathbf{g}_{i_l}] = \operatorname{adj} F_i$, for $i = 1, \ldots, \beta$. By Definition 3, $a_i = db_i$, for $i = 1, \ldots, \beta$, where d is a g.c.d. of $a_1, \ldots, a_\beta$.

An $l \times (l+m)$ matrix B_i is now constructed as follows. In columns $i_1, \ldots, i_l$ of B_i, we place $\mathbf{g}_{i_1}, \ldots, \mathbf{g}_{i_l}$. The remaining columns of B_i are filled with zeros. Using the determinant formula [17], it can be easily verified that

$$B_i F = a_i I_l \tag{13}$$

and

$$F B_i = W_i' \tag{14}$$

where any entry of W_i' is either equal to 0 or equal to some elements of $\{\pm a_1, \ldots, \pm a_\beta\}$. Therefore, d is a divisor of any entry of W_i'. Let

$$H = \sum_{i=1}^{\beta} \lambda_i B_i \tag{15}$$

we have

$$\begin{aligned} HF &= \left(\sum_{i=1}^{\beta} \lambda_i B_i\right) F \\ &= \sum_{i=1}^{\beta} \lambda_i B_i F \\ &= \sum_{i=1}^{\beta} \lambda_i a_i I_l \\ &= \sum_{i=1}^{\beta} \lambda_i d b_i I_l \\ &= d \sum_{i=1}^{\beta} \lambda_i b_i I_l \\ &= d s I_l \end{aligned} \tag{16}$$

Step 2: Let $W = FH$. From (14) and (15), we have

$$W = F \left(\sum_{i=1}^{\beta} \lambda_i B_i\right)$$

$$= \sum_{i=1}^{\beta} \lambda_i F B_i$$

$$= \sum_{i=1}^{\beta} \lambda_i W_i' \tag{17}$$

Since d is a divisor of any entry of W_i', it is clear that d is also a divisor of any entry of W.

Let $H_1, \ldots, H_\beta$ denote the $l \times l$ submatrices of H, and let $\Delta_1, \ldots, \Delta_\beta$ deote the corresponding minors. Consider an arbitrary submatrix H_p $(1 \leq p \leq \beta)$, and let $W_p = FH_p$. Let $c_1, \ldots, c_\beta$ denote the $l \times l$ minors of W_p. Since W_p is a submatrix of W, it is clear that d is also a divisor of any entry of W_p. It follows that d^l divides the $l \times l$ minors of W_p. i.e.

$$c_i = d^l c_i', \qquad i = 1, \ldots, \beta, \tag{18}$$

where $c_i' \in \mathbf{R}[\mathbf{z}]$. On the other hand, since $W_p = FH_p$, it follows that

$$\begin{aligned} c_i &= \Delta_p a_i \\ &= \Delta_p d b_i \qquad i = 1, \ldots, \beta. \end{aligned} \tag{19}$$

Combining (18) and (19) yields

$$\Delta_p b_i = d^{l-1} c_i', \qquad i = 1, \ldots, \beta. \tag{20}$$

Since $b_1, \ldots, b_\beta$ are factor coprime, d^{l-1} is necessarily a divisor of Δ_p. Because of the arbitrary choice of p, it can be concluded that d^{l-1} is a common divisor of $\Delta_1, \ldots, \Delta_\beta$.

Step 3: Partition H as $H = [X_0 \ Y_0]$ where $X_0 \in \mathbf{R}^{l \times l}[\mathbf{z}]$, $Y_0 \in \mathbf{R}^{l \times m}[\mathbf{z}]$. Let $e_1, \ldots, e_\beta$ denote the generating polynomials of $H = [X_0 \ Y_0]$, i.e.

$$\Delta_i = \hat{d} e_i \qquad i = 1, \ldots, \beta, \tag{21}$$

where $\hat{d}$ is a g.c.d. of $\Delta_1, \ldots, \Delta_\beta$. Since we have shown in Step 2 that d^{l-1} is a common divisor of $\Delta_1, \ldots, \Delta_\beta$, it follows that d^{l-1} is necessarily a divisor of $\hat{d}$, i.e.

$$\hat{d} = d^{l-1} s_2 \tag{22}$$

for some $s_2 \in \mathbf{R}[\mathbf{z}]$. Hence,

$$\Delta_i = d^{l-1} s_2 e_i \qquad i = 1, \ldots, \beta. \tag{23}$$

Recalling (16) gives

$$HF = dsI_l, \tag{24}$$

or

$$X_0 D + Y_0 N = dsI_l. \tag{25}$$

It follows

$$\det(HF) = d^l s^l. \tag{26}$$

On the other hand, by the Cauchy-Binet formula, we have

$$\begin{aligned}\det(HF) &= \sum_{i=1}^{\beta} \Delta_i a_i \\ &= \sum_{i=1}^{\beta} d^{l-1} s_2 e_i d b_i \\ &= d^l s_2 \sum_{i=1}^{\beta} e_i b_i\end{aligned} \tag{27}$$

Combining (26) and (27) gives

$$s_2 \sum_{i=1}^{\beta} e_i b_i = s^l \tag{28}$$

Therefore,

$$\sum_{i=1}^{\beta} e_i b_i = s_1 \tag{29}$$

for some $s_1 \in \mathbf{R}[\mathbf{z}]$ such that $s^l = s_1 s_2$. Since $s \neq 0$ in $\overline{U}^n$, it follows that $s_1 \neq 0$ in $\overline{U}^n$. We have thus shown that a sufficient condition for $e_1, \ldots, e_\beta$ to satisfy (9) is that d^{l-1} is a common divisor of $\Delta_1, \ldots, \Delta_\beta$. On the other hand, if d^{l-1} is not a common divisor of $\Delta_1, \ldots, \Delta_\beta$, then $\sum_{i=1}^{\beta} e_i b_i$ will contain d^{l_1} for some positive integer l_1. Consequently, $e_1, \ldots, e_\beta$ cannot satisfy (9) when d is irreducible and has a zero in $\overline{U}^n$.

