Completeness of Randomized Kinodynamic Planners with State-based Steering

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Abstract— The panorama of probabilistic completeness results for kinodynamic planners is still confusing. Most existing completeness proofs require strong assumptions that are difficult, if not impossible, to verify in practice. To make completeness results more useful, it is thus necessary to estabish a classification of the various types of constraints and planning methods, and then attack each class with specific proofs and hypotheses that can be verified in practice. We propose such a classification, and provide a proof of probabilistic completeness for an important class of planners, namely those whose steering method is based on the interpolation of system trajectories in the state space. We also provide design guidelines for the interpolation function and discuss two criteria arising from our analysis: local boundedness and acceleration compliance.

I. INTRODUCTION

A *deterministic* motion planning algorithm (or planner) is said to be *complete* if it returns a solution to a motion planning problem whenever one exists (see *e.g.*, [1]). A *randomized* planner is said to be *probabilistically complete* if the probability of returning a solution tends to one as execution time goes to infinity. The concepts of completeness and probabilistic completeness, although theoretical by nature, are also of practical interest: proving them requires one to specify what assumptions are needed for a planner to find solutions, *i.e.*, what types of problems can be solved. This provides more general guarantees than empirical results. Experiments can show that a planner works for a given combination of robot, environment, task, (set of tweaks and heuristics), but a proof of completeness is a certificate that the planner works for a whole set of problems, the size of this set being determined by the assumptions required to make the proof (the weaker the assumptions, the larger the set of solvable problems).

While the probabilistic completeness of randomized planners has been well established for systems with *geometric* constraints (such as obstacle avoidance), proofs for systems with *kinodynamic* constraints [2], [3], [4] have not yet reached the same level of generality: in many proofs, the assumptions made are quite strong and difficult to verify on practical systems (as a matter of fact, none of the previously mentioned works verified their hypotheses on non-trivial systems). One of the reasons for this lies in the very large variety of kinodynamic constraints and of planning methods.

To make completeness proofs more useful in practice, it is thus necessary to estabish a classification of the different types of constraints and planning methods, and then attack each class with specific proofs and hypotheses that can be more easily checked. In section II, we propose such a classification of kinodynamic constraints (as non-holonomic or dynamics-bound-based) and of planning methods (based on their underlying steering methods: analytic, control-based, or state-based). We also discuss the shortcomings of existing completeness proofs. Then, in section III, we prove a completeness result for the class of state-based steering planners for systems subject to dynamics bounds. Finally, in section IV, we conclude by discussing the implications of our results as well as future research objectives.

II. CLASSIFICATION OF KINODYAMIC CONSTRAINTS AND STEERING METHODS

A. Classification of Kinodynamic Constraints

Motion planning was first concerned only with *geometric* constraints such as obstacle avoidance or those imposed by the kinematic structures of manipulators [5], [6], [4], [2]. More recently, *kinodynamic* constraints, which stem from the *dynamical equations* the systems are subject to, have also been taken into account (see *e.g.*, [7], [2], [8]).

Kinodynamic constraints are more difficult to deal with than geometric constraints because they cannot in general be expressed using only *configuration-space variables* – such as the joint angles of a manipulator, the position and the orientation of a mobile robot, etc. They indeed involve *higher-order derivatives* of the configuration-space variables. However, the *way* these derivatives appear in the constraints is not uniform, and can in fact be classified into two main classes involving very different types of difficulties:

1) *Non-holonomic constraints* are non-integrable *equality* constraints on higher-order derivatives of the configuration-space variables. They can be of the first-order, as found in wheeled vehicles (see *e.g.*, [9]), or of the second-order, as found in underactuated manipulators (see *e.g.*, [10]) or space robots (see *e.g.*, [11].

2) *Bounds on dynamics quantities* are *inequality* constraints on higher-order derivatives of the configuration-space variables. These include torque bounds for manipulators (see *e.g.*, [12]), ZMP constraints for walking robots (see *e.g.*, [13]), friction constraints in grasp synthesis (see *e.g.*, [14]), etc.

Some authors have considered systems that are subject to both types of constraints, for instance under-actuated manipulators with torque bounds [15].

The completeness results of section III of the present paper concern systems subject to the second type of constraints

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only, more precisely holonomic (fully-actuated) systems with inequality constraints on their first and/or second-order derivatives.

B. Structure of Randomized Planners

The *expansion* step, as given by Algorithm 1, forms the core of most randomized planners, such as Probabilistic Roadmaps (PRM) [6] or Rapidly-exploring Random Trees (RRT) [2]. This step in turn involves three sub-routines (see Fig. 1 for an illustration):

- *Sampling* SAMPLE(S): randomly samples from a set S;
- *Antecedent selection* PARENTS(x', V): returns a set of states x belonging to the roadmap (or the tree) V, from which steering towards x' will be attempted;
- *Local steering* STEER(x, x'): tries to steer the system from x towards x'. If successful, returns a new node x_{steer} ready to be added to the roadmap.

