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Anticipatory Iterative Learning Control for Nonlinear Systems with Arbitrary Relative Degree

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***Abstract*—In this note, the anticipatory iterative learning control is extended to a class of nonlinear continuous-time systems without restriction on relative degree. The learning algorithm calculates the required input action for the next operation cycle based on the pair of input action taken and its resultant variables. The tracking error convergence performance is examined under input saturation being taken into account. The learning algorithm is shown effective even if differentiation of any order from the tracking error is not used.**

***Index Terms*—Convergence, learning control, nonlinear systems, relative degree.**

I. Introduction

Recently, rigorous analyses of continuous-time iterative learning control (ILC) have been developed, see, for example, [2]–[10]. In particular, a fundamental characteristic of a class of learning control design methodologies is examined in [5], which clarifies the necessity of the use of error derivative for systems without direct transmission term. In [6], this characteristic is further clarified for nonlinear continuous-time systems where error derivatives, the highest order is equal to the relative degree of the systems, are used to update the control input. ILC using the highest-order error derivatives only is termed D-type ILC. Numerical calculations might be required to obtain error derivatives for the implementation. However, the signals obtained by numerical differentiation will be very noisy if the measurement is contaminated with noise. ILC without using differentiation is referred to as P-type ILC. Several technical analyzes of P-type ILC are presented for nonlinear continuous-time systems with relative degree one, by imposing somewhat strict restriction on system dynamics, for example, the passivity property [11] and the boundedness of derivative of the input-output coupling matrix [12],[13]. Most recently, in [1], a fundamental concept is introduced in parallel to the two basic schemes: D-type and P-type ILCs. This design approach has the anticipatory characteristic of the D-type ILC and the simplicity like P-type ILC. Results have been developed again for nonlinear continuous-time systems with relative degree one and experimental results are obtained in robotic systems. This approach is also studied in the form of noncausal filtering [9]. In this note, the anticipatory learning algorithm [1] is applied to systems with arbitrary relative degree. A definition of extended relative degree is presented to explore a causal property of the systems under consideration. The tracking error convergence results are established.

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II. Problem Formulation

Consider the class of nonlinear continuous-time systems described by the state-space equations

$$\dot{x}(t) = f(x(t)) + B(x(t))u(t) \tag{1}$$
$$y(t) = g(x(t)) \tag{2}$$

where $x \in R^n$, $u \in R^r$ and $y \in R^m$ denote the state, control input and output of the system, respectively. The functions $f(\cdot) \in R^n$, $B(\cdot) = [b_1(\cdot), \cdots, b_r(\cdot)] \in R^{n\times r}$ and $g(\cdot) = [g_1(\cdot), \cdots, g_m(\cdot)]^T \in R^m$ are smooth in their domain of definition and are known of certain properties only. This system performs repetitive operations within a finite time interval $[0, T]$. For each fixed $x(0)$, S denotes a mapping from $(x(0), u(t), t \in [0, T])$ to $(x(t), t \in [0, T])$ and O a mapping from $(x(0), u(t), t \in [0, T])$ to $(y(t), t \in [0, T])$. In these notations, $x(\cdot) = S(x(0), u(\cdot))$ and $y(\cdot) = O(x(0), u(\cdot))$. The control problem to be solved is formulated as follows. Given a realizable trajectory $y_d(t), t \in [0, T]$ and a tolerance error bound $\varepsilon > 0$, find a control input $u(t), t \in [0, T]$, by applying an ILC technique, so that the error between the output trajectory $y(t)$ and the desired one $y_d(t)$ is within the tolerance error bound, i.e., $\|y_d(t) - y(t)\| < \varepsilon, t \in [0, T]$, where $\|\cdot\|$ is the vector norm defined as $\|a\| = \max_{1\le i\le n} |a_i|$ for an n-dimensional vector $a = [a_1, \cdots, a_n]^T$. Throughout the paper, for a matrix $A = \{a_{ij}\} \in R^{m\times n}$, the induced norm $\|A\| = \max_{1\le i\le m} \Sigma_{j=1}^n |a_{ij}|$. To solve this problem, we use the ILC in the form of the following anticipatory updating law [1]:

$$v_{k+1}(t) = \begin{cases} u_k(t) + \Gamma_k(t)e_k(t+\sigma), & \text{if } t \in [0, T-\sigma] \\ v_k(T-\sigma), & \text{if } t \in (T-\sigma, T] \end{cases} \tag{3}$$
$$u_k(t) = \text{sat}\,(v_k(t)) \tag{4}$$

where

$\sigma > 0$	small number;
k	number of operation cycle;
$e_k(t) = y_d(t) - y_k(t)$	output or tracking error;
$\Gamma_k(t) \in R^{r\times m}$	learning gain matrix piecewise continuous and bounded.