Now If $\det X_0(\mathbf{0}) \neq 0$ and $Y_0(\mathbf{0}) = 0_{l,m}$, it is obvious $\det X_0(\mathbf{z}) \not\equiv 0$. Let $X = X_0$, $Y = Y_0$ and $C = X^{-1}Y$, and the proof is completed. Otherwise, proceed to Step 4.

Step 4: Decompose P into a left MFD

$$P = \tilde{D}^{-1}\tilde{N} \tag{30}$$

where $\tilde{D} \in \mathbf{R}^{m\times m}[\mathbf{z}]$, $\tilde{N} \in \mathbf{R}^{m\times l}[\mathbf{z}]$, with $\det \tilde{D}(\mathbf{0}) \neq 0$.

Note that

$$\begin{aligned} X &= X_0 - S\tilde{N} \\ Y &= Y_0 + S\tilde{D}\end{aligned} \tag{31}$$

is also a solution to (25), i.e., $XD + YN = dsI_l$. Let

$$S = -\frac{d^{l-1}(\mathbf{z})}{d^{l-1}(\mathbf{0})} Y_0(\mathbf{0}) \tilde{D}^{-1}(\mathbf{0}) \tag{32}$$

Now $Y(\mathbf{0}) = 0_{l,m}$, $\det X(\mathbf{0}) = \det\{d(\mathbf{0})s(\mathbf{0})D^{-1}(\mathbf{0})\} \neq 0$.

Let $\hat{\Delta}_1, \ldots, \hat{\Delta}_\beta$ deote the $l \times l$ minors of $\hat{H} = [X \;\; Y]$. We next show that d^{l-1} is a common divisor of $\hat{\Delta}_1, \ldots, \hat{\Delta}_\beta$. Rewrite $\hat{H}$ as a summation of two matrices:

$$\hat{H} = H + H' \tag{33}$$

where $H = [X_0 \; Y_0]$ and $H' = [-S\tilde{N} \;\; S\tilde{D}]$. From the theory of determinant, it is easy to see that for an arbitrary i, $\hat{\Delta}_i$ is a summation of a finite number of determinants of some $l \times l$ matrices which consist of either all columns from H or at least one column from H'. From Step 2, we know d^{l-1} is a divisor of the determinant of an $l \times l$ matrix which consists of all columns from H. On the other hand, from the way S is constructed, it is clear that d^{l-1} is a divisor of any entry of H'. It follows that d^{l-1} is a divisor of the determinant of an $l \times l$ matrix which consists of at least one column from H'. Therefore, d^{l-1} is a divisor of $\hat{\Delta}_i$. Because of the arbitrary choice of i, we conclude that d^{l-1} is a common divisor of $\hat{\Delta}_1, \ldots, \hat{\Delta}_\beta$. Let $\hat{e}_1, \ldots, \hat{e}_\beta$ denote the generating polynomials of $[X \;\; Y]$. Proceeding as in Step 3, it can be shown that

$$\sum_{i=1}^{\beta} \hat{e}_i b_i = \hat{s}_1 \tag{34}$$

for some $\hat{s}_1 \in \mathbf{R}[\mathbf{z}]$ with $\hat{s}_1 \neq 0$ in $\overline{U}^n$. Let $C = X^{-1}Y$, the proof is thus completed. ■

Remark 2. The technique for constructing an n-D polynomial matrix $H(\mathbf{z})$ such that $H(\mathbf{z})F(\mathbf{z}) = s(\mathbf{z})I_l$ has been adopted in [5,11,18,19]. However, the case where the $l \times l$ minors of $F(\mathbf{z})$ have a non-trivial g.c.d. $d(\mathbf{z})$ that may have a zero in $\overline{U}^n$ has not yet been discussed before. The main contribution of Theorem 1 is to show that $d^{l-1}(\mathbf{z})$ is a common divisor of the $l \times l$ minors of $H(\mathbf{z})$ constructed in (15). As shown in Step 3 of the proof of Theorem 1, this property is a necessary and sufficient condition for $e_1, \ldots, e_\beta$ to satisfy (9) when $d(\mathbf{z})$ is irreducible and has a zero in $\overline{U}^n$. The reason for $S(\mathbf{z})$ to contain $d^{l-1}(\mathbf{z})$ in (32) rather than just to be a constant matrix as for the 2-D case [6] is to preserve this property.

In the next theorem, a necessary and sufficient condition for the feedback stabilizability of an MIMO n-D system P is derived in terms of the generating polynomials associated with an MFD of P.

THEOREM 2 *Let $P = ND^{-1} \in \mathbf{R}^{m\times l}(\mathbf{z})$ represent an n-D system, and let $b_{11}, \ldots, b_{1\beta}$ denote the generating polynomials of $[D^T \;\; N^T]^T$, with $\beta = \binom{m+l}{l}$. The following statements* *are equivalent:*

1) P *is output feedback stabilizable.*

2) $b_{11}, \ldots, b_{1\beta}$ *have no common zeros in* $\overline{U}^n$.

3) There exists an n-D compensator $C = X^{-1}Y \in \mathbf{R}^{l\times m}(\mathbf{z})$ *such that the generating polynomials of* $[X\ Y]$*, denoted by* $b_{21}, \ldots, b_{2\beta}$*, satisfy:*

$$\sum_{i=1}^{\beta} b_{1i} b_{2i} = s_1 \tag{35}$$

for some $s_1 \in \mathbf{R}[\mathbf{z}]$ *with* $s_1 \neq 0$ *in* $\overline{U}^n$.

Moreover, if a stabilizable P is causal (not necessarily strictly causal), a strictly causal stabilizing compensator C can be constructed.

Proof: The implication 3) $\Longrightarrow$ 2) is obvious, and the implication 2) $\Longrightarrow$ 3) is the statement of Theorem 1. Therefore, it suffices to show the equivalence of statements 1) and 3).