Algorithm 1 Expansion step in randomized planners

Require: initial node x_{init}, number of iterations N
1: $(V, E) \leftarrow (\{x_{init}\}, \emptyset)$
2: **for** N steps **do**
3:     $x_{rand} \leftarrow$ SAMPLE($\mathcal{X}_{free}$)
4:     $X_{parents} \leftarrow$ PARENTS(x_{rand}, V)
5:     **for** x_{parent} in $V_{parents}$ **do**
6:         $x_{steer} \leftarrow$ STEER(x_{parent}, x_{rand})
7:         **if** x_{steer} is a valid state **then**
8:             $V \leftarrow V \cup \{x_{steer}\}$
9:             $E \leftarrow E \cup \{(x_{parent}, x_{steer})\}$
10:        **end if**
11:     **end for**
12: **end for**
13: **return** (V, E)

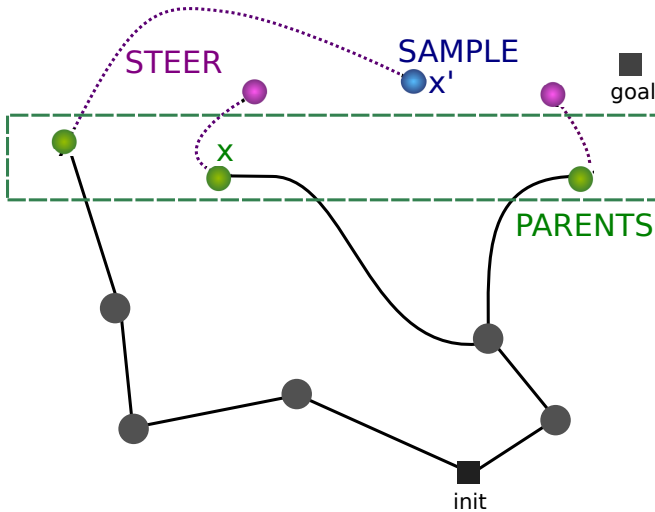

Fig. 1. Illustration of the expansion routine of randomized planners. The sampled node $x' =$ SAMPLE($\mathcal{X}_{free}$) is shown in blue. Purple edges and nodes show steering attempts resulting from the STEER(x, x') method, where antecedents $x \in$ PARENTS(x', V) are pictured in green.

It is easy to see that the design of each sub-routine greatly impacts the quality and even the completeness of the resulting planner.

In the literature, SAMPLE is usually implemented as uniform random sampling. Some authors have suggested to use adaptive sampling to improve the performance of RRT or PRM planners [16], [17].

In geometric planners, PARENTS is usually implemented by defining a metric (*e.g.*, the ℓ_2 norm) in the configuration space, and using nearest-neighbors as antecedents. Such a choice results in the so-called Voronoi bias of RRTs [2]. Both experiments and theoretical analysis support this choice for geometric planning. However, when moving to kinodynamic planning, designing a metric that yields good antecedents becomes as challenging as the motion planning problem itself, and the Euclidean norm becomes highly inefficient (see *e.g.*, [18] for an illustration in the case of the actuated pendulum subject to torque bounds).

The next section discusses the various implementations of the steering sub-routine.

C. Classification of Steering Methods

We propose to classify existing steering methods into three categories.

1) Control-based steering: compute a control function $u : [0, T] \to \mathcal{U}_{adm}$ and apply them to the system, simulating with forward dynamics. Stop after a given duration or if the system reached a state close enough to x'. To compute u, [2], [8] sample functions from a family of primitives (*e.g.*, piecewise constant functions), try a certain number of them, and eventually choose the one that drives the system closest to x'. Linear-Quadratic Regulation (LQR) [19], [20] also falls in this category: in this case, u is computed as the optimal policy for a linear approximation of the system given a quadratic cost function.

2) State-based steering: compute a trajectory $\widetilde{\gamma} : [0, \Delta \widetilde{t}] \to \mathcal{C}$, then try to find a control that follows this trajectory. For instance, one can interpolate third-order polynomials in t verifying $(\widetilde{\gamma}(0), \dot{\widetilde{\gamma}}(0)) = x$ and $(\widetilde{\gamma}(\Delta t), \dot{\widetilde{\gamma}}(\Delta t)) = x'$. Since the interpolated trajectory may not respect control constraints, it needs to be checked (using inverse dynamics) or adapted (*e.g.*, using a dynamics filter). See Algorithm 2.