This updating law is based on the causal relationship between the control input and the system output to be specified in the next section. The time shift ahead in the tracking error installs the anticipatory characteristic in the updating law, where actuator saturation is taken into account. The input saturation function sat : $R^r \to R^r$ is defined as $\text{sat}(v) = [\text{sat}(v_1), \cdots, \text{sat}(v_r)]^T$, where

$$\text{sat}(v_p) = \begin{cases} v_p, & \text{if } |v_p| \le \delta_p \\ \text{sgn}(v_p)\delta_p, & \text{if } |v_p| > \delta_p \end{cases}$$

for the input saturation bound $\delta_p > 0, p = 1, \cdots, r$. Define $\delta = \max_{1 \le p \le r} \{\delta_p\}$.

Assumptions are as follows:

A1) for each fixed $x_k(0)$, the mappings S and O are one to one;

A2) the desired trajectory $y_d(t)), t \in [0, T]$ is achievable by an input within saturation bounds, i.e., $u_d(t) = \text{sat}\ (u_d(t)), t \in [0, T]$;

A3) there exists a compact set $X \subset R^n$ such that the system state $x(t), t \in [0, T]$ produced by any input $u(t) \in U, t \in [0, T]$ belongs to X, where $U \subset R^r$ is a bounded set, i.e., $x(t) \in X, t \in [0, T]$;

A4) the operations start from the initial condition $x_k(0) = x_d(0)$ where $x_d(0)$ is the initial condition corresponding to the desired trajectory.

Remark 2.1: For a realizable trajectory $y_d(t), t \in [0, T]$, (A1) implies that there exists a unique control input $u_d(t), t \in [0, T]$ such that $y_d(t) = g(x_d(t))$ and $\dot{x}_d(t) = f(x_d(t)) + B(x_d(t))u_d(t)$ where $x_d(t), t \in [0, T]$ is the corresponding state. ILC with an input saturator can be still effective by assuming A2) as argued in [1], [14]. The saturation bounds can be set in accordance with actuator limitations. Assumption A3) is reasonable for systems which have no finite escape time on $[0, T]$ and most practical systems driven by a bounded input will not diverge in a finite-time interval due to energy limitation.

The following definition extends the relative degree concept in [15]. Here, the derivative of a scalar function $g(x)$ along a vector $f(x)$ is defined as $L_f g(x) = (\partial g(x)/\partial x) f(x)$. The repeated derivatives along the same vector are $L_f^i g(x) = L_f(L_f^{i-1} g(x)), L_f^0 g(x) = g(x)$. In addition, the derivative of $g(x)$ taken first along $f(x)$ and then along a vector $b(x)$ is $L_b L_f g(x) = (\partial(L_f g(x))/\partial x) b(x)$.

Definition 2.1: Extended relative degree of the system (1) and (2) is associated with a set of integers $\{\eta_1, \cdots, \eta_m\}$ such that

$$L_{b_p} L_f^i g_q(x(t)) = 0, \qquad 0 \le i \le \eta_q - 2, \quad 1 \le p \le r$$
$$1 \le q \le m$$

and the $m \times r$ matrix, shown in the equation at the bottom of the page, has full-column rank for $t \in [0, T]$ and $x(t) \in X$.

Remark 2.2: The relative degree of a continuous-time system is the times of differentiation of the output so that the terms involving the input appear [15]. The extended relative degree of the same system is the integration times of certain terms so that the output $y(t + \sigma)$ is dynamically related with the input $u(t)$. Definition 2.1 allows that for some states at some instants, $L_{b_p} L_f^{\eta_q - 1} g_q(x(t)) = 0, 1 \le p \le r$ and/or $L_{b_p} L_f^{\eta_q - 1} g_q(x(t)) = 0, 1 \le q \le m$, and the number of outputs can be greater than the number of inputs.

III. Convergence Analysis

In this section, we shall examine the convergence performance when the proposed updating law (3)–(4) is applied to systems (1)–(2) with extended relative degree $\{\eta_1, \cdots, \eta_m\}$. For simplicity, the result is presented for the single-input–single-output (SISO) case of the nonlinear systems before it is extended to the multiple-input–multiple-output (MIMO) case.

A. Single-Input–Single-Output Systems

The SISO nonlinear system under consideration takes the form of (1)–(2) with $u(t)$ and $y(t)$ being the scalar input and the scalar output, respectively, $B(\cdot) = b(\cdot) \in R^n$, and $g(\cdot) \in R$ being smooth in their domain of definition. The updating law is (3)–(4) with $\Gamma_k(t) = \gamma_k(t)$ being the scalar learning gain.

Remark 3.1: The relative degree of the SISO nonlinear system is the integer μ such that [15]

$$L_b L_f^i g(x) = 0, 0 \le i \le \mu - 2$$
$$L_b L_f^{\mu - 1} g(x) \ne 0.$$

However, the SISO nonlinear system has extended relative degree η, if

$$L_b L_f^i g(x(t)) = 0, \qquad 0 \le i \le \eta - 2$$

and

$$\int_t^{t+\sigma} \int_t^{t_1} \cdots \int_t^{t_{\eta-1}} L_b L_f^{\eta-1} g(x(t_\eta))\, dt_\eta \cdots dt_1 \ne 0$$

where $L_b L_f^{\eta-1} g(x(t)) = 0$ is allowed for some states at some instants. Obviously, the system has extended relative degree μ if the relative degree of the system is μ.

