resistors of decreasing sensitivity would become unacceptable members of the first pass set. As we have seen before, the chosen resistors will depend on the magnitude of the orthogonal component of first pass sensitivity vectors, with respect to the span of the second pass sensitivity vectors. We know, however, that as long as the linear model is valid, the number of resistors in the second pass set, need grow no larger than the inherent rank of the total set of sensitivity vectors. Whether this rank can be determined directly from the circuit topology without reference to element values is an interesting question.

In [4], Hocevar and Trick have dealt with the question of a minimum tuning resistor set. The difference in their work is that the approach of [4] is motivated by a desire to maintain numerical stability in the calculation. This leads to a conclusion that a QR algorithm to determine the singular values of the matrix of sensitivity vectors is the best approach to selecting the tuning resistor set. This calculation of a spanning set is similar to our calculation of the second pass set. However, an explicit model of the effects of first pass trimming inaccuracy is not created. The inherent rank of the sensitivity vectors mentioned above is bounded by the number of coefficients in the transfer function. For our first example this bound would be 13 $(=2*7-1)$. Because adequate performance was obtained with just five resistors we see that here the desired number is dominated by the level of variation in the first pass resistors rather than the inherent rank limit. Further in [4], an assumption is made which is equivalent to saying that the resistors in the first pass set are set to their design values. Hence the first pass resistor set is not at all used to correct for capacitor errors. On line calculations and adjustments are made to tuning resistors only. The performance error due to the first pass set, now arises because of the deviations of optimum resistor values from nominal design values, as well as from trimming or post-trim random deviations.

An interesting possibility of extension is to consider the problem of tuning frequency selection as a dual to the problem of minimum resistor selection. In this case, one would calculate sensitivity vectors for a deliberately large number of frequencies. The algorithm given above can then be applied to the transpose matrix, (i.e. each one of its rows is a sensitivity vector $\boldsymbol{h}_k$ considered before) to select a minimal frequency set.

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On Matrix Fraction Descriptions of Multivariable Linear n-D Systems

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Abstract —**This paper examines the approach of matrix fraction description to the study of multivariable linear n-D ($n \geqslant 3$) systems. By introducing a new concept called "generating polynomials", several interesting properties of n-D polynomial and rational matrices in connection with MFDs of n-D systems have been obtained. These properties do not occur in the 1-D and 2-D cases, and explain to some extent the difficulties encountered in the analysis of n-D systems. As an application of the generating polynomials, a stability test is presented for multivariable linear discrete n-D systems.**

I. Introduction

During recent years, increasing attention has been directed to the development of n-D systems theory, which has applications in digital filtering, image processing, seismic data processing, some distributed-parameter systems and other areas (see, e.g., [1]–[3]). This paper is concerned with the matrix fraction description (MFD) approach to the study of multiple-input multiple-output (multivariable) linear n-D ($n \geqslant 3$) systems,[1] which may be represented by n-D rational matrices. In the 1-D case, the MFD approach has been extensively treated in the literature (see, e.g., [4]–[7]). For example, the MFD approach has led to a better understanding of various structural properties of 1-D systems [4], [7]. Moreover, the celebrated result giving a parametrization of the class of all stabilizing compensators for a given stabilizable plant was first derived by using the MFD approach [6], [8].

Morf *et al.* [9] are among the first researchers to generalize the MFD approach to the study of 2-D systems. The MFD approach has great potential in the analysis and synthesis of 2-D linear systems. For example, using the MFD approach, Kung *et al.* [10] have studied the relationship between controllability, observability and minimality of 2-D systems; Humes–Jury [11], on the other hand, have obtained a stability test for multivariable 2-D systems. Recently, the problem of feedback stabilization of multivariable 2-D systems has been investigated in detail in [2, chap. 3], [13].

The problem concerning the generalizations of the MFD approach to the study of n-D systems has been suggested and considered by a number of researchers (see, e.g., [1], [11], [14], [15]). However, due to the structural complexity of n-D polynomial matrices [15], [16], it seems that this problem has not yet been fully investigated in the past, and will be studied in some detail in this paper. By introducing a new concept called "generating polynomials," we obtain several interesting properties of n-D polynomial and rational matrices in connection with MFD's of n-D systems. These properties do not occur in the 1-D and 2-D cases, and explain to some extent the difficulties encountered in the analysis of n-D systems. The notion of generating polynomials appears to be useful in the investigation of n-D systems. For example, using the generating polynomials, we derive a stability test for multivariable linear discrete n-D systems.

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[1] In what follows, the term "n-D" implies ($n \geqslant 3$) unless otherwise specified.

II. Preliminaries and Problem Formulation

The state-space model of an n-D system considered in this paper is the generalized Roesser's model given as follows [11], [17]:

$$\begin{bmatrix} x_1(k_1+1,k_2,\cdots,k_n) \\ x_2(k_1,k_2+1,\cdots,k_n) \\ \vdots \\ x_n(k_1,k_2,\cdots,k_n+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1(k_1,k_2,\cdots,k_n) \\ x_2(k_1,k_2,\cdots,k_n) \\ \vdots \\ x_n(k_1,k_2,\cdots,k_n) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} u(k_1,k_2,\cdots,k_n) \quad (2.1a)$$

$$y(k_1,k_2,\cdots,k_n) = [C_1,C_2,\cdots,C_n] \cdot \begin{bmatrix} x_1(k_1,k_2,\cdots,k_n) \\ x_2(k_1,k_2,\cdots,k_n) \\ \vdots \\ x_n(k_1,k_2,\cdots,k_n) \end{bmatrix} + Du(k_1,k_2,\cdots,k_n) \quad (2.1b)$$

where

$k_1,k_2,\cdots,k_n$	integer-valued coordinates,
$x_i(k_1,k_2,\cdots,k_n) \in \mathbb{R}^{n_i\times 1}$	ith $(i=1,2,\cdots,n)$ state vector,
$u(k_1,k_2,\cdots,k_n) \in \mathbb{R}^{l\times 1}$	input vector,
$y(k_1,k_2,\cdots,k_n) \in \mathbb{R}^{m\times 1}$	output vector,
$A_{ij}, B_i, C_j (i,j=1,2,\cdots,n)$ and D	real matrices of appropriate sizes.