Algorithm 2 STEER(x, x')

1: $\widetilde{\gamma} \leftarrow$ INTERPOLATE(x, x')
2: $\widetilde{u} := t \mapsto f(\widetilde{\gamma}(t), \dot{\widetilde{\gamma}}(t), \ddot{\widetilde{\gamma}}(t))$
3: **if** Im($\widetilde{u}$) $\subset \mathcal{U}_{adm}$ **then**
4:     **return** the last state of $\widetilde{\gamma}$
5: **end if**
6: **return** failure

The INTERPOLATE routine computes a trajectory as described previously. It can also include a correction step, like the dynamics filter used in [21] (for balance correction of humanoid motions) or the Admissible Velocity Propagation (AVP) that we introduced in [22].

3) Analytical steering: with control-driven steering, it is easy to respect differential constraints but difficult to control position. Conversely, with state-based steering, it is easy to control position but difficult to enforce differential constraints. However, for some systems, a steering function satisfying both requirements is known, *e.g.*, Reeds and Shepp curves for cars. When it is the case, the problem can either be reduced to path planning [23] or additional optimality guarantees can be provided [24].

D. Previous Proofs of Probabilistic Completeness

Randomized planners such as Rapidly-exploring Random Trees (RTT) and Probabilistic Roadmaps (PRM) are popular because they are grounded on an intuitive idea and simple to implement. Proofs of probabilistic completeness come as another indicator in their favor. However, one should beware of general conceptions such as "RRT is probabilistically complete": as we will see, it is not always true for kino-dynamic planning.

Probabilistic completeness is a scalability property: if the local planners used for extension are suitable, then the overall planner will explore its environment exhaustively. The assumptions made to achieve this scalability play a crucial role: as they become stronger, proofs of completeness become easier, but the link with the motion planning problem vanishes. In particular, it is not possible to check some assumptions made in the literature on actual systems.

Completeness of RRT planners has been established for *path planning* [2], [3], [4]. In their proof, Hsu et al. quantified the problem of narrow passages in configuration space with the notion of (α, β)-expansiveness [4]. The two constants α and β express a geometric lower bound on the rate of expansion of reachability areas. The authors later extended their solution to kinodynamic planning [8], using the same notion of expansiveness, but this time in the $\mathcal{X} \times \mathcal{T}$ (state and time) space with control-based steering. They established that, when $\alpha > 0$ and $\beta > 0$, their planner is probabilistically complete. However, whether $\alpha > 0$ or $\alpha = 0$ in the $\mathcal{X} \times \mathcal{T}$ space remains undiscussed, and the problem of evaluating (α, β) is deemed as difficult as the initial planning problem [4].

LaValle et al. provided a completeness argument for kinodynamic planning [25]. In their proof, they assumed the existence of an *attraction sequence*, which is a covering of the state space where two major problems of kinodynamic planning, namely steering and antecedent selection (see Section II-C), are already solved. However, conditions of existence of such a sequence are not discussed.

These two examples highlight our concern about completeness proofs: in both cases, probabilistic completeness is established under assumptions whose verification is at least as difficult as the motion planning problem itself. This observation does not question the quality of the associated planners, which have also been checked experimentally. Rather, it hints that too much of the complexity of kino-dynamic planning has been abstracted into hypotheses. As a result, these completeness proof do not help us understand why these planners work (or don't work) in practice.

Karaman et al. introduced their path planning algorithm RRT* in [3] and extended it to kinodynamic planning with differential constraints in [24], providing a sketch of proof for the completeness of their solution. However, they assumed that their planner had access to the optimal cost metric and optimal local steering (*i.e.*, STEER(x_1, x_2) returns the optimal trajectory starting from x_1 and ending at x_2), which restricts the analysis to systems for which these ideal solutions are known.

The same authors tackled the problem from a slightly different perspective in [26]. They now assumed that the PARENTS function computes *w-weighted boxes*, which are abstractions of the system's local controlability. It remains unclear to us how these boxes can be computed or approximated in practice, given that their definition involves the joint flow of vector fields spanning the tangent space of the system's manifold. Although their set of assumptions is of primary concern to us since we follow a similar approach in Section III, they did not prove their theorem, arguing that the reasoning was similar to the one in [3] for kinematic systems.

To the best of our knowledge, as of yet there is no completeness proof for kinodynamic planners using state-based steering. We will establish such a result in the following section.