Decompose an n-D compensator $C \in \mathbf{R}^{l\times m}(\mathbf{z})$ into a right MFD,

$$C = \tilde{Y}\tilde{X}^{-1}. \tag{36}$$

Let

$$\begin{aligned} F_1 &= \begin{bmatrix} D \\ N \end{bmatrix} \in \mathbf{R}^{(m+l)\times l}[\mathbf{z}], \\ \tilde{F}_2 &= \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix} \in \mathbf{R}^{(m+l)\times m}[\mathbf{z}], \\ F_2 &= [X\ Y] \in \mathbf{R}^{l\times(m+l)}[\mathbf{z}], \end{aligned}$$

and let

$a_{11}, \ldots, a_{1\beta}$ denote the $l \times l$ minors of F_1;
$\tilde{a}_{21}, \ldots, \tilde{a}_{2\beta}$ denote the $m \times m$ minors of $\tilde{F}_2$;
$a_{21}, \ldots, a_{2\beta}$ denote the $l \times l$ minors of F_2.

By Definition 3, we have

$$a_{pi} = d_p b_{pi} \qquad i = 1, \ldots, \beta;\ p = 1, 2 \tag{37}$$

where d_p is a g.c.d. of $a_{p1}, \ldots, a_{p\beta}$ $(p = 1, 2)$. By Definition 3 and Proposition 2, we have

$$\tilde{a}_{2i} = \pm\tilde{d}_2 b'_{2i} \qquad i = 1, \ldots, \beta, \tag{38}$$

where $\tilde{d}_2$ is a g.c.d. of $\tilde{a}_{21}, \ldots, \tilde{a}_{2\beta}$, and $b'_{21}, \ldots, b'_{2\beta}$ are obtained by re-ordering $b_{21}, \ldots, b_{2\beta}$

appropriately. In particular, we have

$$\tilde{a}_{21} = \tilde{d}_2 b_{21} \tag{39}$$

Next,

$$\begin{aligned} H_{eu} &= \begin{bmatrix} (I_m + PC)^{-1} & -P(I_l + CP)^{-1} \\ C(I_m + PC)^{-1} & (I_l + CP)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I_m & P \\ -C & I_l \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I_m & ND^{-1} \\ -\tilde{Y}\tilde{X}^{-1} & I_l \end{bmatrix}^{-1} \\ &= \left\{ \begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & D \end{bmatrix} \begin{bmatrix} \tilde{X}^{-1} & 0_{m,l} \\ 0_{l,m} & D^{-1} \end{bmatrix} \right\}^{-1} \\ &= \underbrace{\begin{bmatrix} \tilde{X} & 0_{m,l} \\ 0_{l,m} & D \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & D \end{bmatrix}}_{A}{}^{-1} \end{aligned} \tag{40}$$

where $A,\ B \in \mathbf{R}^{(m+l)\times(m+l)}[\mathbf{z}]$. Let

$$F_3 = \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & D \\ \tilde{X} & 0_{m,l} \\ 0_{l,m} & D \end{bmatrix} \tag{41}$$

and let $a_{31}, \ldots, a_{3\mu}$ denote the $(m+l)\times(m+l)$ minors of F_3, where $\mu = \binom{2(m+l)}{(m+l)}$. Suppose that $b_{31}, \ldots, b_{3\mu}$ are the generating polynomials of F_3, i.e.,

$$a_{3i} = d_3 b_{3i} \qquad i = 1, \ldots, \mu, \tag{42}$$

where d_3 is a g.c.d. of $a_{31}, \ldots, a_{3\mu}$.

Direct calculation gives

$$\begin{aligned} \det A &= \det \begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & D \end{bmatrix} \\ &= \det \tilde{X} (\det X)^{-1} \det(XD + YN) \end{aligned} \tag{43}$$

By the Cauchy-Binet formula,

$$\begin{aligned}\det(XD+YN) &= \det\left\{[X\ Y]\begin{bmatrix} D \\ N \end{bmatrix}\right\} \\ &= \sum_{i=1}^{\beta} a_{2i}a_{1i} \\ &= d_1 d_2 \sum_{i=1}^{\beta} b_{1i}b_{2i} \end{aligned} \tag{44}$$

Thus,

$$\begin{aligned}\det A &= \det\tilde{X}(\det X)^{-1} d_1 d_2 \sum_{i=1}^{\beta} b_{1i}b_{2i} \\ &= \tilde{a}_{21}\frac{1}{a_{21}} d_1 d_2 \sum_{i=1}^{\beta} b_{1i}b_{2i} \\ &= \tilde{d}_2 b_{21}\frac{1}{d_2 b_{21}} d_1 d_2 \sum_{i=1}^{\beta} b_{1i}b_{2i} \\ &= \tilde{d}_2 d_1 \sum_{i=1}^{\beta} b_{1i}b_{2i} \end{aligned} \tag{45}$$

Let

$$U = \begin{bmatrix} 0_{l,m} & 0_l & 0_{l,m} & I_l \\ I_m & 0_{m,l} & -I_m & 0_{m,l} \\ 0_m & 0_{m,l} & I_m & 0_{m,l} \\ 0_{l,m} & -I_l & 0_{l,m} & I_l \end{bmatrix} \tag{46}$$

Then,

$$UF_3 = \underbrace{\begin{bmatrix} 0_{l,m} & D \\ 0_m & N \\ \tilde{X} & 0_{m,l} \\ \tilde{Y} & 0_l \end{bmatrix}}_{F_4} \tag{47}$$