We can define the n-D z-transform [14] of $x(k_1,k_2,\cdots,k_n)$:

$$\hat{x}(z_1,z_2,\cdots,z_n) \triangleq \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} x(i_1,i_2,\cdots,i_n) z_1^{i_1} z_2^{i_2}\cdots z_n^{i_n}. \quad (2.2)$$

By applying the n-D z-transform to (2.1), and assuming zero initial conditions, we obtain

$$\hat{y}(z_1,z_2,\cdots,z_n) = P(z_1,z_2,\cdots,z_n)\hat{u}(z_1,z_2,\cdots,z_n) \quad (2.3)$$

where

$$P(z_1,z_2,\cdots,z_n) \triangleq P(z) = [C_1,C_2,\cdots,C_n] \cdot \begin{bmatrix} (z_1^{-1}I_{n_1}-A_{11}) & -A_{12} & \cdots & -A_{1n} \\ -A_{21} & (z_2^{-1}I_{n_2}-A_{22}) & \cdots & -A_{2n} \\ \vdots & \vdots & & \vdots \\ -A_{n1} & -A_{n2} & \cdots & (z_n^{-1}I_{nn}-A_{nn}) \end{bmatrix}^{-1} \cdot \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} + D. \quad (2.4)$$

This is the transfer function matrix description of the n-D system (2.1).

The input-output relation of the n-D system (2.1) can also be represented by a convolution equation [3]:

$$y(k_1,k_2,\cdots,k_n) = \sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2}\cdots\sum_{i_n=0}^{k_n} G(k_1-i_1,k_2-i_2,\cdots,k_n-i_n) \cdot u(i_1,i_2,\cdots,i_n) \quad (2.5)$$

where $G(k_1,k_2,\cdots,k_n)$ is the impulse response of the system.

The concept of bounded-input bounded-output (BIBO) stability of n-D systems is defined as follows.

Definition 2.1 (BIBO stability): The n-D system (2.5) is said to be BIBO stable provided that for every $r_1>0$, there exists some $r_2>0$, such that, if $\|u(k_1,k_2,\cdots,k_n)\| \leqslant r_1$ for all $(k_1,k_2,\cdots,k_n)$, then $\|y(k_1,k_2,\cdots,k_n)\| \leqslant r_2$ for all $(k_1,k_2,\cdots,k_n)$, where $\|v\|$ is the Euclidean norm of a vector v. □

In the study of n-D linear systems, it is convenient to introduce the notion of structural stability [14], which is slightly stronger than that of BIBO stability.

Definition 2.2 (Structural stability): A scalar n-D system $p(z) = n(z)/d(z)$ is said to be structurally stable if and only if $d(z)$ has no zeros in the region $\bar{U}^n \triangleq \{(z_1,\cdots,z_n)\colon |z_1|\leqslant 1,\cdots, |z_n|\leqslant 1\}$, where $d(z)$ and $n(z)$ are factor coprime; A multivariable n-D system $P\in\mathbb{R}^{m\times l}(z)$ is said to be structurally stable if and only if every entry of $P(z)$ corresponds to a scalar system which is structurally stable. □

Some definitions and results concerning the coprimeness of n-D $(n\geqslant 1)$ polynomial matrices due to Youla–Gnavi are reproduced here.

Definition 2.3 [16]: Let $D\in\mathbb{R}^{l\times l}[z]$, $N\in\mathbb{R}^{m\times l}[z]$, and $F \triangleq [D^T \ \ N^T]^T$, where D^T denotes the transposed matrix of D. Then D and N are said to be:

(i) zero right coprime (ZRC) if there exists no n-tuple $z^0\in\mathbb{C}^n$ which is a common zero of the $l\times l$ minors of $F(z)$;
(ii) minor right coprime (MRC) if the above minors are factor coprime;
(iii) factor right coprime (FRC) if in any polynomial decomposition $F(z)=F_1(z)F_2(z)$, the $l\times l$ matrix $F(z)$ is a unimodular matrix, i.e., $\det F_2(z) = k\in\mathbb{R}^*$.[2]

In a dual manner, $\tilde{D}\in\mathbb{R}^{m\times m}[z]$ and $\tilde{N}\in\mathbb{R}^{m\times l}[z]$ are said to be zero left coprime (ZLC) *etc.*, if $\tilde{D}^T$ and $\tilde{N}^T$ are ZRC, *etc.* □

Theorem 2.1 [16]: For $n=1$, the three definitions of zero, minor and factor right coprimeness are equivalent, i.e., ZRC ≡ MRC ≡ FRC; for $n=2$, ZRC ≢ MRC = FRC; for $n\geqslant 3$, ZRC ≢ MRC ≢ FRC; for all $n\geqslant 1$, ZRC ⇒ MRC ⇒ FRC. □

Remark 2.1: Because of the above result, in this paper whenever we say that D, N are FRC, we mean that D, N are FRC but *not* MRC. □

Now consider an n-D rational matrix $P\in\mathbb{R}^{m\times l}(z)$ having the following MFD's:

$$P(z) = \tilde{D}^{-1}(z)\tilde{N}(z) = N(z)D^{-1}(z)$$

where $\tilde{D},\tilde{N}$ are MLC, and D, N are FRC.