Remark 3.2: If the SISO system has extended relative degree η, the system output at the instant $t + \sigma, t \in [0, T - \sigma]$ can be written as

$$y(t + \sigma) = g(x(t)) + \int_t^{t+\sigma} L_f g(x(t_1))\, dt_1$$

where the second term can be expressed as, keeping in mind of the extended relative degree of the system

$$\int_t^{t+\sigma} L_f g(x(t_1))\, dt_1$$
$$= \sigma L_f g(x(t)) + \int_t^{t+\sigma} \int_t^{t_1} L_f^2 g(x(t_2))\, dt_2\, dt_1.$$

$$\begin{bmatrix} \int_t^{t+\sigma} \int_t^{t_1} \cdots \int_t^{t_{\eta_1 - 1}} [L_{b_1} L_f^{\eta_1 - 1} g_1(x(t_{\eta_1})), \cdots, L_{b_r} L_f^{\eta_1 - 1} g_1(x(t_{\eta_1}))]\, dt_{\eta_1} \cdots dt_1 \\ \vdots \\ \int_t^{t+\sigma} \int_t^{t_1} \cdots \int_t^{t_{\eta_m - 1}} [L_{b_1} L_f^{\eta_m - 1} g_m(x(t_{\eta_m})), \cdots, L_{b_r} L_f^{\eta_m - 1} g_m(x(t_{\eta_m}))]\, dt_{\eta_m} \cdots dt_1 \end{bmatrix}$$

Similarly, the second term can be rewritten repetitively until $\eta - 2$ differentiation as

$$\begin{aligned}&\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-3}} L_f^{\eta-2}g(x(t_{\eta-2}))\,dt_{\eta-2}\cdots\,dt_1\\&\quad=\frac{\sigma^{\eta-2}}{(\eta-2)!}L_f^{\eta-2}g(x(t))+\int_t^{t+\sigma}\int_t^{t_1}\\&\qquad\cdots\int_t^{t_{\eta-2}} L_f^{\eta-1}g(x(t_{\eta-1}))\,dt_{\eta-1}\cdots\,dt_1.\end{aligned}$$

Then, the control appears in the integration as follows:

$$\begin{aligned}&\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-2}} L_f^{\eta-1}g(x(t_{\eta-1}))\,dt_{\eta-1}\cdots\,dt_1\\&\quad=\frac{\sigma^{\eta-1}}{(\eta-1)!}L_f^{\eta-1}g(x(t))\\&\qquad+\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-1}}[L_f^{\eta}g(x(t_\eta))\\&\qquad+L_bL_f^{\eta-1}g(x(t_\eta))u(t_\eta)]\,dt_\eta\cdots\,dt_1.\end{aligned}$$

Thus, the system output at the instant $t+\sigma$ can be finally written as

$$\begin{aligned}y(t+\sigma)&=g(x(t))+\sigma L_fg(x(t))\\&\quad+\cdots+\frac{\sigma^{\eta-1}}{(\eta-1)!}L_f^{\eta-1}g(x(t))\\&\quad+\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-1}}\{L_f^{\eta}g(x(t_\eta))\\&\quad+L_bL_f^{\eta-1}g(x(t_\eta))u(t_\eta)\}\,dt_\eta\cdots\,dt_1.\end{aligned}\tag{5}$$

Equation (5) shows that $\{u(t), y(t+\sigma)\}$ is a pair of dynamically related cause and effect. Thus, the updating law (3)–(4) is effect-driven, and it has the anticipatory nature capturing the trend/directional information. However, no error differentiation is required in the updating law.

Theorem 3.1: Given a desired trajectory $y_d(t), t \in [0, T]$ for the SISO system (1)–(2) with extended relative degree η, let the system satisfy assumptions A1)–A4) and use the updating law (3)–(4). If

$$\left\|1-\gamma_k(t)\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-1}} L_bL_f^{\eta-1}g(x_k(t_\eta))\,dt_\eta\cdots\,dt_1\right\|\le\rho<1\tag{6}$$

the system output converges to the desired trajectory in the sense of

$$\limsup_{k\to\infty}\sup_{t\in[0,T]}\|e_k(t)\|\le\beta\delta\max\left\{\frac{\sigma^\eta}{\eta!},\sigma\right\}$$

where β is a positive constant to be defined.