Let $a_{41}, \ldots, a_{4\mu}$ denote the $(m+l)\times(m+l)$ minors of F_4. Due to the special structure of F_4, by appropriately re-ordering $a_{41}, \ldots, a_{4\mu}$ as $a'_{41}, \ldots, a'_{4\mu}$, we can obtain

$$\begin{array}{llll} a'_{41} = a_{11}\tilde{a}_{21}, & a'_{4,\beta+1} = a_{12}\tilde{a}_{21}, & \cdots, & a'_{4,\beta(\beta-1)+1} = a_{1\beta}\tilde{a}_{21}; \\ a'_{42} = a_{11}\tilde{a}_{22}, & a'_{4,\beta+2} = a_{12}\tilde{a}_{22}, & \cdots, & a'_{4,\beta(\beta-1)+2} = a_{1\beta}\tilde{a}_{22}; \\ \vdots & \vdots & \ddots & \vdots \\ a'_{4\beta} = a_{11}\tilde{a}_{2\beta}, & a'_{4,2\beta} = a_{12}\tilde{a}_{2\beta}, & \cdots, & a'_{4,\beta^2} = a_{1\beta}\tilde{a}_{2\beta}, \end{array} \tag{48}$$

and

$$a'_{4,\beta^2+1} = \cdots = a'_{4\mu} = 0.$$

It is convenient to express (48) in a more compact form:

$$a'_{4,\beta(i-1)+j} = a_{1i}\tilde{a}_{2j} \qquad i = 1, \ldots, \beta;\ j = 1, \ldots, \beta.$$

Recalling (37) and (38) gives

$$\begin{array}{l} a'_{4,\beta(i-1)+j} = \pm(d_1\tilde{d}_2)b_{1i}b'_{2j} \qquad i = 1, \ldots, \beta;\ j = 1, \ldots, \beta, \\ a'_{4,\beta^2+1} = \cdots = a'_{4\mu} = 0. \end{array} \tag{49}$$

Since $b_{11}, \ldots, b_{1\beta}$ are factor coprime, and $b'_{21}, \ldots, b'_{2\beta}$ are factor coprime, it is clear from (49) that $d_1\tilde{d}_2$ is a g.c.d. of $a'_{41}, \ldots, a'_{4\mu}$ and hence is a g.c.d. of $a_{41}, \ldots, a_{4\mu}$. Since $F_4 = UF_3$ and U is a unimodular matrix, it follows from Lemma 1 that

$$d_3 = r_1 d_1 \tilde{d}_2 \tag{50}$$

for some $r_1 \in \mathbf{R}^*$. By Definition 3, we know that

$$\det A = a_{31} = d_3 b_{31} \tag{51}$$

From (45), (50) and (51), it follows easily that

$$b_{31} = r_1^{-1} \sum_{i=1}^{\beta} b_{1i} b_{2i} \tag{52}$$

Therefore, by Definition 1 and Proposition 3, the n-D system P is feedback stabilizable if and only if

$$b_{31} = r_1^{-1} s_1, \tag{53}$$

or

$$\sum_{i=1}^{\beta} b_{1i} b_{2i} = s_1 \tag{54}$$

for some $s_1 \in \mathbf{R}[\mathbf{z}]$ with $s_1 \neq 0$ in $\overline{U}^n$. Thus the equivalence of statements 1) and 3) has been shown.

Finally, when a stabilizable plant P is causal (not necessarily strictly causal), from the above proof procedure and by using Theorem 1, we can find a stabilizing compensator C which is strictly causal. The proof is thus completed. ■

We are now in a position to compare the method presented in this paper for testing the output feedback stabilizability and obtaining a stabilizing compensator for a given causal MIMO n-D linear plant $P \in \mathbf{R}^{m\times l}(\mathbf{z})$ with the method by Sule in [8].

The method presented in this paper may be summarized in the following steps:

1) Decompose P into a right and a left MFD, $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$.

2) Obtain the $l \times l$ minors of $F = [D^T \ N^T]^T$, denoted by $a_1, \ldots, a_\beta$, and the generating polynomials $b_1, \ldots, b_\beta$, where $a_i = db_i$ $(i = 1, \ldots, \beta)$, and d is a g.c.d. of $a_1, \ldots, a_\beta$.

3) If $b_1, \ldots, b_\beta$ have a common zero in $\overline{U}^n$, P cannot be output feedback stabilized, stop here. Otherwise, proceed to the next step.

4) Find polynomials $\lambda_1, \ldots, \lambda_\beta$ such that $\sum_{i=1}^{\beta} \lambda_i b_i = s$ with $s \neq 0$ in $\overline{U}^n$.

5) For $i = 1, \ldots, \beta$, construct polynomial matrix B_i from F such that $B_i F = a_i I_l$, and then construct $H = \sum_{i=1}^{\beta} \lambda_i B_i$ such that $HF = dsI_l$.

6) Partition $H = [X_0 \ Y_0]$. Let $X = X_0 - S\tilde{N}$, $Y = Y_0 + S\tilde{D}$, where $S = -\frac{d^{l-1}(\mathbf{z})}{d^{l-1}(\mathbf{0})} Y_0(\mathbf{0})\tilde{D}^{-1}(\mathbf{0})$. Then $C = X^{-1}Y$ is a strictly causal stabilizing compensator for P.

The method by Sule in [8] may be summarized in the following steps:

1) Decompose P into a right and a left MFD, $P = Nd^{-1} = d^{-1}N$. Let $T = [(dI)^T \ N^T]^T$, and $W = [N \ (dI)]$.

2) Obtain the $l \times l$ minors of T, denoted by $a_1, \ldots, a_\beta$, and the reduced minors (or the generating polynomials) $b_1, \ldots, b_\beta$, where $a_i = db_i$ $(i = 1, \ldots, \beta)$, and d is a g.c.d. of $a_1, \ldots, a_\beta$.

3) If $b_1, \ldots, b_\beta$ have a common zero in $\overline{U}^n$, P cannot be output feedback stabilized, stop here. Otherwise, proceed to the next step.

4) Obtain the family of elementary factors of T, denoted by $\{f_1, \ldots, f_r\}$, $r \leq \beta$, the family of elementary factor of W, denoted by $\{g_1, \ldots, g_l\}$, $l \leq \beta$, and the family of elementary factors of P, denoted by $H = \{h_1, \ldots, h_k\} = \{f_i g_j, i = 1, \ldots, r; \ j = 1, \ldots, l\}$.