The following questions arise:

(i) If $P(z)$ is decomposed into another right MFD, i.e. $P(z) = N_1(z)D^{-1}(z)$, does there always exist a $W_1\in\mathbb{R}^{l\times l}[z]$, such that $D_1 = DW_1$, $N_1 = NW_1$?

[2] $\det A(z) \triangleq \det A(z)$, and $\mathbb{R}^* \triangleq \mathbb{R}\backslash\{0\}$.

(ii) Does $P(z)$ admit a minor right coprime MFD?
(iii) Does there exist a $k \in \mathbb{R}^*$, such that $\det \tilde{D}(z) = k \cdot \det D(z)$?
(iv) Is a zero of $\det D(z)$ necessarily a "pole"[3] of $P(z)$?

The above questions are closely related to some problems in n-D systems theory such as stability test [1], [14]. To answer these questions properly, it is convenient to introduce a new concept called "generating polynomials." This will be discussed in detail in the next section.

III. Main Results

First, some preliminaries regarding the ordering of the submatrices and minors of a matrix are required. Let

$$F \triangleq [f_1 \cdots f_{m+l}]^T \in \mathbb{R}^{(m+l)\times l}[z] \tag{3.1}$$

and consider all the $l \times l$ submatrices of $F(z)$. The number of these submatrices is $\beta \triangleq \binom{m+l}{l}$. If a submatrix $F_i (1 \leqslant i \leqslant \beta)$ is formed by selecting rows $1 \leqslant i_1 < \cdots < i_l \leqslant m+l$, we associate F_i with an l-tuple $(i_1, \cdots, i_l)$. It is easy to see that there exists a one to one correspondence between all the $l \times l$ submatrices of F and the collection of all strictly increasing l-tuples $(i_1, \cdots, i_l)$, where $1 \leqslant i_1 < \cdots < i_l \leqslant m+l$. Now by enumerating the above l-tuples $(i_1, \cdots, i_l)$ in the lexicographic order, the $l \times l$ submatrices of F are ordered accordingly. This ordering of the $l \times l$ submatrices of F will be assumed throughout this paper. Next, the $l \times l$ minors of the matrix $F(z)$, denoted by $a_1, \cdots, a_\beta$, will always be ordered in the same way as $F_1, \cdots, F_\beta$, i.e., $a_i = \det F_i$, $i = 1, \cdots, \beta$.

A new concept called "generating polynomials" is now introduced.

Definition 3.1: Let $F \in \mathbb{R}^{(m+l)\times l}[z]$ be of normal full rank,[4] and let $a_1(z), \cdots, a_\beta(z)$ denote the $l \times l$ minors of $F(z)$, where $\beta \triangleq \binom{m+l}{l}$. Extracting a greatest common divisor (g.c.d.) $d(z)$ of $a_1(z), \cdots, a_\beta(z)$ gives [1]:

$$a_i(z) = d(z) b_i(z), \qquad i = 1, \cdots, \beta. \tag{3.2}$$

Then, $b_1(z), \cdots, b_\beta(z)$ are called "generating polynomials" of $F(z)$. □

Remark 3.1: Since $F(z)$ is of normal full rank and the order of $a_1(z), \cdots, a_\beta(z)$ is fixed, generating polynomials of $F(z)$ are essentially unique (i.e., unique up to the multiplication by a nonzero constant). □

Definition 3.2: Let $P \in \mathbb{R}^{m\times l}(z)$ and consider a right MFD of $P(z)$:

$$P(z) = N_0(z) D_0^{-1}(z) \tag{3.3}$$

where D_0 and N_0 are not necessarily factor right coprime. Let

$$F_0 \triangleq \left[D_0^T N_0^T \right]^T.$$

Then the generating polynomials of F_0, denoted by $b_{01}(z), \cdots, b_{0\beta}(z)$, are called the "MFD-minor's generating polynomials," or in short, the "generating polynomials" of the right MFD $N_0 D_0^{-1}$. □

Remark 3.2: Due to the way in which the $l \times l$ minors of $F_0(z)$ are ordered, it is clear that $b_{01}(z)$ corresponds to $\det D_0(z)$. Moreover, if $D_0(z)$ and $N_0(z)$ in (3.3) are MRC, then the $l \times l$ minors of $F_0(z)$, denoted by $a_{01}(z), \cdots, a_{0\beta}(z)$, may be taken as the generating polynomials of $N_0 D_0^{-1}$. □

The term "generating polynomials" is justified by the following theorem.

Theorem 3.1: The generating polynomials are essentially unique for all the right MFD's of $P(z) \in \mathbb{R}^{m\times l}(z)$, i.e., if

$$P(z) = N_1(z) D_1^{-1}(z) = N_2(z) D_2^{-1}(z) \tag{3.4}$$

$b_{11}(z), \cdots, b_{1\beta}(z)$ are the generating polynomials of the right MFD $N_1 D_1^{-1}$, and $b_{21}(z), \cdots, b_{2\beta}(z)$ are the generating polynomials of the right MFD $N_2 D_2^{-1}$, where $\beta \triangleq \binom{m+l}{l}$, then

$$b_{2i}(z) = k b_{1i}(z), \qquad i = 1, \cdots, \beta$$

for some $k \in \mathbb{R}^*$.