Proof: We first evaluate the error $u_d(t)-v_k(t)$ for $t\in[0,T-\sigma]$. It follows from (3) and (5) that:

$$\begin{aligned}\Delta v_{k+1}(t)&=\left(1-\gamma_k(t)\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-1}} L_bL_f^{\eta-1}g(x_k(t_\eta))\,dt_\eta\cdots\,dt_1\right)\Delta u_k(t)\\&\quad-\gamma_k(t)(\xi_k(t)+\zeta_k(t)+\varpi_k(t))\end{aligned}$$

where $\Delta v_k(t)=u_d(t)-v_k(t), \Delta u_k(t)=u_d(t)-u_k(t)$ and

$$\begin{aligned}\xi_k(t)&=g(x_d(t))-g(x_k(t))\\&\quad+\sigma(L_fg(x_d(t))-L_fg(x_k(t)))+\cdots\\&\quad+\frac{\sigma^{\eta-1}}{(\eta-1)!}(L_f^{\eta-1}g(x_d(t))-L_f^{\eta-1}g(x_k(t)))\\\zeta_k(t)&=\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-1}}[L_f^{\eta}g(x_d(t_\eta))-L_f^{\eta}g(x_k(t_\eta))\\&\quad+(L_bL_f^{\eta-1}g(x_d(t_\eta))\\&\quad-L_bL_f^{\eta-1}g(x_k(t_\eta)))u_d(t_\eta)]\,dt_\eta\cdots dt_1\\\varpi_k(t)&=\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-1}}L_bL_f^{\eta-1}g(x_k(t_\eta))(\Delta u_k(t_\eta)\\&\quad-\Delta u_k(t))\,dt_\eta\cdots dt_1.\end{aligned}$$

Taking norms and applying the bounds yield

$$\|\Delta v_{k+1}(t)\|\le\rho\|\Delta u_k(t)\|+c_\gamma(\|\xi_k(t)\|+\|\zeta_k(t)\|+\|\varpi_k(t)\|)$$

where c_γ is the norm bound for $\gamma_k(t)$. Note that the functions $f(\cdot), b(\cdot), L_f^ig(\cdot), 0\le i\le\eta$ and $L_bL_f^{\eta-1}g(\cdot)$ are local Lipschitz in $x\in X$ since they are smooth functions. Both $L_bL_f^{\eta-1}g(\cdot)$ and $b(\cdot)$ are bounded on X due to the same reason. In the rest of the proof, l_f, l_b, l_{fg}, l_{bfg}, c_{bfg} and c_b denote the Lipschitz constants and the norm bounds, respectively. Therefore

$$\begin{aligned}\|\xi_k(t)\|&\le\left(1+\frac{\sigma}{1!}+\cdots+\frac{\sigma^{\eta-1}}{(\eta-1)!}\right)l_{fg}\|\Delta x_k(t)\|\\\|\zeta_k(t)\|&\le(l_{fg}+l_{bfg}c_{ud})\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-1}}\|\Delta x_k(t_\eta)\|\,dt_\eta\cdots dt_1\\\|\varpi_k(t)\|&\le c_{bfg}\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-1}}\|\Delta u_k(t_\eta)-\Delta u_k(t)\|\,dt_\eta\cdots dt_1\end{aligned}$$

where $\Delta x_k(t)=x_d(t)-x_k(t)$ and $c_{ud}=\sup_{t\in[0,T]}\|u_d(t)\|$. Defining

$$\begin{aligned}c_1&=c_\gamma\left(1+\frac{\sigma}{1!}+\cdots+\frac{\sigma^{\eta-1}}{(\eta-1)!}\right)l_{fg}\\c_2&=c_\gamma(l_{fg}+l_{bfg}c_{ud})\quad\text{and}\quad c_3=c_\gamma c_{bfg}\end{aligned}$$

gives rise to

$$\begin{aligned}&\|\Delta v_{k+1}(t)\|\\&\quad\le\rho\|\Delta u_k(t)\|+c_1\|\Delta x_k(t)\|\\&\qquad+c_2\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-1}}\|\Delta x_k(t_\eta)\|\,dt_\eta\cdots dt_1\\&\qquad+c_3\int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta-1}}\\&\qquad\|\Delta u_k(t_\eta)-\Delta u_k(t)\|\,dt_\eta\cdots dt_1.\end{aligned}\tag{7}$$

To evaluate the state errors in the right-hand side of (7), we integrate the state equations to obtain

$$\begin{aligned}\|\Delta x_k(t)\|\le\int_0^t&(\|f(x_d(s))-f(x_k(s))\|\\&+\|b(x_d(s))-b(x_k(s))\|\|u_d(s)\|\\&+\|b(x_k(s))\|\|\Delta u_k(s)\|)\,ds.\end{aligned}$$

Defining $c_4 = l_f + l_b c_{ud}$ and using the Bellman–Gronwall lemma, we have

$$\|\Delta x_k(t)\| \le c_b \int_0^t e^{c_4(t-s)} \|\Delta u_k(s)\|\, ds. \tag{8}$$

