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Analysis of Nonlinear Discrete-Time Systems with Higher-Order Iterative Learning Control

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Abstract. In this paper, an iterative learning control method is proposed for a class of nonlinear discrete-time systems with well-defined relative degree, which uses the output data from several previous operation cycles to enhance tracking performance. A new analysis approach is developed, by which the iterative learning control is shown to guarantee the convergence of the output trajectory to the desired one within bound and the bound is proportional to the bound on resetting errors. It is further proved effective to overcome initial shifts and the resultant output trajectory can be assessed as iteration increases. Numerical simulation is carried out to verify the theoretical results and exhibits that the proposed updating law possesses good transient behavior of learning process so that the convergence speed is improved.

Keywords: learning control, initial condition problem, relative degree, discrete-time, nonlinear systems

1. Introduction

Discrete-time iterative learning control (ILC) has been an active research area for years. Researches dealt with this method for linear discrete-time systems by applying optimization approaches [17], 2D system theory [5,9], the impulsive response sequences [7], the pulse transfer function [16], and the transfer matrix representation in terms of the system Markov parameters that make up the entries of the transfer matrix [2,10,12]. Recently, discrete-time ILC for certain classes of nonlinear systems has received much attention. The analyses for convergence and robustness have been obtained by using discrete-time λ-norm [1,4,8,13,14,18], and analysis techniques without using such a norm [6,19].

Most of the literature mentioned above are concerned with first-order updating laws. Namely, the current control input is generated by the updating law with the information including the input and the output error at the last cycle. In continuous-time ILC, higher-order updating laws were suggested in [3] to enhance tracking performance. It is already known that the suitably designed higher-order schemes usually achieve fast convergence speed, mainly due to updating actions constituted by the information from several previous operation cycles, and more freedom in selection of the learning gains [15]. However, higher-order ILC has not been well investigated for nonlinear discrete-time systems. In [4], the robustness issue is addressed in the presence of resetting errors, state disturbances and output measurement noises. The result is restricted to systems with relative degree one. Note that the derived sufficient conditions [1,4,13,18] fail to hold for the systems with higher relative degree. Further efforts are still needed for more extensive applicability. The relative degree of discrete-time dynamic systems is the time delay between the input and the output of the system which is inherent in many practical applications. In [8], a first-order updating law was proposed using relative degree of the controlled systems. The updating law uses the multi-step output errors in one operation cycle which can approximate derivatives of continuous-time updating laws. The convergence was established without considering uncertainties and disturbances. In particular, the initial condition at each cycle should be reset to the same as the desired one.

In this paper, higher-order iterative learning control is applied to a class of nonlinear discrete-time systems with well-defined relative degree. The updating law uses the control inputs and the output errors from several previous operation cycles but adopts only one-step output error in one operation cycle based on the pair of action taken and its resulting variable. With the aid of an inequality, we develop a new systematic technique for the analysis purpose, instead of applying discrete-time λ-norm and also different from [6,19]. Boundedness of the output error is established in the presence of resetting errors and in the absence of resetting errors convergence of the output trajectory to the desired one is ensured. The terms in the derived sufficient condition reflect all of the learning gains which can be responsible for the performance improvement. This iterative learning control is also proved effective against initial shifts and the resultant output trajectory can be assessed by the initial shifts as iteration number increases. Finally, the performance improvement of the higher-order ILC is demonstrated by a numerical simulation.

2. Problem Formulation and Preliminaries

Consider a class of nonlinear discrete-time systems described by the state space equations

$$x(t+1) = f\bigl(x(t)\bigr) + B\bigl(x(t)\bigr)u(t), \tag{1}$$

$$y(t) = g\bigl(x(t)\bigr) \tag{2}$$

where t is the discrete time index, $x \in \mathbb{R}^n$, $u = [u_1, \ldots, u_r]^T \in \mathbb{R}^r$ and $y = [y_1, \ldots, y_m]^T \in \mathbb{R}^m$ denote the state, the control input and the output of the system, respectively. The functions $f(\cdot) \in \mathbb{R}^n$, $B(\cdot) \in \mathbb{R}^{n\times r}$ and $g(\cdot) = [g_1(\cdot), \ldots, g_m(\cdot)]^T \in \mathbb{R}^m$ are smooth in their domain of definition.

Before stating the control problem under consideration, it is necessary to introduce some notations and the definition for relative degree to characterize system (1)–(2). The composite function of $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ is denoted by $g \circ f$, and f^i is the ith composite function of f satisfying

$$f^i(x) = f^{i-1} \circ \bigl(f(x)\bigr),$$
$$f^0(x) = x.$$

The partial derivative of the function $h = [h_1, \ldots, h_m]^T$ with respect to a vector $u = [u_1, \ldots, u_r]^T$ is denoted by

$$\frac{\partial h}{\partial u} = \left\{ \frac{\partial h_i}{\partial u_j} \right\}.$$

Definition 2.1. The (vector) relative degree of system (1)–(2) is associated with vector $\mu = \{\mu_1, \ldots, \mu_m\}$ satisfying, for $x \in \mathbb{R}^n$,

$$\frac{\partial}{\partial u} g_q \circ f^i\big(f(x) + B(x)u\big) = 0, \quad 0 \leqslant i \leqslant \mu_q - 2,\ 1 \leqslant q \leqslant m,$$

and the $m \times r$ matrix

$$D(x, u) = \begin{bmatrix} \dfrac{\partial}{\partial u} g_1 \circ f^{\mu_1 - 1}\big(f(x) + B(x)u\big) \\ \vdots \\ \dfrac{\partial}{\partial u} g_m \circ f^{\mu_m - 1}\big(f(x) + B(x)u\big) \end{bmatrix}$$

is of full column rank.