Proof: Decompose $P(z)$ into a left MFD:

$$P(z) = \tilde{D}^{-1}(z) \tilde{N}(z). \tag{3.5}$$

Let $F_1 \triangleq [D_1^T \ \ N_1^T]^T$, $F_2 \triangleq [D_2^T \ \ N_2^T]^T$, and $\tilde{F} \triangleq [-\tilde{N} \ \ \tilde{D}]$. From (3.4) and (3.5), it follows that $\tilde{F}F_1 = 0$ and $\tilde{F}F_2 = 0$. Now consider the field of n-D rational functions $\mathbb{R}(z)$. Clearly,

$$\tilde{F} \in \mathbb{R}^{m\times(m+l)}(z); \; F_1, F_2 \in \mathbb{R}^{(m+l)\times l}(z).$$

Since $\det \tilde{D}(z) \not\equiv 0$ and $\det D_q(z) \not\equiv 0$ $(q = 1,2)$, $\tilde{F}$ and F_q $(q = 1,2)$ are of normal full rank. So F_1 qualifies as a basis of the right null space of $\tilde{F}$ with respect to $\mathbb{R}(z)$. Therefore

$$F_2 = F_1 G_1 \tag{3.6}$$

for some $G_1 \in \mathbb{R}^{l\times l}(z)$. Let $a_{q1}(z), \cdots, a_{q\beta}(z)$ denote the $l \times l$ minors of $F_q(z)$, for $q = 1,2$. By Definition 3.2

$$a_{qi}(z) = d_q(z) b_{qi}(z), \qquad i = 1, \cdots, \beta; \; q = 1,2$$

for some $d_q \in \mathbb{R}[z]$. From (3.6)

$$a_{2i}(z) = a_{1i}(z) \cdot \det G_1(z)$$

or

$$d_2(z) b_{2i}(z) = \det G_1(z) \cdot d_1(z) b_{1i}(z), \qquad i = 1, \cdots, \beta.$$

Next, $\det D_2(z) \not\equiv 0$ implies $d_2(z) \not\equiv 0$. Thus

$$b_{2i}(z) = \frac{\det G_1(z) \cdot d_1(z)}{d_2(z)} b_{1i}(z)$$

$$\triangleq \frac{t_n(z)}{t_d(z)} b_{1i}(z), \qquad i = 1, \cdots, \beta \tag{3.7}$$

where $t_n, t_d \in \mathbb{R}[z]$, with t_n and t_q being factor coprime.

From (3.7)

$$t_d(z) b_{2i}(z) = t_n(z) b_{1i}(z), \qquad i = 1, \cdots, \beta. \tag{3.8}$$

The assumption that $b_{q1}, \cdots, b_{q\beta}$ $(q = 1,2)$ are factor coprime, together with the fact that $\mathbb{R}[z]$ is a unique factorization domain, implies that $t_d(z), t_n z)$ are nonzero constants, i.e., $t_n(z)/t_d(z) = k \in \mathbb{R}^*$, and (3.8) gives

$$b_{2i}(z) = k b_{1i}(z), \qquad i = 1, \cdots, \beta. \qquad \square$$

So far only the generating polynomials of right MFD's have been defined and discussed. The generating polynomials of left MFD's can be defined and considered analogously. Let $\tilde{F} \in \mathbb{R}^{m\times(m+l)}[z]$. Suppose that $\tilde{F}_1^T, \cdots, \tilde{F}_\beta^T$ are the ordered $m \times m$ submatrices of $\tilde{F}^T(z)$. Then $\tilde{F}_1, \cdots, \tilde{F}_\beta$ are taken as the ordered $m \times m$ submatrices of $\tilde{F}(z)$, where $\beta \triangleq \binom{m+l}{m}$. The $m \times m$ minors of $\tilde{F}(z)$ can be similarly ordered, and when $\tilde{F}(z)$ is of normal full rank, the generating polynomials of $\tilde{F}(z)$ can be defined analogously as in Definition 3.1. For convenience of exposition, the

[3] z^0 is a "pole" of $P(z)$ if z^0 is a zero of the denominator of some entry of $P(z)$ [11]. It should be noted that "poles" as defined here include both nonessential singularities of the first and second kinds.

[4] A $p \times q$ matrix $A(x)$ is of normal full rank if there exists an $r \times r$ minor of $A(z)$ that is not identically zero, where $r = \min\{p, q\}$.

definition for the generating polynomials of a left MFD of an n-D rational matrix is given as follows.

Definition 3.3: Let $P \in \mathbb{R}^{m\times l}(z)$ and consider a left MFD of $P(z)$:

$$P(z) = \tilde{D}_0^{-1}(z)\tilde{N}_0(z) \tag{3.9}$$

where $\tilde{D}_0(z)$ and $\tilde{N}_0(z)$ are not necessarily factor left coprime. Let

$$\tilde{F}_0 \triangleq [\tilde{D}_0 \quad \tilde{N}_0].$$

Then the generating polynomials of $\tilde{F}_0(z)$, denoted by $\tilde{b}_{01}(z), \cdots, \tilde{b}_{0\beta}(z)$, are called the generating polynomials of the left MFD $\tilde{D}_0^{-1}\tilde{N}_0$. □

Using the argument similar to the one given in the proof of Theorem 3.1, it can be shown that the generating polynomials are essentially unique for all the left MFD's of a given n-D rational matrix $P(z)$. Furthermore, the following theorem establishes a close relationship between the generating polynomials of left and right MFD's of $P(z)$.