5) For each h_i in H $(i = 1, \ldots, k)$, obtain rational matrices X_i, Y_i, U_i, V_i such that $X_i N = U_i d$, $Y_i N = V_i d$ and $NY_i = (I - X_i)d$. The denominators of entries of X_i, Y_i, U_i, V_i are integer power of h_i.

6) Find a sufficiently large integer n_i such that $h_i^{n_i} X_i$, $h_i^{n_i} Y_i$, $h_i^{n_i} U_i$ and $h_i^{n_i} V_i$ are polynomial matrices.

7) Find rational function α_i whose denominator is not equal to zero in $\overline{U}^n$ such that $\sum_{i=1}^{k} \alpha_i h_i^{n_i} = 1$, or equivalently, find polynomials $\lambda_1, \ldots, \lambda_\beta$ such that $\sum_{i=1}^{k} \lambda_i h_i^{n_i} = s$ with $s \neq 0$ in $\overline{U}^n$.

8) Let $X = \sum_{i=1}^{k} \alpha_i h_i^{n_i} X_i$, $Y = \sum_{i=1}^{k} \alpha_i h_i^{n_i} Y_i$. Then $C = Y X^{-1}$ is a stabilizing compensator for P.

From the above summary of two different methods, it is clear that the procedure for testing the feedback stabilizability is the same for both methods, while for obtaining a stabilizing compensator, our method is computationally simpler than the one given by Sule in [8]. As can be seen, the most difficult part for obtaining a stabilizing compensator in our method is in step 4) for finding polynomials $\lambda_1, \ldots, \lambda_\beta$ such that $\sum_{i=1}^{\beta} \lambda_i b_i = s$ with $s \neq 0$ in $\overline{U}^n$. Sule's method also requires in step 7) to obtain polynomials $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i h_i^{n_i} = s$ with $s \neq 0$ in $\overline{U}^n$. The other computationally more involved steps in Sule's method are to obtain the family of elementary factors (which are more difficult to obtain than the generating polynomials, as pointed out by Sule himself in [8]) and the construction of rational matrices X_i, Y_i, U_i, V_i such that $X_i N = U_i d$, $Y_i N = V_i d$ and $N Y_i = (I - X_i) d$, for $i = 1, \ldots, k$. Since a sufficiently larger integer n_i is required to convert X_i, Y_i, U_i, V_i into polynomial matrices using Sule's method, the resultant stabilizing compensator given in step 8) is in general more complicated than the one using our method. Furthermore, for a causal but not strictly causal system, we are able to construct a strictly causal compensator, while Sule's method cannot guarantee to give a causal compensator (p. 1694 in [8]). An illustrative example will be given in the next section.

4. Example

Consider an unstable 3-D system represented by:

$$P(z_1, z_2, z_3) = \frac{1}{\Delta_p} \begin{bmatrix} 2(z_1 + z_2) & (2z_1 + 3)(2z_2 + 2z_3 + 3) \\ (2z_2 - 1)(z_3 + 2) & 2(2z_1 + 2z_2 z_3 + 4z_2 + 2z_3^2 + 7z_3 + 7) \end{bmatrix}, \quad (55)$$

where $\Delta_p = (2z_1 + 1)(z_2 + 2)(z_3 - 2)$.

Decompose $P(z_1, z_2, z_3)$ into a left and a right MFD:

$$\begin{aligned} P(z_1, z_2, z_3) &= \tilde{D}^{-1}(z_1, z_2, z_3) \tilde{N}(z_1, z_2, z_3) \\ &= N(z_1, z_2, z_3) D^{-1}(z_1, z_2, z_3) \end{aligned} \quad (56)$$

where

$$\tilde{D} = D = \begin{bmatrix} (2z_1 + 1)(z_2 + 2)(z_3 - 2) & 0 \\ 0 & (2z_1 + 1)(z_2 + 2)(z_3 - 2) \end{bmatrix},$$

and

$$\tilde{N} = N = \begin{bmatrix} 2(z_1 + z_2) & (2z_1 + 3)(2z_2 + 2z_3 + 3) \\ (2z_2 - 1)(z_3 + 2) & 2(2z_1 + 2z_2 z_3 + 4z_2 + 2z_3^2 + 7z_3 + 7) \end{bmatrix}.$$

Let

$$F = \begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} (2z_1+1)(z_2+2)(z_3-2) & 0 \\ 0 & (2z_1+1)(z_2+2)(z_3-2) \\ 2(z_1+z_2) & (2z_1+3)(2z_2+2z_3+3) \\ (2z_2-1)(z_3+2) & 2(2z_1+2z_2z_3+4z_2+2z_3^2+7z_3+7) \end{bmatrix}$$

The 2×2 minors of F are:

$$a_1 = d\, b_1, \quad a_2 = d\, b_2,$$
$$a_3 = d\, b_3, \quad a_4 = d\, b_4,$$
$$a_5 = d\, b_5, \quad a_6 = d\, b_6.$$

where $d = (2z_1 + 1)$, and $b_1, \ldots, b_6$ are the generating polynomials of F:

$$b_1 = (2z_1+1)(z_2+2)^2(z_3-2)^2,$$
$$b_2 = (z_2+2)(z_3-2)(2z_1+3)(2z_2+2z_3+3),$$
$$b_3 = 2(z_2+2)(z_3-2)(2z_1+2z_2z_3+4z_2+2z_3^2+7z_3+7),$$
$$b_4 = -2(z_1+z_2)(z_2+2)(z_3-2),$$
$$b_5 = -(2z_2-1)(z_2+2)(z_3^2-4),$$
$$b_6 = 4\,z_1 + 18 - 8\,z_2\,z_3 - 4\,{z_2}^2 z_3 + 4\,z_2 + 21\,z_3 + 6\,{z_3}^2 - 8\,{z_2}^2 - 4\,z_2\,{z_3}^2.$$