III. COMPLETENESS OF STATE-BASED STEERING KINODYNAMIC PLANNERS

A. Terminology

A function is *smooth* when all its derivatives exist and are continuous. A function $f : A \rightarrow B$ between metric spaces is Lipschitz when there exists a constant K_f such that

$$\forall (x, y) \in A, \ \|f(x) - f(y)\| \leq K_f \|x - y\|.$$

Throughout the present paper, we will work within normed vector spaces and $\| \cdot \|$ will refer to the Euclidean norm $\| \cdot \|_2$. We will also consistently denote by K_f the (smallest possible) Lipschitz constant of any Lipschitz function f.

Let $\mathcal{C}$ denote n-dimensional configuration space, where n is the number of degrees of freedom of the robot. We will call *state space* the $2n$-dimensional manifold $\mathcal{X}$ of configuration and velocity coordinates. In the present paper, we only consider fully actuated systems. Let the control input space ("control space" for short) be an n-dimensional manifold $\mathcal{U}$. The dynamics of the robot follow the equations of motion, which can be written in generalized coordinates as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u. \tag{1}$$

Equivalently, the robot's dynamics follow the time-invariant differential system

$$\dot{x}(t) = f(x(t), u(t)), \tag{2}$$

where $x(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$. We will assume that f is Lipschitz continuous in both of its arguments. The set $\mathcal{U}_{\text{adm}}$

of admissible controls is assumed to be a compact subset of $\mathcal{U}$.

A trajectory is a continuous function $\gamma : [0, T] \to \mathcal{C}$. A path is the image of a trajectory. An admissible trajectory is a solution to the differential system 2. The kinematic motion planning problem is to find a path in the collision-free subset $\mathcal{C}_{free} \subset \mathcal{C}$ from an initial configuration q_{init} to any configuration q_{goal} in a set of goals. Meanwhile, the kinodynamic motion planning problem is to find an *admissible trajectory* from q_{init} to q_{goal}, both avoiding obstacles and following the system's dynamics.

A control function $t \mapsto u(t)$ is said to have δ-clearance when its image is in the δ-interior of the set of admissible controls, *i.e.*, for any time t, $\mathcal{B}(u(t), \delta) \subset \mathcal{U}_{\text{adm}}$.

We define the distance between a state $x \in \mathcal{X}$ and the curve γ as:

$$\text{dist}_\gamma(x) := \min_{t \in [0,T]} \|(\gamma, \dot{\gamma})(t) - x\|$$

1) Notations: whenever considering two states x and x', we will write:

$$\left\{ \begin{array}{rcl} x &=:& (q, \dot{q}) \\ x' &=:& (q', \dot{q}') \end{array} \right. \qquad \left\{ \begin{array}{rcl} \Delta x &:=& x' - x \\ \Delta q &:=& q' - q \\ \Delta \dot{q} &:=& \dot{q}' - \dot{q} \end{array} \right.$$

Similarly, for two time instants $t < t'$, we will write $\Delta t := t' - t$ and $\Delta g := g(t') - g(t)$ for any function g.

B. Completeness theorem

Our model for an $\mathcal{X}$-state randomized planner is given by Algorithm 1 using the state-based steering described in Algorithm 2. We make the following three assumptions on the system:

Assumption 1: The system is fully actuated.

Assumption 2: The set of admissible controls $\mathcal{U}_{\text{adm}}$ is compact.

Assumption 3: The inverse of the differential constraint f from Equation (2), *i.e.*, the function f^{-1} s.t. $u = f^{-1}(x, \dot{x})$, is Lipschitz in both of its arguments.

Assumption 1 is a pre-requisite for the function f^{-1} used in Assumption 3 to be well-defined. The latter assumption is satisfied when f is given by the dynamics equations 1 as long as the matrices $M(q)$ and $C(q, \dot{q})$ have bounded norm, and the gravity term $g(q)$ is Lipschitz. Indeed, for a small displacement between x and x',

$$\|u' - u\| \le \|M\| \|\ddot{q}' - \ddot{q}\| + \|C(q, \dot{q})\| \|\dot{q}' - \dot{q}\| + K_g \|q' - q\|$$

Example: Assumption 3 is satisfied for the fully-actuated double pendulum shown in Figure 2. When links have mass m and length l, the gravity term $g(\theta_1, \theta_2) = \frac{mgl}{2} [\sin \theta_1 + \sin(\theta_1 + \theta_2) \ \sin(\theta_1 + \theta_2)]$ is Lipschitz with constant $K_g = 2mgl$. Meanwhile, $\|M\| (\theta_1, \theta_2) \le 3ml^2$ and, when joint angular velocities are bounded by ω, the norm of the Coriolis tensor is bounded by $2\omega ml^2$.