Theorem 3.2: Let $P \in \mathbb{R}^{m\times l}(z)$ and decompose $P(z)$ into the following MFD's:

$$P(z) = \tilde{D}^{-1}(z)\tilde{N}(z) = N(z)D^{-1}(z).$$

Denote by $\tilde{b}_1(z), \cdots, \tilde{b}_\beta(z)$ the generating polynomials of $\tilde{D}^{-1}\tilde{N}$, and by $b_1(z), \cdots, b_\beta(z)$ the generating polynomials of ND^{-1}, respectively. Then

$$b_i(z) = \pm k\tilde{b}_i'(z), \qquad i = 1, \cdots, \beta$$

for some $k \in \mathbb{R}^*$, where $\tilde{b}_1'(z), \cdots, \tilde{b}_\beta'(z)$ are obtained by reordering $\tilde{b}_1(z), \cdots, \tilde{b}_\beta(z)$ appropriately, with $\tilde{b}_1'(z) = \tilde{b}_1(z)$. □

The proof is rather involved and is given in the Appendix.

Let us call the generating polynomials of all the right MFD's of a given n-D rational matrix $P(z)$ the generating polynomials of $P(z)$. We are now in a position to answer the questions (i)–(iii) raised in Section III.

Theorem 3.3: Suppose that $P \in \mathbb{R}^{m\times l}(z)$ has the following MFD's:

$$P(z) = \tilde{D}^{-1}(z)\tilde{N}(z) = N(z)D^{-1}(z) \tag{3.10}$$

where $\tilde{D}, \tilde{N}$ are MLC, and D, N are FRC. Then:

(i) There exists a right MFD of $P(z)$, $P(z) = N_1(z)D_1^{-1}(z)$, such that $D_1 \neq DW_1$ and $N_1 \neq NW_1$ for any $W_1 \in \mathbb{R}^{l\times l}[z]$.
(ii) There does not exist a minor right coprime MFD of $P(z)$.
(iii) $\det \tilde{D}(z) \neq k \cdot \det D(z)$ for any $k \in \mathbb{R}^*$. □

The proof will be given after the following result due to Youla–Gnavi [16].

Lemma 3.1 [16]: Let $A_{11}(z)$, $A_{12}(z)$ and $A_{21}(z)$ be n-D polynomial matrices of compatible sizes such that $A_{21}(z)A_{11}^{-1}(z)A_{12}(z)$ is a polynomial matrix. Then, if $A_{11}(z)$ and $A_{21}(z)$ are minor right coprime, $A_{11}^{-1}(z)A_{12}(z)$ is a polynomial matrix. □

Proof of Theorem 3.3: Let $b_1(z), \cdots, b_\beta(z)$ denote the generating polynomials of $P(z)$. Since $\tilde{D}(z)$ and $\tilde{N}(z)$ are MLC, applying Theorem 3.2 gives

$$\det \tilde{D}(z) = k_1 b_1(z)$$

for some $k_1 \in \mathbb{R}^*$.

Next, let $a_1(z), \cdots, a_\beta(z)$ denote the $l\times l$ minors of $[D^T \quad N^T]^T$. Since $D(z)$ and $N(z)$ are FRC, applying Theorem 3.1 yields

$$a_i(z) = d(z)b_i(z), \qquad i = 1, \cdots, \beta$$

for some nontrivial polynomial $d(z)$ (i.e., $d(z)$ is not a constant). Without loss of generality, it may be assumed that $d(z)$ depends on z_1. By considering $P(z)$ as a matrix in $\mathbb{R}(z_2, \cdots, z_n)^{m\times l}(z_1)$, $P(z)$ has a right MFD $N_1D_1^{-1}$ such that a g.c.d. $d_1(z)$ of the $l\times l$ minors of the matrix $F_1 \triangleq [D_1^T \quad N_1^T]^T$ is independent of the variable z_1 (see, e.g., [7], and also the proof of Theorem 3.2 of this paper). Clearly, $d(z)$ is not a divisor of $d_1(z)$, since by assumption $d(z)$ depends on z_1. Let $a_{11}(z), \cdots, a_{1\beta}(z)$ denote the $l\times l$ minors of F_1. By Theorem 3.1

$$a_{1i}(z) = d_1(z)b_i(z), \qquad i = 1, \cdots, \beta.$$

Now the following properties can be readily shown.

(i) $D_1 \neq DW_1$ and $N_1 \neq NW_1$ for any $W_1 \in \mathbb{R}^{l\times l}[z]$, since $\det D(=db_1)$ is not a divisor of $\det D_1(=d_1b_1)$.
(ii) We show by contradiction that $P(z)$ does not admit a minor right coprime MFD. Suppose that $P(z)$ admits a MRC MFD, i.e., $P = N_cD_c^{-1}$, where D_c, N_c are MRC. Then $N_cD_c^{-1} = ND^{-1}$, or $N_cD_c^{-1}D = N$. Since D_c and N_c are MRC, by Lemma 3.1, $W \triangleq D_c^{-1}D$ is a polynomial matrix. Hence $D = D_cW$ and $N = N_cW$. The assumption that D and N are FRC implies that W is a unimodular matrix. This in turn implies that D and N are MRC, which is a contradiction.
(iii) $\det \tilde{D}(z) \neq k \det D(z)$ for any $k \in \mathbb{R}^*$, since $\det \tilde{D}(z) = k_1b_1(z)$ and $\det D(z) = d(z)b_1(z)$. □

Remark 3.3: Lévy [15] has constructed a 3-D rational matrix $P(z_1, z_2, z_3)$, such that $P = \tilde{D}^{-1}\tilde{N} = ND^{-1}$, where $\tilde{D}, \tilde{N}$ are MLC, and D, N are FRC, but $\det \tilde{D} \neq k \det D$ for any $k \in \mathbb{R}^*$, and furthermore, shown that there does not exist a minor right coprime MFD for P. His example turns out to be a special case of parts (ii) and (iii) of Theorem 3.3. □

The remaining question (iv) is closely related to the so called "determinant test" for structural (or BIBO) stability of multivariable linear shift-invariant (LSI) discrete causal n-D$(n \geq 1)$ systems [14]. This question can be answered by the following example.