Remark 2.1. Comparing with the definitions in [8,11], Definition 2.1 allows that the number of outputs is greater than the number of inputs. As system (1)–(2) has relative degree $\mu = \{\mu_1, \ldots, \mu_m\}$, the qth component of output can be written as, for $1 \leqslant q \leqslant m$,

$$y_q(t+i) = g_q \circ f^i\big(x(t)\big), \quad 0 \leqslant i \leqslant \mu_q - 1, \tag{3}$$

$$y_q(t+\mu_q) = g_q \circ f^{\mu_q - 1}\Big(f\big(x(t)\big) + B\big(x(t)\big)u(t)\Big) \tag{4}$$

which implies that μ_q is exactly the steps of delay in the qth output $y_q(t)$ in order to have at least one component of the control input $u(t)$ appearing. In this sense the defined relative degree indicates the inherent level between the input and the output.

Remark 2.2. If we assume that, as discussed in [8], the scalar function $g_q \circ f^{\mu_q - 1}(f(x) + B(x)u)$ is linear in u, the function $(\partial/\partial u)(g_q \circ f^{\mu_q - 1}(f(x) + B(x)u))$ will be independent of u and the matrix $D(x, u)$ is thus independent of u so that it can be denoted by $D(x)$. At the same time, the qth component of output is evaluated as

$$\begin{aligned} & y_q(t+\mu_q) \\ &\quad = g_q \circ f^{\mu_q - 1}\Big(f\big(x(t)\big) + B\big(x(t)\big)u\Big)\Big|_{u=0} \\ &\qquad + \int_0^{u(t)} \frac{\partial}{\partial u} g_q \circ f^{\mu_q - 1}\Big(f\big(x(t)\big) + B\big(x(t)\big)u\Big)\,\mathrm{d}u \\ &\quad = g_q \circ f^{\mu_q - 1}\Big(f\big(x(t)\big)\Big) + \frac{\partial}{\partial u} g_q \circ f^{\mu_q - 1}\Big(f\big(x(t)\big) + B\big(x(t)\big)u\Big)\bigg|_{u=u(t)} u(t) \end{aligned} \tag{5}$$

and the output can be written in the following simple form:

$$y(t+\mu) = \big[y_1(t+\mu_1), \ldots, y_m(t+\mu_m)\big]^T = \hat{g}\big(x(t)\big) + D\big(x(t)\big)u(t) \tag{6}$$

where $\hat{g}(x(t)) = [g_1 \circ f^{\mu_1}(x(t)), \ldots, g_m \circ f^{\mu_m}(x(t))]^T$.

Throughout the paper, the vector norm is defined as $\|x\| = \max_{1\leqslant i\leqslant n} |x_i|$ for an n-dimensional vector $x = [x_1, \ldots, x_n]^T$ and the matrix norm as the induced norm by the vector norm, namely, $\|A\| = \max_{1\leqslant i\leqslant m} \sum_{j=1}^{n} |a_{ij}|$ for a matrix $A = \{a_{ij}\} \in \mathbb{R}^{m\times n}$.

Now, the control problem to be solved in this paper is formulated as follows:

For system (1)–(2) with relative degree $\mu = \{\mu_1, \ldots, \mu_m\}$, given a realizable trajectory $y_d(t) = [y_{1,d}(t), \ldots, y_{m,d}(t)]^T$ with $y_{q,d}(t)$, $0 \leqslant t \leqslant N + \mu_q$, and a tolerance error bound ε, find an input profile $u(t)$, $0 \leqslant t \leqslant N$, so that the error between the resultant output $y(t)$ and the desired trajectory $y_d(t)$ is within the tolerance error bound ε, i.e., $\|y_{q,d}(t) - y_q(t)\| < \varepsilon$, $\mu_q \leqslant t \leqslant N + \mu_q$, $1 \leqslant q \leqslant m$.

The following properties for system (1)–(2) are assumed, where S denotes a mapping from $(x(0), u(t),\ 0 \leqslant t \leqslant N)$ to $(x(t),\ 0 \leqslant t \leqslant N+1)$ and O a mapping from $(x(0), u(t), 0 \leqslant t \leqslant N)$ to $y(t)$ with $y_q(t), 0 \leqslant t \leqslant N + \mu_q, 1 \leqslant q \leqslant m$.

(A1) The mappings S and O are one to one.

(A2) The system has relative degree $\mu = \{\mu_1, \ldots, \mu_m\}$ for $x \in \mathbb{R}^n$.

(A3) For $1 \leqslant q \leqslant m$,

$$\frac{\partial^2}{\partial u_i \partial u_j} g_q \circ f^{\mu_q - 1}\big(f(x) + B(x)u\big) = 0, \quad 1 \leqslant i, j \leqslant r. \tag{7}$$

(A4) The functions $f(\cdot)$, $B(\cdot)$ and $g(\cdot)$ and $D(\cdot)$ are Lipschitz in $\mathbb{R}^n$. That is,

$$\begin{aligned} \big\|f(x') - f(x'')\big\| &\leqslant l_f \big\|x' - x''\big\|, \\ \big\|B(x') - B(x'')\big\| &\leqslant l_B \big\|x' - x''\big\|, \\ \big\|g(x') - g(x'')\big\| &\leqslant l_g \big\|x' - x''\big\|, \\ \big\|D(x') - D(x'')\big\| &\leqslant l_D \big\|x' - x''\big\| \end{aligned}$$

for $x', x'' \in \mathbb{R}^n$, and positive constants l_f, l_B, l_g and l_D.

(A5) The operators $B(\cdot)$ and $D(\cdot)$ are bounded in $\mathbb{R}^n$.

(A1) implies the existence of a unique control input for a realizable trajectory. (A2) restricts the system class being with well-defined relative degree. (A3) implies that the function $g_q \circ f^{\mu_q - 1}(f(x) + B(x)u)$ is linear in u. Note the fact that if $a(\cdot)$ is Lipschitz in its argument and $b(x)$ is Lipschitz in x, $a \circ b(x)$ is Lipschitz in x. (A4) and (A5) implies that $\hat{g}(x)$ is Lipschitz in $x \in \mathbb{R}^n$. That is,

$$\big\|\hat{g}(x') - \hat{g}(x'')\big\| \leqslant l_{\hat{g}} \big\|x' - x''\big\|$$

for $x', x'' \in \mathbb{R}^n$ and positive constant $l_{\hat{g}}$.