It can be checked using a criterion developed in [14] that $F(z_1, z_2, z_3)$ does not admit a primitive factorization, and thus $D(z_1, z_2, z_3)$ and $N(z_1, z_2, z_3)$ are FRC. Since b_1 has zeros in $\overline{U}^3$, by Proposition 3, the plant P is unstable. This agrees with the fact that all entry of P have poles in $\overline{U}^3$. It is easy to test that $b_1, \ldots, b_6$ have no common zeros in $\overline{U}^3$. By Proposition 4, it is possible to find $\lambda_1, \ldots, \lambda_6 \in \mathbf{R}[z_1, z_2, z_3]$ such that

$$\sum_{i=1}^{6} \lambda_i(z_1, z_2, z_3)\, b_i(z_1, z_2, z_3) = s(z_1, z_2, z_3)$$

for some $s \in \mathbf{R}[z_1, z_2, z_3]$, with $s(z_1, z_2, z_3) \neq 0$ in $\overline{U}^3$. In fact, if we choose $\lambda_1 = \lambda_3 = \lambda_4 = \lambda_6 = 0$, $\lambda_2 = (z_3 + 2)$, and $\lambda_5 = (2z_1 + 3)$, then

$$\begin{aligned} \sum_{i=1}^{6} \lambda_i b_i &= \lambda_2 b_2 + \lambda_5 b_5 \\ &= (z_2+2)\{(2z_1+3)(z_2+2)(z_3-2)(2z_2+2z_3+3)\} + \\ &\quad (2z_1+3)\{-(2z_2-1)(z_2+2)(z_3^2-4)\} \\ &= 2(2z_1+3)(z_2+2)(z_3-2)(z_3+2)^2 \\ &\neq 0 \text{ in } \overline{U}^3 \end{aligned}$$

Let F_2' and F_5' denote the 2×2 submatrices of F, corresponding to b_2 and b_5 respectively, i.e.

$$F_2' = \begin{bmatrix} (2z_1+1)(z_2+2)(z_3-2) & 0 \\ 2(z_1+z_2) & (2z_1+3)(2z_2+2z_3+3) \end{bmatrix}$$

and

$$F_5' = \begin{bmatrix} 0 & (2z_1+1)(z_2+2)(z_3-2) \\ (2z_2-1)(z_3+2) & 2(2z_1+2z_2z_3+4z_2+2z_3^2+7z_3+7) \end{bmatrix}.$$

Then

$$\text{adj } F_2' = \begin{bmatrix} (2z_1+3)(2z_2+2z_3+3) & 0 \\ -2(z_1+z_2) & (2z_1+1)(z_2+2)(z_3-2) \end{bmatrix}$$

and

$$\text{adj } F_5' = \begin{bmatrix} 2(2z_1+2z_2z_3+4z_2+2z_3^2+7z_3+7) & -(2z_1+1)(z_2+2)(z_3-2) \\ -(2z_2-1)(z_3+2) & 0 \end{bmatrix}.$$

Let $\mathbf{g}_{2_1}$, $\mathbf{g}_{2_2}$ denote columns 1 and 2 of adj F_2', and let $\mathbf{g}_{5_1}$, $\mathbf{g}_{5_2}$ denote columns 1 and 2 of adj F_5'. Let:

$$B_2 = [\mathbf{g}_{2_1},\ 0_{2,1},\ \mathbf{g}_{2_2},\ 0_{2,1}], \quad B_5 = [0_{2,1},\ \mathbf{g}_{5_1},\ 0_{2,1},\ \mathbf{g}_{5_2}].$$

Then

$$H = \lambda_2\, B_2 + \lambda_5\, B_5 = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \end{bmatrix}$$

where

$$\begin{aligned}
h_{11} &= (z_3+2)(2z_1+3)(2z_2+2z_3+3), \\
h_{12} &= 2(2z_1+3)(2z_1+2z_2z_3+4z_2+2z_3^2+7z_3+7), \\
h_{13} &= 0, \\
h_{14} &= -(2z_1+3)(2z_1+1)(z_2+2)(z_3-2), \\
h_{21} &= -2(z_3+2)(z_1+z_2), \\
h_{22} &= -(2z_1+3)(2z_2-1)(z_3+2), \\
h_{23} &= (z_3+2)(2z_1+1)(z_2+2)(z_3-2), \\
h_{24} &= 0.
\end{aligned}$$

Direct calculation yields

$$\begin{aligned} HF &= 2(2z_1+3)(2z_1+1)(z_2+2)(z_3-2)(z_3+2)^2\, I_2 \\ &= d(z_1,z_2,z_3)\, s(z_1,z_2,z_3)\, I_2. \end{aligned}$$

The 2×2 minors of H are:

$$\Delta_1 = \hat{d}\, e_1, \quad \Delta_2 = \hat{d}\, e_2,$$

$$\Delta_3 = \hat{d}\, e_3, \quad \Delta_4 = \hat{d}\, e_4,$$

$$\Delta_5 = \hat{d}\, e_5, \quad \Delta_6 = \hat{d}\, e_6.$$

where $\hat{d} = (2z_1+1)(2z_1+3)(z_3+2)$, and $e_1, \ldots, e_6$ are the generating polynomials of H:

$$e_1 = 4z_1 + 18 - 8z_2z_3 - 4{z_2}^2z_3 + 4z_2 + 21z_3 + 6{z_3}^2 - 8{z_2}^2 - 4z_2{z_3}^2,$$

$$e_2 = (z_2+2)(z_3^2-4)(2z_2+2z_3+3),$$

$$e_3 = -2(z_1+z_2)(z_2+2)(z_3-2),$$

$$e_4 = 2(z_2+2)(z_3-2)(2z_1+2z_2z_3+4z_2+2z_3^2+7z_3+7),$$

$$e_5 = -(2z_1+3)(2z_2-1)(z_2+2)(z_3-2),$$

$$e_6 = (2z_1+1)(z_2+2)^2(z_3-2)^2.$$

Direct calculation gives:

$$\sum_{i=1}^{6} e_i b_i = 4(2z_1+3)(z_2+2)^2(z_3-2)^2(z_3+2)^3 \neq 0 \text{ in } \overline{U}^3.$$