Example 3.1: Let

$$P(z_1, z_2, z_3) = \begin{bmatrix} \dfrac{z_3^2 + z_3 + 0.25}{(z_2+2)(z_3+2.5)} & \dfrac{1}{(z_2+2)(z_3+4.5)} \\ \dfrac{z_3+0.5}{(z_1+3)(z_3+2.5)} & \dfrac{1}{(z_1+3)(z_3+4.5)} \end{bmatrix}.$$

Clearly, $P(z_1, z_2, z_3)$ corresponds to a (structurally) stable LSI discrete causal 3-D system, since $P(z_1, z_2, z_3)$ has no poles in $\bar{U}^3$. Decompose $P(z_1, z_2, z_3)$ into a right MFD:

$$P(z_1, z_2, z_3) \triangleq N(z_1, z_2, z_3)D^{-1}(z_1, z_2, z_3)$$

where

$$D(z_1, z_2, z_3) = \begin{bmatrix} (z_2+2)(z_3+2.5) & -(z_1+3)(z_3+2.5) \\ -(z_3+0.5)(z_2+2)(z_3+4.5) & (z_3+0.5)^2(z_1+3)(z_3+4.5) \end{bmatrix}$$

and

$$N(z_1,z_2,z_3)=\begin{bmatrix}(z_3+0.5)(z_3-0.5) & 0\\ 0 & (z_3+0.5)(z_3-0.5)\end{bmatrix}.$$

It can be checked [18] that $D(z_1,z_2,z_3)$ and $N(z_1,z_2,z_3)$ are FRC. But

$$\det D=(z_3+0.5)(z_3-0.5)(z_1+3)(z_2+2)(z_3+2.5)(z_3+4.5)$$

which has zeros in the unstable region $\bar{U}^3$. □

The above example shows that the determinant test for structural stability of 2-D systems due to Humes–Jury [11] may not be extended to the n-D case when $P(z)$ does not admit a MRC MFD. Nevertheless, a similar test for structural stability of multivariable LSI discrete causal n-D systems can be derived in terms of the generating polynomials. This will be discussed in the following two theorems.

Theorem 3.4: Let $b_1(z),\cdots,b_\beta(z)$ denote the generating polynomials of $P\in\mathbb{R}^{m\times l}(z)$, where $\beta\triangleq\binom{m+l}{l}$. Then $z^0\in\mathbb{C}^n$ is a pole of $P(z)$ if and only if z^0 is a zero of $b_1(z)$.

Proof (Necessity): Decompose $P(z)$ into a right MFD:

$$P(z)\triangleq N(z)D^{-1}(z). \tag{3.11}$$

Let $F\triangleq[D^T\quad N^T]^T$, and let $a_1(z),\cdots,a_\beta(z)$ denote the $l\times l$ minors of F. By Theorem 3.1,

$$a_i(z)=d(z)b_i(z),\qquad i=1,\cdots,\beta \tag{3.12}$$

where $d\in\mathbb{R}[z]$ is a g.c.d. of $a_1(z),\cdots,a_\beta(z)$. From (3.11),

$$P(z)=\underbrace{N(z)\operatorname{adj}D(z)}_{M(z)}/\det D(z). \tag{3.13}$$

Denote by $m_{i,j}$ the entry of M at the position (i,j). It can be seen by using Cramer's rule that $m_{i,j}=\det D_{i,j}$, where $D_{i,j}$ is the matrix D with the jth row replaced by the ith row of N. By appropriately permuting some of its rows, $D_{i,j}$ becomes an $l\times l$ submatrix of F. Thus $m_{i,j}=\pm a_k$ for some $k\in\{1,\cdots,\beta\}$. It is then clear from (3.12) that $d(z)$ divides every entry of $M(z)$, i.e.,

$$M(z)=d(z)N'(z)$$

for some $N'\in\mathbb{R}^{m\times l}[z]$. But

$$\det D(z)=d(z)b_1(z).$$

Hence, after cancelling $d(z)$ from $M(z)$ and $\det D(z)$, (3.13) becomes

$$P(z)=N'(z)/b_1(z). \tag{3.14}$$

Therefore, if z^0 is a pole of $P(z)$, it must be a zero of $b_1(z)$.

Sufficiency: let $d_0(z)$ denote a least common multiple of the denominator of all the entries of $P(z)$, i.e., $P(z)=N_0(z)\{d_0(z)I_l\}^{-1}$, where $N_0\in\mathbb{R}^{m\times l}[z]$. Clearly, if z^0 is a zero of $\det(d_0(z)I_l)$, then z^0 must be a pole of $P(z)$. By Theorem 3.1, $\det(d_0(z)I_l)=d'(z)b_1(z)$ for some $d'\in\mathbb{R}[z]$. Therefore, if z^0 is a zero of $b_1(z)$, z^0 is necessarily a zero of $\det(d_0(z)I_l)$, and hence is a pole of $P(z)$. □

As a direct consequence of Theorem 3.4, a test for structural stability of n-D systems is given as follows.

Theorem 3.5: A LSI discrete causal n-D system characterized by $P(z)\in\mathbb{R}^{m\times l}(z)$ is structurally stable if and only if $b_1(z)\neq 0$ in $\bar{U}^n$, where $b_1(z),\cdots,b_\beta(z)$ are the generating polynomials of $P(z)$. □

To conclude this paper, let us reconsider Example 3.1.