Regarding Assumption 2, since torque constraints are our main concern, we will make our proof of completeness for
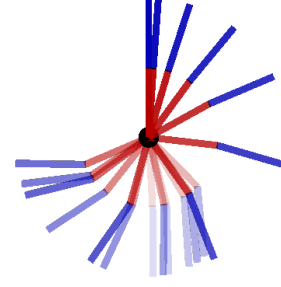

Fig. 2. A fully-actuated double pendulum with torque bounds must swing back and forth several times in order to reach the inverted position (lighter images represent earlier states in time).

(note that the comparison is component-wise)

$$\mathcal{U}_{\text{adm}} := \{u \in \mathcal{U}, \ |u| \le \tau_{\text{max}}\},$$

which is indeed compact. The generalization to an arbitrary compact set presents no technical difficulty.

Let us now turn to the design of the interpolation routine. We make the following three hypotheses:

Assumption 4: Interpolated trajectories $\widetilde{\gamma}$ are smooth Lipschitz functions, and their time-derivatives $\dot{\widetilde{\gamma}}$ (*i.e.*, interpolated velocities) are also Lipschitz.

Assumption 5 (Local boundedness): We suppose that there exists a constant η such that, for any $(x, x') \in \mathcal{X}^2$, the interpolated trajectory $\widetilde{\gamma} : [0, \Delta \tilde{t}] \to \mathcal{C} = \text{INTERPOLATE}(x, x')$ is such that

$$\forall \tau \in [0, \Delta \tilde{t}], \ (\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau)) \in \mathcal{B}(x, \eta \|x' - x\|).$$

Assumption 6 (Acceleration compliance): The acceleration of interpolated trajectories uniformly converges to the discrete velocity derivative, *i.e.*, there exists some $\nu > 0$ such that, if $\widetilde{\gamma} : [0, \Delta \tilde{t}] \to \mathcal{C}$ results from $\text{INTERPOLATE}(x, x')$, then

$$\forall \tau \in [0, \Delta \tilde{t}], \ \left\| \ddot{\widetilde{\gamma}}(\tau) - \frac{\|\dot{q}\|}{\|\Delta q\|} \Delta \dot{q} \right\| \le \nu \|\Delta x\|$$

Assumption 4 is is easy to satisfy in practice. Assumption 5 bounds the position and velocity of interpolated trajectories with respect to the neighborhood of x and x'. Meanwhile, Assumption 6 bounds their acceleration with respect to the discrete derivative of the velocity between x and x'.

Note that these three assumptions rely only on the topology of the state space. They are independent from the instance of the motion planning problem $(f, \mathcal{U}_{\text{adm}}, ...)$. We can therefore consider them as design guidelines for the interpolation functions.

We can now state our main theorem:

Theorem 1: Consider a time-invariant differential system (2) with Lipschitz-continuous f and full actuation over a compact set of admissible controls $\mathcal{U}_{\text{adm}}$. Suppose that the kinodynamic planning problem between two states x_{init} and x_{goal} admits a smooth Lipschitz solution $\gamma : [0, T] \to \mathcal{C}$ with δ-clearance in control space. Let $\mathcal{K}$ denote a randomized motion planner (Algorithm 1) using state-based steering (Algorithm 2) and a locally bounded, Lipschitz, acceleration-compliant interpolation primitive. $\mathcal{K}$ is probabilistically com-

plete.

C. Preliminary lemmas

Let us start with a simple upper bound on the difference between the variation rate and derivative of a Lipschitz.

Lemma 1: Let $g : [0, T] \to \mathbb{R}^k$ denote a smooth Lipschitz function. Then, for any $(t, t') \in [0, T]^2$,

$$\left\| \dot{g}(t) - \frac{g(t') - g(t)}{|t' - t|} \right\| \leq \frac{K_g}{2} |t' - t|.$$

Proof: We can suppose without loss of generality that $t' > t$. Then, $\left\| \dot{g}(t) - \frac{g(t') - g(t)}{t' - t} \right\| \leq \frac{1}{t' - t} \left\| \int_t^{t'} (\dot{g}(t) - \dot{g}(w)) \, dw \right\| \leq \frac{1}{t' - t} \int_t^{t'} \| \dot{g}(t) - \dot{g}(w) \| \, dw \leq \frac{K_g}{t' - t} \int_t^{t'} |t - w| \, dw \leq \frac{K_g}{2} (t' - t).$ ∎

As an important first step, we show that, given the existence of a solution γ with δ-clearance in control space, we can suppose without loss of generality that velocities $\|\dot{\gamma}\|$ and accelerations $\|\ddot{\gamma}\|$ along the curve are lower bounded by a strictly positive constant. This observation is formalized by the two following lemmas. Detailed proofs are provided in the supplementary material.

Lemma 2: If there exists a trajectory γ with δ-clearance in control space, then there exists $\delta' < \delta$ and a trajectory γ' with δ'-clearance in control space such that $\inf_t \|\ddot{\gamma}'(t)\| > 0$.