Substituting (8) into (7) produces

$$\begin{aligned}\|\Delta v_{k+1}(t)\| \le{}& \rho \|\Delta u_k(t)\| \\ &+ c_1 c_b \int_0^t e^{c_4(t-s)} \|\Delta u_k(s)\|\, ds \\ &+ c_2 c_b \int_t^{t+\sigma} \int_t^{t_1} \cdots \int_t^{t_{\eta-1}} \int_0^{t_\eta} e^{c_4(t_\eta - s)} \\ &\cdot \|\Delta u_k(s)\|\, ds\, dt_\eta \cdots dt_1 \\ &+ c_3 \int_t^{t+\sigma} \int_t^{t_1} \cdots \int_t^{t_{\eta-1}} \\ &\|\Delta u_k(t_\eta) - \Delta u_k(t)\|\, dt_\eta \cdots dt_1. \end{aligned} \tag{9}$$

Multiplying both sides of (9) by $e^{-\lambda t} (\lambda > 0)$ gives

$$\begin{aligned} e^{-\lambda t} & \|\Delta v_{k+1}(t)\| \\ \le{}& \rho e^{-\lambda t} \|\Delta u_k(t)\| \\ &+ c_1 c_b \int_0^t e^{(c_4-\lambda)(t-s)} e^{-\lambda s} \|\Delta u_k(s)\|\, ds \\ &+ e^{-\lambda t} c_2 c_b \int_t^{t+\sigma} \int_t^{t_1} \cdots \int_t^{t_{\eta-1}} \int_0^{t_\eta} \\ &\|\Delta u_k(s)\|\, ds\, dt_\eta \cdots dt_1 \\ &+ e^{-\lambda t} c_3 \int_t^{t+\sigma} \int_t^{t_1} \cdots \int_t^{t_{\eta-1}} \\ &\|\Delta u_k(t_\eta) - \Delta u_k(t)\|\, dt_\eta \cdots dt_1. \end{aligned}$$

The saturation feature leads to $\|\Delta u_k(s) \le 2\delta$ and $\|\Delta u_k(t_\eta) - \Delta u_k(t)\| \le 4\delta$, and under the assumption A2), $\|\Delta u_k(t)\| \le \|\Delta v_k(t)\|$. Thus

$$\begin{aligned} \sup_{t\in[0,T-\sigma]} & \{e^{-\lambda t} \|\Delta v_{k+1}(t)\|\} \\ &\le \overline{\rho} \sup_{t\in[0,T-\sigma]} \{e^{-\lambda t}\|\Delta v_k(t)\|\} + c_5 \delta \frac{\sigma^\eta}{\eta!} \end{aligned} \tag{10}$$

where

$$\begin{aligned} \lambda &> c_4 \\ \overline{\rho} &= \rho + c_1 c_b \frac{1 - e^{(c_4-\lambda)T}}{\lambda - c_4} \\ c_5 &= 2 c_2 c_b \frac{e^{c_4 T} - 1}{c_4} + 4 c_3. \end{aligned}$$

Since $0 \le \rho < 1$, it is possible to find a $\lambda > c_4$ sufficiently large such that $\overline{\rho} < 1$. Then, (10) is a contraction in $\sup_{t\in[0,T-\sigma]} \{e^{-\lambda t}\|\Delta v_k(t)\|\}$ which leads to

$$\limsup_{k\to\infty} \sup_{t\in[0,T-\sigma]} \{e^{-\lambda t}\|\Delta v_k(t)\|\} \le \frac{c_5 \delta}{1-\overline{\rho}} \frac{\sigma^\eta}{\eta!}. \tag{11}$$

From (8), and using the similar manipulations, we obtain

$$\begin{aligned} \limsup_{k\to\infty} & \sup_{t\in[0,T-\sigma]} \{e^{-\lambda t}\|\Delta x_k(t)\|\} \\ &\le c_b \frac{1 - e^{(c_4-\lambda)T}}{\lambda - c_4} \frac{c_5\delta}{1-\overline{\rho}} \frac{\sigma^\eta}{\eta!}. \end{aligned} \tag{12}$$

For $t \in (T-\sigma, T]$, (8) still holds so that

$$\begin{aligned} \|\Delta x_k(t)\| \le{}& c_b \int_0^{T-\sigma} e^{c_4(t-s)} \|\Delta v_k(s)\|\, ds \\ &+ c_b \int_{T-\sigma}^t e^{c_4(t-s)} \|\Delta u_k(s)\|\, ds \end{aligned}$$

which results in

$$\begin{aligned} \sup_{t\in(T-\sigma,T]} & \{e^{-\lambda t}\|\Delta x_k(t)\|\} \\ &\le c_b \frac{1 - e^{(c_4-\lambda)T}}{\lambda - c_4} \sup_{t\in[0,T-\sigma]} \{e^{-\lambda t}\|\Delta v_k(t)\|\} + 2 c_b \delta \sigma \end{aligned}$$

for $\lambda > c_4$. Therefore

$$\begin{aligned} \limsup_{k\to\infty} & \sup_{t\in(T-\sigma,T]} \{e^{-\lambda t}\|\Delta x_k(t)\|\} \\ &\le c_b \frac{1 - e^{(c_4-\lambda)T}}{\lambda - c_4} \frac{c_5\delta}{1-\overline{\rho}} \frac{\sigma^\eta}{\eta!} + 2 c_b \delta \sigma. \end{aligned} \tag{13}$$