Example 3.1 (continued): The 2×2 minors of $[D^T\quad N^T]$ are

$$a_1=db_1\quad a_2=db_2$$
$$a_3=db_3\quad a_4=db_4$$
$$a_5=db_5\quad a_6=db_6$$

where $d=(z_3+0.5)(z_3-0.5)$, and $b_1,\cdots,b_6$ are the generating polynomials of P:

$$b_1=(z_1+3)(z_2+2)(z_3+2.5)(z_3+4.5)$$
$$b_2=(z_1+3)(z_3+2.5)$$
$$b_3=(z_2+2)(z_3+2.5)$$
$$b_4=-(z_3+0.5)^2(z_1+3)(z_3+4.5)$$
$$b_5=-(z_3+0.5)(z_2+2)(z_3+4.5)$$
$$b_6=(z_3+0.5)(z_3-0.5).$$

Clearly, $b_1(z_1,z_2,z_3)$ has no zeros in $\bar{U}^3$. By Theorem 3.5, the 3-D system $P(z_1,z_2,z_3)$ is structurally stable, which agrees with the fact that $P(z_1,z_2,z_3)$ has no poles in $\bar{U}^3$. □

Appendix

Proof of Theorem 3.2: The proof is divided into two steps. In *Step 1*, we show that $b_1=k\tilde{b}_1$ for some $k\in\mathbb{R}^*$. It will then be proved in *Step 2* that for each $i\in\{2,\cdots,\beta\}$, there exists some $t\in\{2,\cdots,\beta\}$ such that $b_i=\pm k\tilde{b}_t$, and when i ranges over the set $\{2,\cdots,\beta\}$, t also ranges over the set $\{2,\cdots,\beta\}$.

Step 1: For convenience of exposition, let

$$\mathbb{R}_q\triangleq\mathbb{R}[z_1,\cdots,z_{q-1},z_{q+1},\cdots,z_n];$$
$$\mathbb{F}_q\triangleq\mathbb{R}(z_1,\cdots,z_{q-1},z_{q+1},\cdots,z_n);\qquad q=1,2,\cdots,n.$$

Clearly, $\mathbb{R}_q$ is a ring, while $\mathbb{F}_q$ is a field. Moreover,

$$\begin{aligned}\mathbb{R}[z]&=\mathbb{R}[z_1,\cdots,z_{q-1},z_q,z_{q+1},\cdots,z_n]\\&=\mathbb{R}[z_1,\cdots,z_{q-1},z_{q+1},\cdots,z_n][z_q]=\mathbb{R}_q[z_q]\\ \mathbb{R}(z)&=\mathbb{R}(z_1,\cdots,z_{q-1},z_q,z_{q+1},\cdots,z_n)\\&=\mathbb{R}(z_1,\cdots,z_{q-1},z_{q+1},\cdots,z_n)(z_q)=\mathbb{F}_q(z_q).\end{aligned}$$

Thus, $P(z)\in\mathbb{R}^{m\times l}(z)$ can also be considered as $P(z)\in\mathbb{F}_q^{m\times l}(z_q)$, for $q=1,2,\cdots,n$. By using a well-known 1-D technique (see, e.g., [7]), $P(z)$ can be decomposed into coprime MFD's in $\mathbb{F}_q[z_q]$, i.e.

$$P=\tilde{D}_q'^{-1}\tilde{N}_q'=N_q'D_q'^{-1} \tag{A.1a}$$

such that

$$\begin{bmatrix}\tilde{X}_q' & \tilde{Y}_q'\\ -\tilde{N}_q' & \tilde{D}_q'\end{bmatrix}\begin{bmatrix}D_q' & Y_q'\\ N_q' & X_q'\end{bmatrix}=\begin{bmatrix}I_l & 0\\ 0 & I_m\end{bmatrix},\qquad q=1,2,\cdots,n \tag{A.1b}$$

where $\tilde{D}_q'$, $\tilde{N}_q'$, $\tilde{X}_q'$, $\tilde{Y}_q'$, D_q', N_q', X_q', and Y_q' are matrices of appropriate sizes with entries in $\mathbb{F}_q[z_q]$. In view of the notation for $\mathbb{F}_q$, $f\in\mathbb{F}_q[z_q]$ implies that $f(z)$ is in $\mathbb{R}(z)$ with a denominator independent of the variable z_q. Simple manipulation of (A.1) gives the following MFDs of $P(z)$:

$$P=\tilde{D}_q^{-1}\tilde{N}_q=N_qD_q^{-1} \tag{A.2a}$$

such that

$$\underbrace{\begin{bmatrix} \tilde{X}_q & \tilde{Y}_q \\ -\tilde{N}_q & \tilde{D}_q \end{bmatrix}}_{\tilde{S}_q} \underbrace{\begin{bmatrix} D_q & Y_q \\ N_q & X_q \end{bmatrix}}_{S_q} = \underbrace{\begin{bmatrix} E_q & 0 \\ 0 & \tilde{E}_q \end{bmatrix}}_{T_q}, \qquad q=1,2,\cdots,n \tag{A.2b}$$

where $\tilde{D}_q$, $\tilde{N}_q$, $\tilde{X}_q$, $\tilde{Y}_q$, D_q, N_q, X_q, Y_q, $\tilde{E}_q$, and E_q are matrices of appropriate sizes with entries in $\mathbb{R}_q[z_q]=\mathbb{R}[z]$. Moreover, $\det \tilde{S}_q$, $\det S_q$ and $\det T_q$ are independent of the variable z_q. From (A.2),

$$\begin{bmatrix} \tilde{X}_q & \tilde{Y}_q \\ -\tilde{N}_q & \tilde{D}_q \end{bmatrix} \begin{bmatrix} D_q & 0 \\ N_q & I_m \end{bmatrix} = \begin{bmatrix} E_q & \tilde{Y}_q \\ 0 & \tilde{D}_q \end{bmatrix}, \qquad q=1,2,\cdots,n.$$