3. Systems with Resetting Errors

In this section, we shall prove a lemma which plays a crucial role to the developed analysis technique, and study the tracking performance of a higher-order ILC in the case where repeatability of the initial condition at each cycle is not perfect but resetting errors for all cycles are bounded by

$$\left\| x_d(0) - x_k(0) \right\| \leqslant c_{x_{d0}}. \tag{8}$$

Consider an updating law in the form of

$$\begin{aligned} u_{k+1}(t) &= \sum_{j=1}^{M} \Psi_j u_{k-j+1}(t) \\ &\quad + \sum_{j=1}^{M} \Phi_{k-j+1}(t) \begin{bmatrix} y_{1,d}(t+\mu_1) - y_{1,k-j+1}(t+\mu_1) \\ \vdots \\ y_{m,d}(t+\mu_m) - y_{m,k-j+1}(t+\mu_m) \end{bmatrix} \end{aligned} \tag{9}$$

where k indicates the number of operation cycle. $M \geqslant 1$ is the order of the updating law, $\Psi_j \in \mathbb{R}^{r\times r}$ and $\Phi_{k-j+1}(t) \in \mathbb{R}^{r\times m}$ are the learning gain matrices.

The updating law uses the data of previous cycles including control inputs and output errors, which provides more freedom in selection of the learning gains. The output errors $y_{q,d}(t+\mu_q) - y_{q,k-j+1}(t+\mu_q)$, $1 \leqslant q \leqslant m$, are used to produces $u_{k+1}(t)$, instead of the output errors at other time, because in the previous cycles the output errors $y_{q,d}(t+\mu_q) - y_{q,k-j+1}(t+\mu_q)$, $1 \leqslant q \leqslant m$, are due to the control action $u_{k-j+1}(t)$. In practical implementation, the initial control inputs $u_0(t), \ldots, u_{M-1}(t)$, can be chosen zero, for convenience.

The following theorem presents one of our main results on applying updating law (9) to systems with relative degree $\{\mu_1, \ldots, \mu_m\}$.

THEOREM 3.1 *Given a realizable trajectory* $y_d(t) = [y_{1,d}(t), \ldots, y_{m,d}(t)]^T$ *with* $y_{q,d}(t)$, $0 \leqslant t \leqslant N + \mu_q$, *let system* (1)–(2) *satisfy Assumptions* (A1)–(A5)*, and updating law* (9) *be applied. If the initial condition at each cycle satisfies* (8) *and for* $0 \leqslant t \leqslant N$,

$$\sum_{j=1}^{M} \Psi_j = I, \tag{10}$$

$$\left\| \Psi_j - \Phi_{k-j+1}(t) D\big(x_{k-j+1}(t)\big) \right\| \leqslant \rho_j, \quad 1 \leqslant j \leqslant M, \tag{11}$$

$$\sum_{j=1}^{M} \rho_j < 1, \tag{12}$$

the asymptotic bound of output error $y_{q,d}(t) - y_{q,k}(t)$, $1 \leqslant q \leqslant m$, *is proportional to* $c_{x_{d0}}$ *for* $\mu_q \leqslant t \leqslant N + \mu_q$ *as* $k \to \infty$. *Moreover, in the absence of resetting errors, i.e.,* $c_{x_{d0}} = 0$, *output error* $y_{q,d}(t) - y_{q,k}(t)$, $1 \leqslant q \leqslant m$, *converges to zero for* $\mu_q \leqslant t \leqslant N + \mu_q$ *as* $k \to \infty$.

The following lemma is needed to aid the proof of Theorem 3.1.

LEMMA 3.1 *Let* $\{a_k\}$ *be a real sequence defined as*

$$a_k \leqslant \rho_1 a_{k-1} + \rho_2 a_{k-2} + \cdots + \rho_M a_{k-M} + d_k, \quad k \geqslant M+1, \tag{13}$$

with initial conditions

$$a_1 = \bar{a}_1, \quad a_2 = \bar{a}_2, \quad \ldots, \quad a_M = \bar{a}_M$$

where d_k *is a specified real sequence. If* $\rho_1, \ldots, \rho_M$ *are nonnegative numbers satisfying*

$$\rho \triangleq \sum_{j=1}^{M} \rho_j < 1$$

then

(i) $d_k \leqslant \bar{d}, k \geqslant M+1$ *implies that*

$$a_k \leqslant \max\{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_M\} + \frac{\bar{d}}{1-\rho}, \quad k \geqslant M+1, \tag{14}$$

and

(ii) $\limsup_{k\to\infty} d_k \leqslant d_\infty$ *implies that*

$$\limsup_{k\to\infty} a_k \leqslant \frac{d_\infty}{1-\rho}. \tag{15}$$

Proof of this lemma can be found in Appendix.

Proof of Theorem 3.1: For the desired initial condition $x_d(0)$, let us denote u_d as a control input satisfying

$$y_d(t) = g\big(x_d(t)\big) \tag{16}$$

where $x_d(t)$ is the corresponding state such that

$$x_d(t+1) = f\big(x_d(t)\big) + B\big(x_d(t)\big)u_d(t). \tag{17}$$

It follows from (6) and (9) that

$$\begin{aligned}\Delta u_{k+1}(t) = {} & \sum_{j=1}^{M}\big[\Psi_j - \Phi_{k-j+1}(t)D\big(x_{k-j+1}(t)\big)\big]\Delta u_{k-j+1}(t) \\ & - \sum_{j=1}^{M}\Phi_{k-j+1}(t)\big\{\hat{g}\big(x_d(t)\big) - \hat{g}\big(x_{k-j+1}(t)\big) \\ & \qquad + \big[D\big(x_d(t)\big) - D\big(x_{k-j+1}(t)\big)\big]u_d(t)\big\}\end{aligned}$$

where $\Delta u_k(t) = u_d(t) - u_k(t)$. Taking norms, and applying the bounds and the Lipschitz conditions, we have

$$\|\Delta u_{k+1}(t)\| \leqslant \sum_{j=1}^{M} \rho_j \|\Delta u_{k-j+1}(t)\| + \sum_{j=1}^{M} c_\Phi c_1 \|\Delta x_{k-j+1}(t)\| \tag{18}$$

where $\Delta x_k(t) = x_d(t) - x_k(t)$, c_Φ is the norm bound for $\Phi_{k-j+1}(\cdot)$, $c_1 = l_{\hat{g}} + l_D c_{u_d}$, and $c_{u_d} = \sup_{0\leqslant t\leqslant N} \|u_d(t)\|$.