Proof: [Sketch of proof] If there is a time interval $[t, t']$ on which $\ddot{\gamma} \equiv 0$, one can leverage full actuation and δ-clearance in control to increment each coordinate with a small wave function $\delta\ddot{\gamma}_i$ of amplitude $\delta\ddot{q}_i$ and zero integral over $[t, t']$. The amplitude $\delta\ddot{q}_i$ is chosen so as to guarantee δ'-clearance in control space, for some $\delta' < \delta$.[1]

We can therefore assume that w.l.o.g. that the roots of $\ddot{\gamma}$ form a discrete set. Let t_0 be such a root. Again, δ-clearance in control and full actuation can be leveraged into adding a small perturbation $\delta\ddot{\gamma}_i$ to each coordinate around t_0. To ensure that $\ddot{\gamma}(t_0)$ becomes non-zero without creating new roots at other time instants, one needs to ensure that the coordinate perturbations are not time-correlated, which is easy to do, for instance using sine waves with different different periods. Special care needs to be taken if the root is at the first (or last) time instants of the trajectory. However, since we do not require accelerations (nor controls) to be continuous, one can simply shift the wave so as to start with (resp. end on) a non-zero value. ∎

Lemma 3: If there exists a trajectory γ with δ-clearance in control space, then there exists $\delta' < \delta$ and a trajectory γ' with δ'-clearance in control space such that $\inf_t \|\dot{\gamma}'(t)\| > 0$.

Proof: [Sketch of proof] The argument is the same as in the proof for Lemma 2: add a small perturbation wave of controlled amplitude to the velocity coordinates. However, the system is controlled in acceleration and not velocity. To overcome this, one can use sine waves as a basis family for the perturbations: their derivatives are cosine waves of controlled amplitude, which can be added to the acceleration

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[1] In the presence of $\mathcal{C}$-space obstacles or velocity limits, one can refine this wave as $\delta\ddot{\gamma}_i(w) = \delta\ddot{q}_i \sin\left(\frac{kw}{t'-t}\right)$, where the period $\frac{t'-t}{k}$ is chosen so as to bound the deviation in velocity and position incurred by the perturbation.

coordinates using full actuation and reducing the δ-clearance in control. Boundary values for these perturbations will be non-zero, which is not a problem since we do not require acceleration nor control to be continuous. ∎

D. Proof of Theorem 1

Let $\gamma : [0, T] \to \mathcal{C}, t \mapsto \gamma(t)$ denote a smooth Lipschitz admissible trajectory from x_{init} to x_{goal} with δ-clearance in control space. We can define:

$$\left\{ \begin{array}{lll} \dot{M} & := & \max_t \|\dot{\gamma}(t)\| \\ \dot{m} & := & \min_t \|\dot{\gamma}(t)\| \end{array} \right. \qquad \left\{ \begin{array}{lll} \ddot{M} & := & \max_t \|\ddot{\gamma}(t)\| \\ \ddot{m} & := & \min_t \|\ddot{\gamma}(t)\| \end{array} \right.$$

From lemmas 2 and 3, $\dot{m} > 0$ and $\ddot{m} > 0$. Consider two states x and x' and the corresponding time instants on the trajectory

$$\left\{ \begin{array}{lll} t & := & \arg\min_t \|(\gamma(t), \dot{\gamma}(t)) - x\|, \\ t' & := & \arg\min_t \|(\gamma(t), \dot{\gamma}(t)) - x'\|. \end{array} \right.$$

We can suppose w.l.o.g. that $t < t'$. First, note that there exists $\delta t_1 > 0$ such that, for any $\Delta t \leq \delta t_1$,

$$\frac{\|\Delta\gamma\|}{\Delta t} \geq \frac{\dot{m}}{2}, \quad \frac{\|\Delta\dot{\gamma}\|}{\Delta t} \geq \frac{\ddot{m}}{2}, \quad \frac{\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|} \leq \frac{2\ddot{M}}{\dot{m}}.$$

Proof: The three functions $\Delta t \mapsto \frac{\|\Delta\gamma\|}{\Delta t}$, $\Delta t \mapsto \frac{\|\Delta\dot{\gamma}\|}{\Delta t}$ and $\Delta t \mapsto \frac{\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|}$. are continuous over the compact set $[0, T]$, hence uniformly continuous, and their limits when $\Delta t \to 0$ are respectively $\|\dot{\gamma}(t)\| \geq \dot{m}$, $\|\ddot{\gamma}(t)\| \geq \ddot{m}$ and $\frac{\|\ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|} \leq \frac{\ddot{M}}{\dot{m}}$. ∎

In what follows, we will suppose that $\Delta t < \delta t_1$.