Therefore, by Theorem 2, $P(z_1,z_2,z_3)$ is output feedback stabilizable. To obtain a stabilizing compensator, the matrix $H(z_1,z_2,z_3)$ is partitioned as

$$H(z_1,z_2,z_3) = [X_0(z_1,z_2,z_3) \quad Y_0(z_1,z_2,z_3)]$$

where $X_0 = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$, and $Y_0 = \begin{bmatrix} h_{13} & h_{14} \\ h_{23} & h_{24} \end{bmatrix}$. Let

$$\begin{aligned} C(z_1,z_2,z_3) &= X_0^{-1}(z_1,z_2,z_3)Y_0(z_1,z_2,z_3) \\ &= \frac{1}{\Delta_c}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \end{aligned} \tag{57}$$

where

$$\begin{aligned} \Delta_c &= 4z_1 + 18 - 8z_2z_3 - 4{z_2}^2z_3 + 4z_2 + 21z_3 + 6{z_3}^2 - 8{z_2}^2 - 4z_2{z_3}^2, \\ c_{11} &= -2(z_2+2)(z_3-2)(2z_1+2z_2z_3+4z_2+2z_3^2+7z_3+7), \\ c_{12} &= (2z_1+3)(2z_2-1)(z_2+2)(z_3-2), \\ c_{21} &= (2z_2+2z_3+3)(z_2+2)(z_3+2)(z_3-2), \\ c_{22} &= -2(z_1+z_2)(z_2+2)(z_3-2). \end{aligned}$$

We have thus obtained a stabilizing compensator $C(z_1, z_2, z_3)$ for $P(z_1, z_2, z_3)$. To verify that the feedback system is indeed stable, we directly obtain $H_{eu}(z_1, z_2, z_3)$ as follows:

$$
\begin{aligned}
H_{eu} &= \begin{bmatrix} I_m & P \\ -C & I_l \end{bmatrix}^{-1} \\
&= \frac{1}{\Delta_h} \begin{bmatrix} h'_{11} & h'_{12} & h'_{13} & h'_{14} \\ h'_{21} & h'_{22} & h'_{23} & h'_{24} \\ h'_{31} & h'_{32} & h'_{33} & h'_{34} \\ h'_{41} & h'_{42} & h'_{43} & h'_{44} \end{bmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_h &= 2\ (2z_1+3)(z_3+2)^2(z_3-2)(z_2+2)\,, \\
h'_{11} &= -(2z_1+3)(z_2+2)(z_3-2)(2z_2-1)(z_3+2)\,, \\
h'_{12} &= 2\ (z_1+z_2)(2z_1+3)(z_2+2)(z_3-2)\,, \\
h'_{13} &= 0, \\
h'_{14} &= -\left(4z_1+18-8z_2z_3-4z_2{}^2z_3+4z_2+21z_3+6z_3{}^2-8z_2{}^2-4z_2z_3{}^2\right) \\
&\quad \times\ (2z_1+3)\,, \\
h'_{21} &= -2\ \left(2z_1+2z_2z_3+4z_2+2z_3{}^2+7z_3+7\right)(z_2+2)(z_3-2)(z_3+2)\,, \\
h'_{22} &= (z_3+2)(z_2+2)(z_3-2)(2z_1+3)(2z_2+2z_3+3)\,, \\
h'_{23} &= \left(4z_1+18-8z_2z_3-4z_2{}^2z_3+4z_2+21z_3+6z_3{}^2-8z_2{}^2-4z_2z_3{}^2\right) \\
&\quad \times\ (z_3+2)\,, \\
h'_{24} &= 0, \\
h'_{31} &= 0, \\
h'_{32} &= -(2z_1+1)(z_2+2)^2(z_3-2)^2(2z_1+3)\,, \\
h'_{33} &= (z_3+2)(z_2+2)(z_3-2)(2z_1+3)(2z_2+2z_3+3)\,, \\
h'_{34} &= 2\ \left(2z_1+2z_2z_3+4z_2+2z_3{}^2+7z_3+7\right)(2z_1+3)(z_2+2)(z_3-2)\,, \\
h'_{41} &= (2z_1+1)(z_2+2)^2(z_3-2)^2(z_3+2)\,, \\
h'_{42} &= 0, \\
h'_{43} &= -2\ (z_1+z_2)(z_2+2)(z_3-2)(z_3+2)\,, \\
h'_{44} &= -(2z_1+3)(z_2+2)(z_3-2)(2z_2-1)(z_3+2)\,.
\end{aligned}
$$

Clearly, Δ_h has no zeros in $\overline{U}^3$ and hence by Definition 1 the feedback system is stable. Since $\det X_0(0,0,0) = 108$, the constructed stabilizing compensator $C(z_1, z_2, z_3)$ is causal.

However, since $Y_0(0,0,0) = \begin{bmatrix} 0 & 12 \\ -8 & 0 \end{bmatrix}$, $C(z_1,z_2,z_3)$ is not strictly causal. To obtain a strictly causal compensator, let

$$\begin{aligned} S(z_1,z_2,z_3) &= -\frac{d(z_1,z_2,z_3)}{d(0,0,0)} Y_0(0,0,0)\tilde{D}^{-1}(0,0,0) \\ &= -(2z_1+1)\begin{bmatrix} 0 & 12 \\ -8 & 0 \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & 3(2z_1+1) \\ -2(2z_1+1) & 0 \end{bmatrix}. \end{aligned} \tag{58}$$