From (1) and (17), the state error at the kth iteration can be written as

$$\begin{aligned}\Delta x_k(t) \;=\; & f\big(x_d(t-1)\big) - f\big(x_k(t-1)\big) + \big[B\big(x_d(t-1)\big) - B\big(x_k(t-1)\big)\big]u_d(t-1) \\ & + B\big(x_k(t-1)\big)\Delta u_k(t-1), \quad 1 \leqslant t \leqslant N+1.\end{aligned}$$

Taking norms and using their properties, we have

$$\|\Delta x_k(t)\| \leqslant c_2 \|\Delta x_k(t-1)\| + c_B \|\Delta u_k(t-1)\|, \quad 1 \leqslant t \leqslant N+1, \tag{19}$$

where c_B is the norm bound for $B(\cdot)$ and $c_2 = l_f + l_B c_{u_d}$. By using the lemma in [18], it follows from (19) that

$$\|\Delta x_k(t)\| \leqslant \sum_{i=0}^{t-1} c_2^{t-1-i} c_B \|\Delta u_k(i)\| + c_2^t c_{x_{d0}}, \quad 1 \leqslant t \leqslant N+1. \tag{20}$$

Substituting (20) into (18) yields

$$\begin{aligned}\|\Delta u_{k+1}(t)\| \leqslant & \sum_{j=1}^{M} \rho_j \|\Delta u_{k-j+1}(t)\| + \sum_{j=1}^{M}\sum_{i=0}^{t-1} c_3 \|\Delta u_{k-j+1}(i)\| + c_4 c_{x_{d0}}, \\ & 1 \leqslant t \leqslant N,\end{aligned} \tag{21}$$

where $c_3 = c_\Phi c_B c_1 \varpi$, $c_4 = M c_\Phi c_1 \varpi$, and $\varpi = \max\{1, c_2, \ldots, c_2^N\}$.

To estimate the control input errors, let us define

$$\begin{aligned}\vartheta \;&=\; \max_{0\leqslant t\leqslant N} \max\big\{\|\Delta u_0(t)\|, \ldots, \|\Delta u_{M-1}(t)\|\big\}, \\ \alpha \;&=\; \frac{c_4}{1-\rho}\, c_{x_{d0}}, \\ \beta \;&=\; \frac{M c_3}{1-\rho} + 1.\end{aligned}$$

For the first instant $t = 0$, we apply Lemma 3.1 to (18) to yield

$$\begin{aligned}\|\Delta u_k(0)\| \;&\leqslant\; \max\big\{\|\Delta u_0(0)\|, \ldots, \|\Delta u_{M-1}(0)\|\big\} + \frac{c_4}{1-\rho}\, c_{x_{d0}} \\ &\leqslant\; \vartheta + \alpha, \quad k \geqslant M, \\ \limsup_{k\to\infty} \|\Delta u_k(0)\| \;&\leqslant\; \frac{c_4}{1-\rho}\, c_{x_{d0}} = \alpha.\end{aligned}$$

For the second instant $t = 1$, we apply Lemma 3.1 to (21) to yield

$$\begin{aligned}\left\|\Delta u_k(1)\right\| &\leqslant \max\left\{\left\|\Delta u_0(1)\right\|, \ldots, \left\|\Delta u_{M-1}(1)\right\|\right\} + \frac{Mc_3}{1-\rho}(\vartheta + \alpha) + \frac{c_4}{1-\rho} c_{x_{d0}} \\ &\leqslant (\vartheta + \alpha)\beta, \quad k \geqslant M,\end{aligned}$$

$$\limsup_{k\to\infty}\left\|\Delta u_k(1)\right\| \leqslant \frac{Mc_3}{1-\rho}\alpha + \frac{c_4}{1-\rho} c_{x_{d0}} = \alpha\beta.$$

Now, by induction, we assume that for the instants $t = 0, 1, \ldots, l-1$,

$$\begin{aligned}\left\|\Delta u_k(t)\right\| &\leqslant (\vartheta + \alpha)\beta^t, \quad k \geqslant M, \\ \limsup_{k\to\infty}\left\|\Delta u_k(t)\right\| &\leqslant \alpha\beta^t.\end{aligned}$$

For the instant $t = l$, we apply Lemma 3.1 to (21) to yield

$$\begin{aligned}\left\|\Delta u_k(l)\right\| &\leqslant \max\left\{\left\|\Delta u_0(l)\right\|, \ldots, \left\|\Delta u_{M-1}(l)\right\|\right\} \\ &\quad + \frac{Mc_3}{1-\rho}\left[\vartheta + \alpha + \cdots + (\vartheta + \alpha)\beta^{l-1}\right] + \frac{c_4}{1-\rho} c_{x_{d0}} \\ &\leqslant (\vartheta + \alpha)\left[1 + (\beta - 1)\left(1 + \beta + \cdots + \beta^{l-1}\right)\right] \\ &= (\vartheta + \alpha)\beta^l, \quad k \geqslant M, \\ \limsup_{k\to\infty}\left\|\Delta u_k(l)\right\| &\leqslant \frac{Mc_3}{1-\rho}\left[\alpha + \cdots + \alpha\beta^{l-1}\right] + \frac{c_4}{1-\rho} c_{x_{d0}} \\ &= \alpha\left[1 + (\beta - 1)\left(1 + \beta + \cdots + \beta^{l-1}\right)\right] \\ &= \alpha\beta^l.\end{aligned}$$

Therefore, for $0 \leqslant t \leqslant N$,

$$\left\|\Delta u_k(t)\right\| \leqslant (\vartheta + \alpha)\beta^t, \quad k \geqslant M, \tag{22}$$

$$\limsup_{k\to\infty}\left\|\Delta u_k(t)\right\| \leqslant \alpha\beta^t. \tag{23}$$