Let $\widetilde{\gamma} : [0, \Delta\widetilde{t}] \to \mathcal{C}$ denote the result of INTERPOLATE(x, x'). For $\tau \in [0, \Delta\widetilde{t}]$, the torque required to follow the trajectory $\widetilde{\gamma}$ is $\widetilde{u}(\tau) := f(\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau), \ddot{\widetilde{\gamma}}(\tau))$. Since $\text{Im}(u) \subset \text{int}_\delta(\mathcal{T})$,

$$\begin{aligned} |\widetilde{u}(\tau)| & \leq |\widetilde{u}(\tau) - u(t)| + |u(t)| \\ & \leq \left| f(\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau), \ddot{\widetilde{\gamma}}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t)) \right| \\ & \quad + (1 - \delta)\, \tau_{\mathsf{max}}, \end{aligned}$$

where the comparison here is component-wise. If the first term in this upper bound is $\leq \delta\, \tau_{\mathsf{max}}$, then the system will be able to track $\widetilde{\gamma}$ at time τ. We can rewrite it as follows:

$$\begin{aligned} & \left| f(\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau), \ddot{\widetilde{\gamma}}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t)) \right| \\ & \leq \left\| f(\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau), \ddot{\widetilde{\gamma}}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t)) \right\|_\infty \\ & \leq K_f \left\| (\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau)) - (\gamma(t), \dot{\gamma}(t)) \right\| + K_f \left\| \ddot{\widetilde{\gamma}}(\tau) - \ddot{\gamma}(t) \right\| \\ & \leq \underbrace{K_f \left[(\eta + \nu) \|\Delta x\| + \text{dist}_\gamma(x) \right]}_{\text{distance term (D)}} + \underbrace{K_f \left\| \frac{\|\dot{q}\|}{\|\Delta q\|} \Delta\dot{q} - \ddot{\gamma}(t) \right\|}_{\text{acceleration term (A)}}, \; (\star) \end{aligned}$$

where we used the triangular inequality, the Lipschitz condition on f, as well as local boundedness (Assumption 5) and acceleration compliance (Assumption 6) of the interpolated trajectory. The transition from the norm $\|\cdot\|_\infty$ to $\|\cdot\|$ is possible because all norms of $\mathbb{R}^n$ are equivalent (a change in norm will be reflected by a different constant K_f).

1) Bounding the acceleration term: the discrete velocity derivative $\frac{\|\dot{q}\|}{\|\Delta q\|}\Delta \dot{q}$ can be further decomposed into:

$$\left\| \frac{\|\dot{q}\|}{\|\Delta q\|}\Delta \dot{q} - \ddot{\gamma}(t) \right\| \leq \left\| \Delta \dot{q}\frac{\|\dot{q}\|}{\|\Delta q\|} - \Delta \dot{\gamma}\frac{\|\dot{\gamma}(t)\|}{\|\Delta \gamma\|} \right\|$$
$$+ \frac{\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|}\left| \|\dot{\gamma}(t)\| - \frac{\|\Delta \gamma\|}{\Delta t} \right|$$
$$+ \left\| \frac{\Delta \dot{\gamma}}{\Delta t} - \ddot{\gamma}(t) \right\|.$$

Let us call these three terms (A1), (A2) and (A3). From Lemma 1,

$$(A2) \leq \frac{K_{\dot{\gamma}}}{2}\frac{\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|}\Delta t \leq \frac{K_{\dot{\gamma}}\ddot{M}}{\dot{m}}\Delta t$$
$$(A3) \leq \frac{K_{\dot{\gamma}}}{2}\Delta t.$$

Then, defining $\delta t_2 := \min \left(\delta t_1, \frac{\delta \tau_{\max}}{2K_{\dot{\gamma}}}, \frac{\delta \dot{m}\tau_{\max}}{4\ddot{M}K_{\dot{\gamma}}} \right)$, we have that, for any $\Delta t < \delta t_2$, (A2) and (A3) are upper bounded by $\frac{\delta \tau_{\max}}{4K_f}$.