Now, the result is established for the output error $e_k(t)$, $t \in [0, T]$, following (12) and (13)

$$\limsup_{k\to\infty} \sup_{t\in[0,T]} \|e_k(t)\| \le \beta \delta \max\left\{\frac{\sigma^\eta}{\eta!}, \sigma\right\} \tag{14}$$

where

$$\beta = e^{\lambda T} l_{fg} c_b \left(\frac{1 - e^{(c_4-\lambda)T}}{\lambda - c_4} \frac{c_5}{1-\overline{\rho}} + 2 \right).$$

This completes the proof. ■

Remark 3.3: Theorem 3.1 gives an explicit sufficient condition guaranteeing the convergence of tracking error, and provides a guide for the learning control design. The design needs a system model, but model discrepancy is allowed. Thus, this major advantage of iterative learning control methodology remains in the proposed learning algorithm. For example, we consider the case where σ is set to be small enough so that the condition (6) can be approximately written as

$$\begin{aligned} \Bigg\| 1 - \gamma_k(t) & \int_t^{t+\sigma} \int_t^{t_1} \cdots \int_t^{t_{\eta-1}} L_b L_f^{\eta-1} g(x_k(t_\eta))\, dt_\eta \\ & \cdots dt_1 \Bigg\| \approx \left\| 1 - \gamma_k(t) d_k(t) \frac{\sigma^\eta}{\eta!} \right\| \end{aligned} \tag{15}$$

where $d_k(t) = L_b L_f^{\eta-1} g(x_k(t))$. If $d_k(t)$ is modeled to be $\hat{d}_k(t)$ and we assume that $\hat{d}_k(t) = \alpha_k(t) d_k(t) (\alpha_k(t) > 0)$. We choose $\gamma_k(t) = \overline{\gamma} \hat{d}_k^{-1}(t)$ so that

$$\begin{aligned} & \left\| 1 - \gamma_k(t) d_k(t) \frac{\sigma^\eta}{\eta!} \right\| \\ &= \left\| 1 - \overline{\gamma} \hat{d}_k^{-1}(t) d_k(t) \frac{\sigma^\eta}{\eta!} \right\| \\ &= \left\| 1 - \overline{\gamma} \alpha_k^{-1}(t) \frac{\sigma^\eta}{\eta!} \right\| \end{aligned}$$

where $\overline{\gamma}$ is an adjustable parameter. The condition $\|1 - \overline{\gamma} \alpha_k^{-1}(t)(\sigma^\eta/\eta!)\| \le \rho < 1$ will hold if $0 < \overline{\gamma} < 2\alpha_k(t)(\eta!/\sigma^\eta)$.

B. Multi-Input–Multi-Output Systems

The obtained convergence result for the SISO systems can be extended to the MIMO systems.

Theorem 3.2: Given a desired trajectory $y_d(t), t \in [0, T]$ for the system (1)–(2) with extended relative degree $\{\eta_1, \cdots, \eta_m\}$, let the system satisfy assumptions A1)–A4) and be under the action of the updating law (3)–(4). The system output converges to the desired trajectory in the sense of

$$\limsup_{k\to\infty} \sup_{t\in[0,T]} \|e_k(t)\| \le \beta\delta \max\left\{ \max_{1\le q\le m} \left\{ \frac{\sigma^{\eta_q}}{\eta_q!} \right\}, \sigma \right\}$$

with a positive constant β to be defined, if

$$\|I - \Gamma_k(t)D_k(t)\| \le \rho < 1 \tag{16}$$

where the equation shown at the bottom of the page holds true.

Proof: Parallel to (5), the qth output component at the instant $t+\sigma$ can be written as

$$\begin{aligned} y_{q,k}(t+\sigma) &= g_q(x_k(t)) + \sigma L_f g_q(x_k(t)) + \cdots \\ &\quad + \frac{\sigma^{\eta_q-1}}{(\eta_q-1)!} L_f^{\eta_q-1} g_q(x_k(t)) \\ &\quad + \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_q-1}} \{L_f^{\eta_q} g_q(x_k(t_{\eta_q})) \\ &\quad + [L_{b_1}L_f^{\eta_q-1} g_q(x_k(t_{\eta_q}), t_{\eta_q}), \\ &\quad \cdots, L_{b_r}L_f^{\eta_q-1} g_q(x_k(t_{\eta_q}), t_{\eta_q})] u_k(t_{\eta_q})\} \\ &\quad\quad dt_{\eta_q}\cdots dt_1. \end{aligned} \tag{17}$$