To evaluate the state error $\Delta x_k(t)$, we use (20), (22) and (23) to obtain

$$\left\|\Delta x_k(t)\right\| \leqslant \frac{\beta^t - 1}{\beta - 1}\varpi c_B(\vartheta + \alpha) + \varpi c_{x_{d0}}, \quad 1 \leqslant t \leqslant N + 1, \tag{24}$$

$$\limsup_{k\to\infty}\left\|\Delta x_k(t)\right\| \leqslant \frac{\beta^t - 1}{\beta - 1}\varpi c_B\alpha + \varpi c_{x_{d0}}, \quad 1 \leqslant t \leqslant N + 1. \tag{25}$$

Note that (24) and (25) are also true for $t = 0$ since $\|\Delta x_k(0)\| \leqslant c_{x_{d0}}$. To obtain the result for the output error we can use the relation

$$\left\|y_{q,d}(t + \mu_q) - y_{q,k}(t + \mu_q)\right\| \leqslant c_1\left\|\Delta x_k(t)\right\| + c_D\left\|\Delta u_k(t)\right\|$$

where $\mu_q \leqslant t \leqslant N + \mu_q$. This completes the proof. ■

Remark 3.1. For the case $M = 1$, inequality (11) reduces to

$$\left\| I - \Phi_k(t) D\big(x_k(t)\big) \right\| \leqslant \rho < 1.$$

Only the learning gain matrix $\Phi_k(t)$ is the design parameter. For the Mth order ILC, the order M and the parameter matrices Ψ_j, and $\Phi_{k-j+1}(t)$ are needed to choose. It indicates that the Mth order ILC provides more freedom for the control design, which can be responsible for the convergence speed. The design issue based on the derived sufficient condition highly depends on knowledge of the system dynamics. Obviously, the difficulty of the design task increases with the learning order. To deal with resetting errors, one way to avoid the limitation is to apply it in the selective learning manner [14].

Remark 3.2. The control input $u_k(t)$ has no effect on the outputs $y_{q,k}(t)$ for $0 \leqslant t \leqslant \mu_q - 1$, $1 \leqslant q \leqslant m$. The robustness performance of the outputs are ensured by the repositioning requirement $\|\Delta x_k(0)\| \leqslant c_{x_{d0}}$. Namely,

$$\left\| y_{q,d}(t) - y_{q,k}(t) \right\| \leqslant \left\| g_q \circ f^t\big(x_d(0)\big) - g_q \circ f^t\big(x_k(0)\big) \right\| \leqslant l_{fg} c_{x_{d0}}$$

for some positive constant l_{fg}.

Remark 3.3. The discrete-time λ-norm plays a crucial role in the ILC theoretical analysis, e.g., see [1,4,13,14,18]. In the above proof, a new analysis technique is developed based on Lemma 3.1, which is also different from that developed by [6,19].

4. Extension to Systems with Initial Shifts

In practice, there exists the case where the controlled system does not reset the initial condition to the desired one. Instead, there is an initial shift such that $\|x_d(0) - x_k(0)\| \geqslant c_{x_{d0}}$. One example is that the system output is expected to track a step function from the resetting position. In this section, we shall study the tracking performance in the presence of initial shifts described by

$$\left\| x_0 - x_k(0) \right\| \leqslant c_{x_0} \tag{26}$$

where x_0 is a fixed point. As c_{x_0} tends to zero, we shall show that the iterative learning control will produce a transient trajectory from the system resetting position to the desired trajectory. This transient trajectory can be specified by the initial shift, x_0, which joins the desired trajectory by the delay time.

THEOREM 4.1 *Let system* (1)–(2) *satisfy Assumptions* (A1)–(A5) *and updating law* (9) *be applied. If the initial condition at each cycle satisfies* (26) *and the learning gain is chosen satisfying* (10)–(12)*, the asymptotic bound of error $y_q^*(t) - y_{q,k}(t)$, $1 \leqslant q \leqslant m$, is proportional to c_{x_0} for $0 \leqslant t \leqslant N + \mu_q$ as $k \to \infty$. Moreover, in the absence of initial shifts, i.e., $c_{x_0} = 0$, the output error $y_q^*(t) - y_{q,k}(t)$, $1 \leqslant q \leqslant m$, converges to zero for*

$0 \leqslant t \leqslant N + \mu_q$ *as* $k \to \infty$, *where* $y^*(t) = [y_1^*(t), \ldots, y_m^*(t)]^T$ *and for* $1 \leqslant q \leqslant m$,

$$y_q^*(t) = \begin{cases} g_q \circ f^t(x_0), & 0 \leqslant t \leqslant \mu_q - 1, \\ y_{q,d}(t), & \mu_q \leqslant t \leqslant N + \mu_q. \end{cases} \tag{27}$$

Proof: Given an initial condition x_0, let u^* denote the control input satisfying

$$y^*(t) = g\big(x^*(t)\big), \tag{28}$$

$$x^*(t+1) = f\big(x^*(t)\big) + B\big(x^*(t)\big)u^*(t), \quad x^*(0) = x_0, \tag{29}$$

where $x^*(t)$ is the corresponding state. In view of $y^*(t)$ defined in (27), (9) can be written as

$$u_{k+1}(t) = \sum_{j=1}^{M} \Psi_j u_{k-j+1}(t) + \sum_{j=1}^{M} \Phi_{k-j+1}(t) \begin{bmatrix} y_1^*(t+\mu_1) - y_{1,k-j+1}(t+\mu_1) \\ \vdots \\ y_m^*(t+\mu_m) - y_{1,k-j+1}(t+\mu_m) \end{bmatrix}$$