Thus

$$\det \tilde{S}_q \cdot \det D_q = \det E_q \cdot \det \tilde{D}_q, \qquad q=1,2,\cdots,n. \tag{A.3}$$

By Theorem 3.1, $\det D_q = d_q b_1$, where d_q is a g.c.d. of the $l \times l$ minors of the matrix $[D_q^T \quad N_q^T]^T$. In view of (A.2b), d_q must be independent of z_q. Similarly, $\det \tilde{D}_q = \tilde{d}_q \tilde{b}_1$, where $\tilde{d}_q$ is a g.c.d. of the $m \times m$ minors of the matrix $[\tilde{D}_q \quad \tilde{N}_q]$, and $\tilde{d}_q$ is independent of z_q. Hence, (A.3) becomes

$$\det \tilde{S}_q \cdot d_q b_1 = \det E_q \cdot \tilde{d}_q \tilde{b}_1, \qquad q=1,2,\cdots,n \tag{A.4}$$

or

$$\frac{b_1}{\tilde{b}_1} = \frac{\det E_q \cdot \tilde{d}_q}{\det \tilde{S}_q \cdot d_q} \in \mathbb{F}_q, \qquad q=1,2,\cdots,n.$$

Thus $b_1/\tilde{b}_1$ is independent of z_q $(q=1,2,\cdots,n)$, and must be a non-zero constant $k \in \mathbb{R}^*$.

Step 2: Denote by $a_{q1},\cdots,a_{q\beta}$ the $l \times l$ minors of $[D_q^T N_q^T]^T$, and by $\tilde{a}_{q1},\cdots,\tilde{a}_{q\beta}$ the $m \times m$ minors of $[\tilde{D}_q \quad \tilde{N}_q]$. By Theorem 3.1,

$$a_{qi} = d_q b_i; \; \tilde{a}_{qi} = \tilde{d}_q \tilde{b}_i, \qquad i=1,\cdots,\beta.$$

Let $\tilde{a}'_{q1},\cdots,\tilde{a}'_{q\beta}$ denote the $m \times m$ minors of $[-\tilde{N}_q \quad \tilde{D}_q]$. It is easy to see that $\tilde{a} = \tilde{a}'_{q\beta}$, and for each $t \in \{2,\cdots,\beta\}$, there exists $j \in \{1,\cdots,\beta-1\}$, such that

$$\tilde{a}_{qt} = \pm \tilde{a}'_{qj}. \tag{A.5}$$

Moreover, when t ranges over the set $\{2,\cdots,\beta\}$, j also ranges over the set $\{1,\cdots,\beta-1\}$. Next, for each $i \in \{2,\cdots,\beta\}$, there exists a row permutation matrix Q_i $(\det Q_i = \pm 1)$ such that

$$Q_i \begin{bmatrix} D_q \\ N_q \end{bmatrix} = \begin{bmatrix} D_{qi} \\ N_{qi} \end{bmatrix}, \qquad q=1,2,\cdots,n$$

with $\det D_{qi} = a_{qi} = b_i d_q$. Recalling (A.2b) gives

$$\tilde{S}_q Q_i^{-1} Q_i S_q = T_q, \qquad q=1,2,\cdots,n. \tag{A.6}$$

Let $S_{qi} \triangleq Q_i S_q$, $\tilde{S}_{qi} \triangleq \tilde{S}_q Q_i^{-1}$, and $[-\tilde{N}_{qi} \quad \tilde{D}_{qi}] \triangleq [-\tilde{N}_q \quad \tilde{D}_q] Q_i^{-1}$. Then $\det \tilde{D}_{qi} = \tilde{a}'_{qj}$ for some $j \in \{1,\cdots,\beta-1\}$. Recalling (A.5), there exists $t \in \{2,\cdots,\beta\}$, such that $\det \tilde{D}_{qi} = \pm \tilde{a}_{qt}$, or, $\det \tilde{D}_{qi} = \pm \tilde{b}_t \tilde{d}_q$, for $q=1,2,\cdots,n$. Rewrite (A.6) as

$$\tilde{S}_{qi} S_{qi} = T_q, \qquad q=1,2,\cdots,n.$$

By using an argument similar to the one used in *Step 1*, it follows that

$$\det \tilde{S}_{qi} \cdot \det D_{qi} = \det E_q \cdot \det \tilde{D}_{qi}$$

or

$$\det \tilde{S}_{qi} \cdot b_i d_q = \pm \det E_q \cdot \tilde{b}_t \tilde{d}_q, \qquad q=1,2,\cdots,n.$$

Thus

$$\begin{aligned} b_i &= \pm \frac{\det E_q \cdot \tilde{d}_q}{\det \tilde{S}_{qi} \cdot d_q} \tilde{b}_t \\ &= \pm \frac{\det E_q \cdot \tilde{d}_q}{\det \tilde{S}_q \cdot d_q} \tilde{b}_t \\ &= \pm k \tilde{b}_t \end{aligned}$$

since $\det \tilde{S}_{qi} = \det \tilde{S}_q \det Q_i^{-1} = \pm \det \tilde{S}_q$.

It can be easily seen that when i ranges over $\{2,\cdots,\beta\}$, j ranges over $\{1,\cdots,\beta-1\}$, and thus t ranges over $\{2,\cdots,\beta\}$.

Therefore, by appropriately reordering $b_1(z),\cdots,b_\beta(z)$ as $b'_1(z),\cdots,b'_\beta(z)$, with $b_1(z) = b'_1(z)$, we have

$$b_i(z) = \pm k b'_i(z), \qquad i=2,\cdots,\beta. \qquad \square$$

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