We then have

$$\begin{aligned} X(z_1,z_2,z_3) &= X_0(z_1,z_2,z_3) - S(z_1,z_2,z_3)\tilde{N}(z_1,z_2,z_3) \\ &= \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \end{aligned} \tag{59}$$

where

$$\begin{aligned} x_{11} &= -2\,(z_3+2)(-2z_1z_3 - 3z_3 + 4z_1z_2 - 6 - 6z_1), \\ x_{12} &= -8z_1\left(2z_1 + 2z_2z_3 + 4z_2 + 2z_3{}^2 + 7z_3 + 7\right), \\ x_{21} &= 2\,(-z_3 + 4z_1)(z_1 + z_2), \\ x_{22} &= (2z_1+3)(8z_1z_2 + 8z_1z_3 + 12z_1 + 8 - 2z_2z_3 + 5z_3). \end{aligned}$$

and

$$\begin{aligned} Y(z_1,z_2,z_3) &= Y_0(z_1,z_2,z_3) + S(z_1,z_2,z_3)\tilde{D}(z_1,z_2,z_3) \\ &= \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \end{aligned} \tag{60}$$

where

$$\begin{aligned} y_{11} &= 0, \\ y_{12} &= 4z_1\,(z_3-2)(z_2+2)(2z_1+1), \\ y_{21} &= -(z_3-2)(2z_1+1)(-z_3+4z_1)(z_2+2), \\ y_{22} &= 0. \end{aligned}$$

Clearly, we now have $\det X(0,0,0) = 576$, and $Y(0,0,0) = 0_2$. Moreover, it can be easily checked using a symbolic computation software such as Maple that the unstable factor

$(2z_1 + 1)$ is a common divisor of the 2×2 minors of the matrix $[X\ Y]$. The compensator $C = X^{-1}Y$ is thus the required strictly causal stabilizing compensator.

As pointed out in Remark 2, the inclusion of the factor $(2z_1 + 1)$ in $S(z_1, z_2, z_3)$ in (58) is very important. For example, if we choose $S = \begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}$ in (58) and calculate $X(z_1, z_2, z_3)$ and $Y(z_1, z_2, z_3)$ according to (59) and (60), as we did in [6] for the 2-D case, then, it can be verified that the unstable factor $(2z_1 + 1)$ is now *not* a common divisor of the 2×2 minors of the matrix $[X\ Y]$. Consequently, the resultant compensator $C = X^{-1}Y$, although still strictly causal, is no longer a stabilizing compensator since it can be easily checked using Maple that the feedback system with such a compensator is unstable!

Finally, for comparison, we have also worked out this example using Sule's method suggested in [8]. After a laborious computation, we have also obtained a stabilizing n-D compensator. The resultant compensator is, however, much more complicated than the one derived using our method. The details of a stabilizing n-D compensator using Sule's method [8] are omitted here to save space. The reader is encouraged to work out the given example using two different methods.

5. Conclusion

In this paper, we have investigated the output feedback stabilizability of MIMO n-D linear systems. Using the concept of "generating polynomials" introduced by the author in [13], we are able to derive a necessary and sufficient condition for the output feedback stabilizability of MIMO n-D linear systems. This condition turns out to be the same as the one by Sule [8], who investigated the problem of feedback stabilization of linear systems over commutative rings using the theory of commutative algebra and topology. By restricting our study to the important class of MIMO n-D linear systems, we have obtained the same result as in [8] on output feedback stabilizability of n-D systems using only the polynomial matrix theory that is conceptually and technically simpler than the theory of commutative algebra and topology. Our approach may be considered as a non-trivial generalization of related results on MIMO 2-D linear systems [4]–[7].

Besides deriving a criterion for output feedback stabilizability of MIMO n-D systems, we have also shown how to obtain a stabilizing n-D compensator if a given unstable n-D plant is feedback stabilizable. This is accomplished by solving a generalized polynomial matrix Bezout equation. It turns out that using matrix manipulations, our method is computationally more efficient than the method by Sule, who uses the theory of commutative algebra and topology [8]. Moreover, using the method proposed in this paper, a strictly causal stabilizing n-D compensator can always be constructed for a stabilizable causal (not necessarily strictly causal) n-D plant. In contrast, using Sule's method [8], one can only obtain a causal stabilizing n-D compensator for a strictly causal n-D plant, or a compensator (may not be causal) for a causal plant. A non-trivial example is illustrated. The example has clearly demonstrated the validity and advantages of the new results developed in this paper.

As mentioned at the end of Section 3, a very important and difficult part for the design problem of stabilizing n-D compensator for both our method and Sule's method [8] is the construction of $\lambda_1, \ldots, \lambda_\beta$ such that $\sum_{i=1}^{\beta} \lambda_i b_i = s$, with $s \neq 0$ in $\overline{U}^n$. Although a constructive solution for obtaining $\lambda_1, \ldots, \lambda_\beta$ has been suggested in [16], a computationally more tractable solution is desirable, possibly by exploiting the Gröbner basis [7], [20].

The question of whether an unstable but stabilizable n-D system P admits a right MFD $P = ND^{-1}$ where the maximal order minors of the matrix $[D^T \;\; N^T]^T$ have no common zeros in $\overline{U}^n$ remains unsolved at this stage. The same open problem has also been raised in [8]. Based on the results developed in this paper, we conjecture that if P is output feedback stabilizable, P admits an MFD $P = ND^{-1}$ where the maximal order minors of the matrix $[D^T \;\; N^T]^T$ have no common zeros in $\overline{U}^n$. We feel that the Gröbner basis [20] is likely to be the right tool for this open problem.

Notes

1. In this paper, stability means structural stability rather than BIBO stability [9].
2. $\mathbf{R}^* = \mathbf{R}\backslash\{0\}$, the set of non-zero real numbers.
3. A $p \times q$ matrix $A(\mathbf{z})$ is of normal full rank if there exists an $r \times r$ minor of $A(\mathbf{z})$ that is not identically zero, where $r = \min\{p, q\}$.
4. Denote $0_{l,m}$ an $m \times l$ zero matrix, 0_m an $m \times m$ zero matrix and I_m an $m \times m$ identity matrix.

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