which leads to

$$\begin{aligned} \Delta u_{k+1}^*(t) = & \sum_{j=1}^{M} \big[\Psi_j - \Phi_{k-j+1}(t) D\big(x_{k-j+1}(t)\big)\big] \Delta u_{k-j+1}^*(t) \\ & - \sum_{j=1}^{M} \Phi_{k-j+1}(t) \big\{\hat{g}\big(x^*(t)\big) - \hat{g}\big(x_{k-j+1}(t)\big) \\ & \qquad + \big[D\big(x^*(t)\big) - D\big(x_{k-j+1}(t)\big)\big] u^*(t)\big\} \end{aligned}$$

where $\Delta u_k^*(t) = u^*(t) - u_k(t)$. Taking norms, and applying the bounds and the Lipschitz conditions, we have

$$\big\|\Delta u_{k+1}^*(t)\big\| \leqslant \sum_{j=1}^{M} \rho_j \big\|\Delta u_{k-j+1}^*(t)\big\| + \sum_{j=1}^{M} c_\Phi c_1 \big\|\Delta x_{k-j+1}^*(t)\big\| \tag{30}$$

where $\Delta x_k^*(t) = x^*(t) - x_k(t)$, c_Φ is the norm bound for $\Phi_{k-j+1}(\cdot)$, $c_1 = l_{\hat{g}} + l_D c_{u^*}$, and $c_{u^*} = \sup_{0 \leqslant t \leqslant N} \|u^*(t)\|$.

Through the same derivation parallel to arrive at (19), we have

$$\big\|\Delta x_k^*(t)\big\| \leqslant c_2 \big\|\Delta x_k^*(t-1)\big\| + c_B \big\|\Delta u_k^*(t-1)\big\|, \quad 1 \leqslant t \leqslant N+1, \tag{31}$$

where c_B is the norm bound for $B(\cdot)$ and $c_2 = l_f + l_B c_{u^*}$. By using the lemma in [18], it follows from (31) that

$$\big\|\Delta x_k^*(t)\big\| \leqslant \sum_{i=0}^{t-1} c_2^{t-1-i} c_B \big\|\Delta u_k^*(i)\big\| + c_2^t c_{x_0}, \quad 1 \leqslant t \leqslant N+1. \tag{32}$$

Substituting (32) into (30) yields

$$\|\Delta u_{k+1}^*(t)\| \leqslant \sum_{j=1}^{M} \rho_j \|\Delta u_{k-j+1}^*(t)\| + \sum_{j=1}^{M}\sum_{i=0}^{t-1} c_3 \|\Delta u_{k-j+1}^*(i)\| + c_4 c_{x_0},$$
$$1 \leqslant t \leqslant N, \tag{33}$$

where $c_3 = c_\Phi c_B c_1 \varpi$, $c_4 = M c_\Phi c_1 \varpi$, and $\varpi = \max\{1, c_2, \ldots, c_2^N\}$.

Equation (33) corresponds to Equation (21) in the proof of Theorem 3.1. The rest of the proof is exactly the same as that of Theorem 3.1 after Equation (21). ■

Remark 4.1. Referring to the definition of $y_q^*(t)$ in (27), $y_q^*(t) = y_{q,d}(t)$, $\mu_q \leqslant t \leqslant N + \mu_q$, $1 \leqslant q \leqslant m$. Theorem 4.1 shows that suitable choice of the learning gains leads to the convergence of the system output $y_{q,k}(t)$ to the neighborhood of $y_{q,d}(t)$ for $\mu_q \leqslant t \leqslant N+\mu_q$ even when $\|x_d(0) - x_0\| \gg c_{x_{d0}}$. As c_{x_0} tends to zero, $y_{q,k}(t)$ converges to $y_{q,d}(t)$ is achieved for $\mu_q \leqslant t \leqslant N + \mu_q$. It is interesting to note that this result is true for any value of x_0.

5. Simulation Illustrations

In this section, numerical simulation is conducted to illustrate the theoretical results of this paper. Consider the nonlinear discrete-time system described by

$$\begin{aligned}
x_1(t+1) &= 0.5\sin\big(x_2(t)\big) + \big(1 + 0.5\cos\big(x_1(t)\big)\big)u(t),\\
x_2(t+1) &= x_1(t),\\
x_3(t+1) &= x_2(t),\\
x_4(t+1) &= x_3(t),\\
y(t) &= x_4(t).
\end{aligned}$$

The system has relative degree of four. Let the desired trajectory be given as

$$y_d(t) = \frac{t}{N}\left(1 - \frac{t}{N}\right), \quad 0 \leqslant t \leqslant N+4,$$

where $N = 96$.

Case 1. The simulation is conducted by using

(1) the first-order updating law with

$$\Phi_k(t) = \frac{0.95}{1 + 0.5\cos(x_{1,k}(t))};$$

(2) the second-order updating law with

$$\Psi_1 = 0.8, \quad \Psi_2 = 0.2, \quad \Phi_k(t) = 0.8\,\frac{0.95}{1 + 0.5\cos(x_{1,k}(t))} \quad \text{and}$$

$$\Phi_{k-1}(t) = 0.2\,\frac{0.95}{1 + 0.5\cos(x_{1,k-1}(t))}.$$

For both updating laws, the initial control inputs are chosen as $u_0(t) = 0$, $0 \leqslant t \leqslant N$. Define the performance index $J_k = \max_{0 \leqslant t \leqslant N+4} |y_d(t) - y_k(t)|$. The iteration stops when the tracking index $J_k < 0.001$. Figure 1 shows the resultant output errors when the initial condition at each iteration is chosen as $x_{i,k}(0) = y_d(4 - i)$, $i = 1, 2, 3, 4$, matching the desired initial condition. This figure depicts the faster convergence rate achieved by using the second-order updating law. Then let the initial condition be $x_{i,k}(0) = y_d(4-i) + 0.01\,\text{randn}$, $i = 1, 2, 3, 4$. The randn is a generator of random scalar with normal distribution, mean $= 0$, and variance $= 1$ (white Gaussian noise). The repetitions are conducted until $k = 100$. Figure 2 indicates the output errors by the second-order updating law.