Using (17), the error $\Delta v_k(t)$ for $t\in[0, T-\sigma]$ satisfies

$$\begin{aligned} \Delta v_{k+1}(t) &= (I - \Gamma_k(t)D_k(t))\Delta u_k(t) \\ &\quad - \Gamma_k(t)(\xi_k(t) + \zeta_k(t) + \varpi_k(t)) \end{aligned}$$

where

$$\begin{aligned} \xi_k(t) &= [\xi_{1,k}(t), \cdots, \xi_{m,k}(t)]^T \\ \zeta_k(t) &= [\zeta_{1,k}(t), \cdots, \zeta_{m,k}(t)]^T \\ \varpi_k(t) &= [\varpi_{1,k}(t), \cdots, \varpi_{m,k}(t)]^T, \quad \text{and} \\ \xi_{q,k}(t) &= g_q(x_d(t)) - g_q(x_k(t)) \\ &\quad + \sigma[L_f g_q(x_d(t)) - L_f g_q(x_k(t))] + \cdots \\ &\quad + \frac{\sigma^{\eta_q-1}}{(\eta_q-1)!}[L_f^{\eta_q-1} g_q(x_d(t)) - L_f^{\eta_q-1} g_q(x_k(t))] \\ \zeta_{q,k}(t) &= \int_t^{t+h}\int_t^{t_1}\cdots\int_t^{t_{\eta_q-1}} \\ &\quad \{L_f^{\eta_q} g_q(x_d(t_{\eta_q})) - L_f^{\eta_q} g_q(x_k(t_{\eta_q})) \\ &\quad + ([L_{b_1}L_f^{\eta_q-1} g_q(x_d(t_{\eta_q})) \\ &\quad \cdots, L_{b_r}L_f^{\eta_q-1} g_q(x_d(t_{\eta_q}))] \\ &\quad - [L_{b_1}L_f^{\eta_q-1} g_q(x_k(t_{\eta_q})) \\ &\quad \cdots, L_{b_r}L_f^{\eta_q-1} g_q(x_k(t_{\eta_q}))])u_d(t_{\eta_q})\}\, dt_{\eta_q}\cdots dt_1 \\ \varpi_{q,k}(t) &= \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_q-1}} \\ &\quad [L_{b_1}L_f^{\eta_q-1} g_q(x_k(t_{\eta_q})), \cdots, L_{b_r}L_f^{\eta_q-1} g_q(x_k(t_{\eta_q}))] \\ &\quad \cdot (\Delta u_k(t_{\eta_q}) - \Delta u_k(t))\, dt_{\eta_q}\cdots dt_1. \end{aligned}$$

Taking norms and applying the bounds yield

$$\|\Delta v_{k+1}(t)\| \le \rho\|\Delta u_k(t)\| + c_\Gamma(\|\xi_k(t)\| + \|\zeta_k(t)\| + \|\varpi_k(t)\|)$$

where c_Γ is the norm bound for $\Gamma_k(t)$. To proceed, we need the Lipschitz conditions and the bounds. Denote by l_f, l_B, l_{fg} and l_{bfg} the Lipschitz constants for the functions $f(\cdot), B(\cdot), L_f^i g_q(\cdot), 0\le i\le \eta_q, 1\le q\le m$ and $L_{b_p}L_f^{\eta_q-1}g_q(\cdot), 1\le p\le r, 1\le q\le m$, c_{bfg} and c_B the norm bounds for $[L_{b_1}L_f^{\eta_q-1}g_q(\cdot), \cdots, L_{b_r}L_f^{\eta_q-1}g_q(\cdot)]$, $1\le q\le m$ and $B(\cdot)$, respectively. Therefore

$$\begin{aligned} \|\xi_{q,k}(t)\| &\le \left\{1 + \frac{\sigma}{1!} + \cdots + \frac{\sigma^{\eta_q-1}}{(\eta_q-1)!}\right\} l_{fg}\|\Delta x_k(t)\| \\ \|\zeta_{q,k}(t)\| &\le (l_{fg} + r l_{bfg} c_{ud}) \int_t^{t+\sigma}\int_t^{t_1} \\ &\quad \cdots\int_t^{t_{\eta_q-1}} \|\Delta x_k(t_{\eta_q})\|\, dt_{\eta_q}\cdots dt_1 \\ \|\varpi_{q,k}(t)\| &\le c_{bfg} \int_t^{t+\sigma}\int_t^{t_1} \\ &\quad \cdots\int_t^{t_{\eta_q-1}} \|\Delta u_k(t_{\eta_q}) - \Delta u_k(t)\|\, dt_{\eta_q}\cdots dt_1 \end{aligned}$$