Case 2. To examine convergence performance of the second-order updating law in the presence of an initial shift, let the initial condition at each iteration are chosen as $x_{i,k}(0) = 0.1$, $i = 1, 2, 3, 4$. Define the performance index $J_k = \max_{4 \leqslant t \leqslant N+4} |y_d(t) - y_k(t)|$. The requirement $J_k < 0.001$ of tracking performance is achieved at the 8th iteration. Figure 3

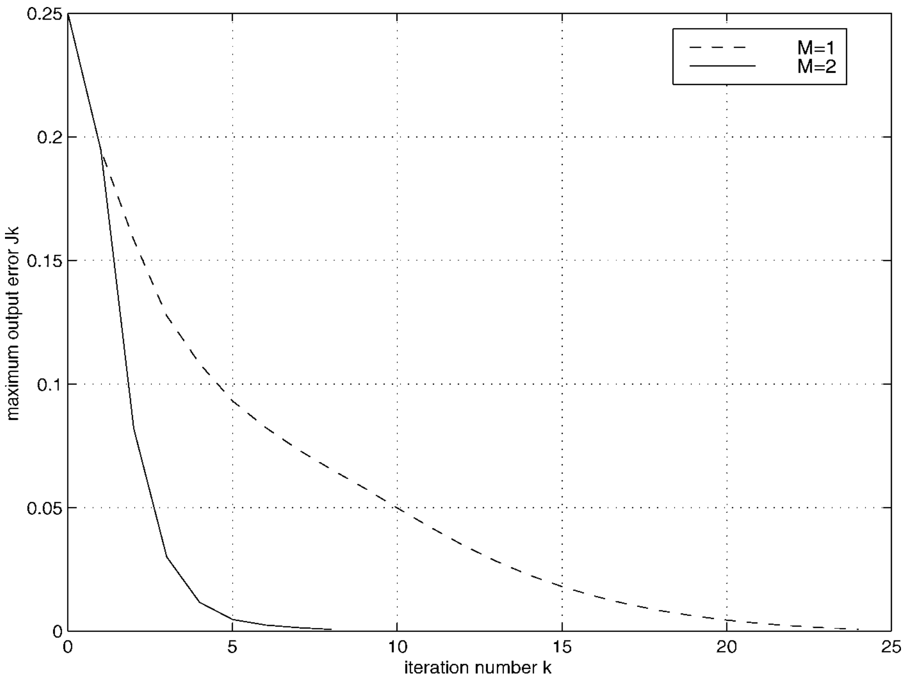

Figure 1. Convergence rate comparison for $M = 1, 2$ in the absence of resetting errors.

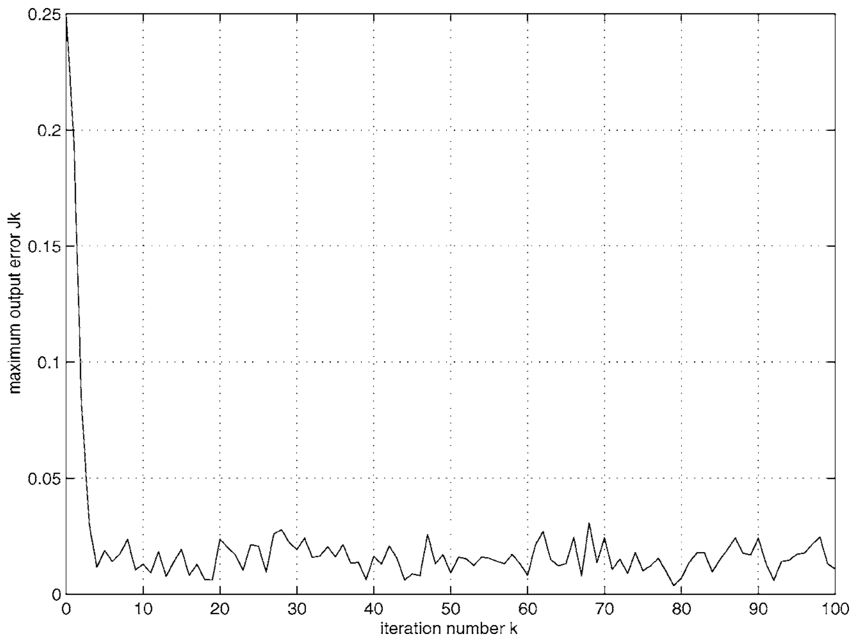

Figure 2. Output errors ($M = 2$) in the presence of resetting errors.

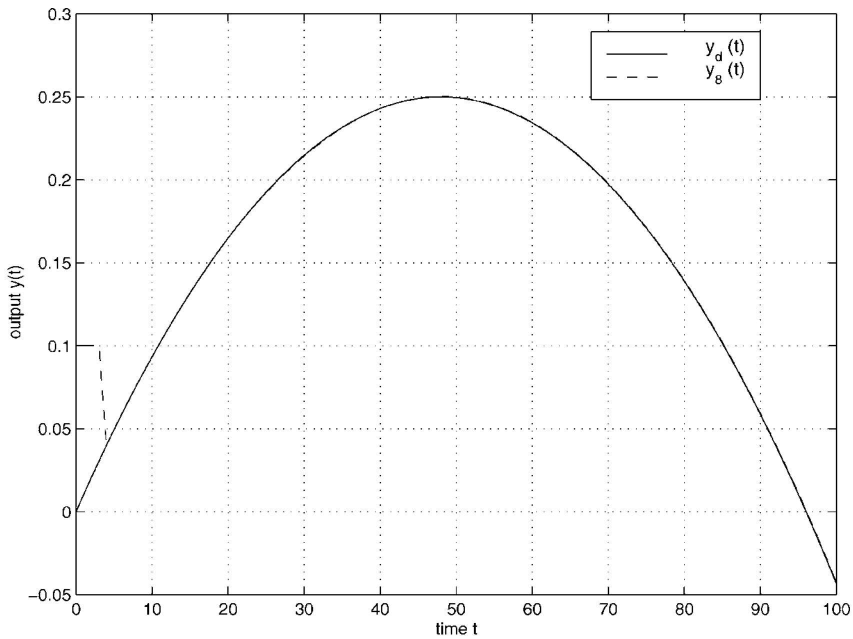

Figure 3. Output trajectory ($M = 2$) in the presence of an initial shift.

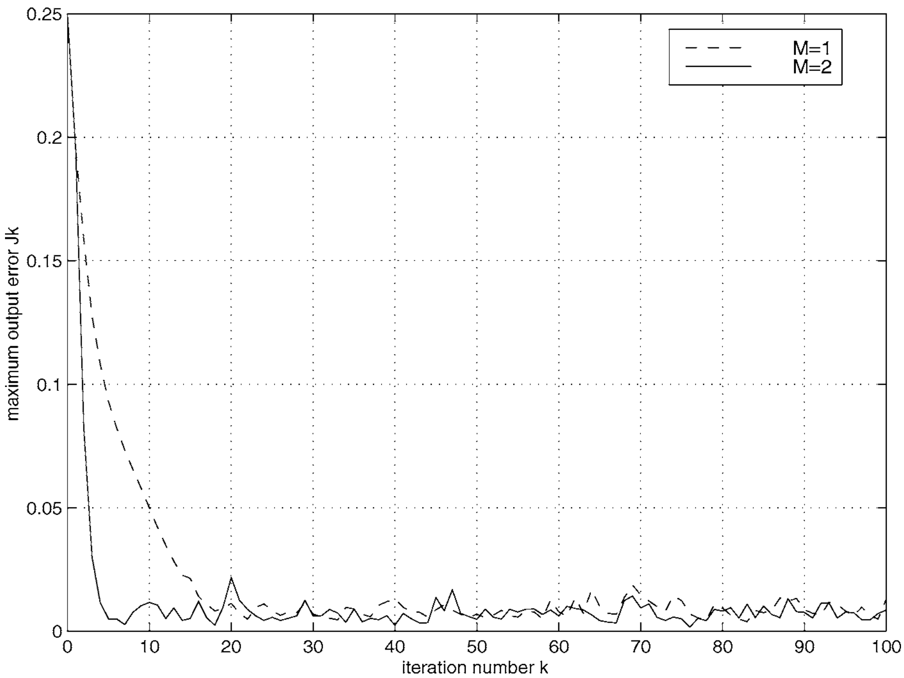

Figure 4. Tracking performance comparison for $M = 1, 2$ in the presence of initial shifts.

shows resulting output trajectory and the desired trajectory and the output trajectory is observed to converge to the desired one for $4 \leqslant t \leqslant N + 4$. For performance comparison of the first-order and the second-order updating laws in the presence of initial shifts, let the initial condition be $x_{i,k}(0) = 0.1 + 0.01\,\text{randn}$, $i = 1, 2, 3, 4$. The repetitions are conducted until $k = 100$. Figure 4 indicates that regardless of the tracking on the interval $0 \leqslant t \leqslant 3$, better tracking performance is achieved by the second-order updating law.

6. Conclusion

The proposed higher-order ILC is shown applicable to a class of nonlinear discrete-time systems with well-defined relative degree. Sufficient condition for the learning gain selection is derived by applying a new analysis technique instead of using discrete-time λ-norm. The bound on the output errors is shown proportional to the bound on resetting errors. The same condition is also proved sufficient to guarantee the boundedness of the output error for the case where initial shifts exist. Furthermore, in the presence of an initial shift, the system output is ensured to converge to the desired trajectory with a transient trajectory governed by the initial shift.

Appendix

Proof of Lemma 3.1: (i) Choosing $a_{k_1} = \max\{a_{k-1}, \dots, a_{k-M}\}$ yields

$$a_k \leqslant \rho a_{k_1} + \bar{d}.$$

Similarly, choosing $a_{k_2} = \max\{a_{k_1-1}, \dots, a_{k_1-M}\}$ yields

$$a_{k_1} \leqslant \rho a_{k_2} + \bar{d}$$

and hence

$$a_k \leqslant \rho^2 a_{k_2} + \rho\bar{d} + \bar{d}.$$

Repeating m times, one obtains

$$a_k \leqslant \rho^m a_{k_m} + \rho^{m-1}\bar{d} + \rho^{m-2}\bar{d} + \cdots + \rho\bar{d} + \bar{d} \leqslant \rho^m a_{k_m} + \frac{1-\rho^m}{1-\rho}\,\bar{d}$$

where

$$k_m \leqslant M, \qquad \left\lceil \frac{k-1}{M} \right\rceil \leqslant m \leqslant k - M.$$

Moreover, let $\bar{a} = \max\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_M\}$, inequality (14) is established by observing

$$a_k \leqslant \rho^m \bar{a} + \frac{1-\rho^m}{1-\rho}\,\bar{d} \leqslant \rho^{\lceil (k-1)/M \rceil}\bar{a} + \frac{1-\rho^{k-M}}{1-\rho}\,\bar{d},$$

(ii) Since $\limsup_{k\to\infty} d_k \leqslant d_\infty$, there exists an integer K such that $d_k \leqslant d_\infty$ as $k > K$. Thus

$$a_k \leqslant \rho_1 a_{k-1} + \rho_2 a_{k-2} + \cdots + \rho_M a_{k-M} + d_\infty, \quad k > K.$$

Similar to the derivation of (i), for $k > K$, one obtains

$$a_k \leqslant \rho^m a_{k_m} + \rho^{m-1} d_\infty + \rho^{m-2} d_\infty + \cdots + \rho d_\infty + d_\infty \leqslant \rho^m a_{k_m} + \frac{1-\rho^m}{1-\rho}\,d_\infty$$

where

$$k_m \leqslant K, \qquad \left\lceil \frac{k-K+M-1}{M} \right\rceil \leqslant m \leqslant k - K.$$

Moreover, let $\bar{a} = \max\{a_{K-M+1}, \dots, a_K\}$, inequality (15) is established by observing

$$a_k \leqslant \rho^m \bar{a} + \frac{1-\rho^m}{1-\rho}\,d_\infty \leqslant \rho^{\lceil (k-K+M-1)/M \rceil}\bar{a} + \frac{1-\rho^{k-M}}{1-\rho}\,d_\infty.$$

This completes the proof. ■

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