Defining

$$\begin{aligned} c_1 &= c_\Gamma \max_{1\le q\le m}\left\{1 + \frac{\sigma}{1!} + \cdots + \frac{\sigma^{\eta_q-1}}{(\eta_q-1)!}\right\} l_{fg} \\ c_2 &= c_\Gamma(l_{fg} + r l_{bfg} c_{ud}), \quad \text{and} \quad c_3 = c_\Gamma c_{bfg} \end{aligned}$$

we have (18), shown at the bottom of the page. Integrating the state

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$$D_k(t) = \begin{bmatrix} \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_1-1}} [L_{b_1}L_f^{\eta_1-1}g_1(x_k(t_{\eta_1})), \cdots, L_{b_r}L_f^{\eta_1-1}g_1(x_k(t_{\eta_1}))]\, dt_{\eta_1}\cdots dt_1 \\ \vdots \\ \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_m-1}} [L_{b_1}L_f^{\eta_m-1}g_m(x_k(t_{\eta_m})), \cdots, L_{b_r}L_f^{\eta_m-1}g_m(x_k(t_{\eta_m}))]\, dt_{\eta_m}\cdots dt_1 \end{bmatrix}.$$

$$\begin{aligned} \|\Delta v_{k+1}(t)\| &\le \rho\|\Delta u_k(t)\| + c_1\|\Delta x_k(t)\| + c_2 \left\| \begin{bmatrix} \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_1-1}} \|\Delta x_k(t_{\eta_1})\|\, dt_{\eta_1}\cdots dt_1 \\ \vdots \\ \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_m-1}} \|\Delta x_k(t_{\eta_m})\|\, dt_{\eta_m}\cdots dt_1 \end{bmatrix} \right\| \\ &\quad + c_3 \left\| \begin{bmatrix} \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_1-1}} \|\Delta u_k(t_{\eta_1}) - \Delta u_k(t)\|\, dt_{\eta_1}\cdots dt_1 \\ \vdots \\ \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_m-1}} \|\Delta u_k(t_{\eta_m}) - \Delta u_k(t)\|\, dt_{\eta_m}\cdots dt_1 \end{bmatrix} \right\|. \end{aligned} \tag{18}$$

$$\|\Delta v_{k+1}(t)\| \le \rho\|\Delta u_k(t)\| + c_1 c_B \int_0^t e^{c_4(t-s)}\|\Delta u_k(s)\|\, ds$$
$$+ c_2 c_B \left\| \begin{bmatrix} \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_1 -1}}\int_0^{t_{\eta_1}} e^{c_4(t_{\eta_1}-s)}\|\Delta u_k(s)\|\, ds\, dt_{\eta_1}\cdots dt_1 \\ \vdots \\ \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_m -1}}\int_0^{t_{\eta_m}} e^{c_4(t_{\eta_m}-s)}\|\Delta u_k(s)\|\, ds\, dt_{\eta_m}\cdots dt_1 \end{bmatrix} \right\|$$
$$+ c_3 \left\| \begin{bmatrix} \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_1 -1}} \|\Delta u_k(t_{\eta_1}) - \Delta u_k(t)\|\, dt_{\eta_1}\cdots dt_1 \\ \vdots \\ \int_t^{t+\sigma}\int_t^{t_1}\cdots\int_t^{t_{\eta_m -1}} \|\Delta u_k(t_{\eta_m}) - \Delta u_k(t)\|\, dt_{\eta_m}\cdots dt_1 \end{bmatrix} \right\|.$$

equations and defining $c_4 = l_f + l_B c_{ud}$ result in

$$\|\Delta x_k(t)\| \le c_B \int_0^t e^{c_4(t-s)}\|\Delta u_k(s)\|\, ds. \tag{19}$$

Substituting (19) into (18) produces the equation shown at the top of the next page. Parallel to the development in the SISO case, we have finally

$$\limsup_{k\to\infty} \sup_{t\in[0,T]} \|e_k(t)\| \le \beta\delta \max\left\{ \max_{1\le q\le m} \left\{\frac{\sigma^{\mu_q}}{\mu_q!}\right\}, \sigma \right\} \tag{20}$$

where

$$\beta = e^{\lambda T} l_{fg} c_B \left(\frac{1-e^{(c_4-\lambda)T}}{\lambda - c_4}\, \frac{c_5}{1-\overline{\rho}} + 2 \right)$$
$$\overline{\rho} = \rho + c_1 c_B \frac{1-e^{(c_4-\lambda)T}}{\lambda - c_4} \quad \text{and}$$
$$c_5 = 2c_2 c_B \frac{e^{c_4 T}-1}{c_4} + 4c_3.$$

This completes the proof. ■

Remark 3.4: Theorems 3.1 and 3.2 show that the extended relative degree of the systems under consideration is not included in the proposed updating law (3)–(4) itself. However, it is required implicitly when we design the parameter σ and the learning gain $\Gamma_k(t)$ following the way specified in Remark 3.3. For D-type ILC, the error derivative with the order being equal to the relative degree is used to update the control input so that the relative degree is required to be known explicitly. There may be points where an extended relative degree cannot be defined for some class of systems and the proposed scheme fails to work. This would be the major limitation on the learning algorithm which is applicable to the systems with well defined extended relative degree.

IV. Conclusion

The concept of extended relative degree is introduced to describe the input–output causality of a class of nonlinear continuous-time systems. The anticipatory iterative learning control method is shown applicable to the systems with such extended relative degree. It is shown to be able to reduce the tracking error iteratively under the derived sufficient condition on the anticipatory parameter and the learning gain. Moreover, this approach does not require any error differentiation.

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