The expression $\Delta \dot{q}\frac{\|\dot{q}\|}{\|\Delta \dot{q}\|}$ in (A1) represents the discrete derivative of the velocity $\dot{q}$ between q and q' (its continuous analog would be $\frac{\|\dot{q}\|}{\|\dot{q}\|}\frac{d\dot{q}}{dt} = \frac{d\dot{q}}{dt}$). Thus, (A1) can be seen as the deviation between the discrete accelerations of $\widetilde{\gamma}$ and γ. Let us decompose this expression in terms of norm and angular deviation:

$$(A1) \leq \left\| \left(\frac{\Delta \dot{\gamma}}{\|\Delta \dot{\gamma}\|} - \frac{\Delta \dot{q}}{\|\Delta \dot{q}\|} \right)\frac{\|\dot{\gamma}\|\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} \right.$$
$$\left. + \frac{\Delta \dot{q}}{\|\Delta \dot{q}\|}\left(\frac{\|\Delta \dot{\gamma}\|\|\dot{\gamma}\|}{\|\Delta \gamma\|} - \frac{\|\Delta \dot{q}\|\|\dot{q}\|}{\|\Delta q\|} \right) \right\|$$
$$\leq 2\frac{\|\dot{\gamma}\|\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|}\underbrace{\left(1 - \cos\left(\widehat{\Delta \dot{q}, \Delta \dot{\gamma}}\right) \right)}_{\text{angular deviation term } (\theta)}$$
$$+ \underbrace{\left| \frac{\|\dot{\gamma}\|\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} - \frac{\|\Delta \dot{q}\|\|\dot{q}\|}{\|\Delta q\|} \right|}_{\text{norm deviation term (N)}}$$

Since the factor $\frac{2\|\dot{\gamma}\|\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|}$ before the angular deviation (θ) is bounded by $\frac{4\ddot{M}\ddot{M}}{\dot{m}}$, $(\widehat{\Delta \dot{q}, \Delta \dot{\gamma}}) \to 0$ is a sufficient condition for $(\theta) \to 0$. We will show that both the norm and angular deviation terms tend to zero as $\Delta t \to 0$.

2) Bounding the norm (N): let us suppose that $\text{dist}_{\gamma}(x)$ and $\text{dist}_{\gamma}(x')$ are $\leq \frac{1}{2}\dot{m}\Delta t^2 =: \delta \rho$. We can expand (N) as follows:

$$(N) \leq \frac{\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|}\left| \|\dot{\gamma}\| - \|\dot{q}\| \right| + \|\dot{q}\|\left| \frac{\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} - \frac{\|\Delta \dot{q}\|}{\|\Delta q\|} \right|$$

$$(N) \leq \frac{2\ddot{M}}{\dot{m}}\delta \rho + \frac{\|\dot{q}\|}{\|\Delta \gamma\|\|\Delta q\|}\left| \|\Delta \dot{\gamma}\|\|\Delta q\| - \|\Delta q\|\|\Delta \dot{q}\| \right|$$
$$\leq \frac{2\ddot{M}}{\dot{m}}\delta \rho + \frac{\|\dot{q}\|\,(\|\Delta \gamma\| + \|\Delta \dot{\gamma}\|)\delta \rho}{\|\Delta \gamma\|\|\Delta q\|}$$
$$\leq \frac{2\ddot{M}}{\dot{m}}\delta \rho + \delta \rho\frac{\|\dot{q}\|}{\|\Delta q\|}\left[1 + \frac{2\ddot{M}}{\dot{m}} \right]$$
$$\leq \frac{2\ddot{M}}{\dot{m}}\delta \rho + \delta \rho\frac{\|\dot{\gamma}\| + \delta \rho}{\|\Delta \gamma\| - \delta \rho}\left[1 + \frac{2\ddot{M}}{\dot{m}} \right]$$
$$\leq \left[\ddot{M}\Delta t + \frac{(\dot{m} + 2\ddot{M})(2\dot{M} + \dot{m}\Delta t^2)}{\dot{m}^2(1 - \Delta t)} \right]\Delta t$$

This last bound is expressed only in terms of Δt and constants $\dot{m}, \dot{M}$ and $\ddot{M}$. Since it tends to zero as $\Delta t \to 0$, there exists some duration $\delta t_3 \leq \delta t_2$ such that, for any $\Delta t \leq \delta t_3$, (N) $\leq \frac{\delta \tau_{\max}}{8K_f}$.

3) Bounding the angular deviation: simple vector geometry shows that

$$\sin\left(\widehat{\Delta \dot{q}, \Delta \dot{\gamma}}\right) \leq \frac{\text{dist}_{\gamma}(x) + \text{dist}_{\gamma}(x')}{\|\Delta \dot{\gamma}\|} \leq \frac{\delta \rho}{\ddot{m}\Delta t} \leq \frac{\dot{m}}{2\ddot{m}}\Delta t.$$

Since $1 - \cos\theta < \sin\theta$ for any $\theta \in [0, \pi/2]$, there exists a duration $\delta t_4 \leq \delta t_3$ such that $\Delta t < \delta t_4 \Rightarrow (\theta) \leq \frac{\delta \tau_{\max}}{8K_f}$. Combining our bounds on terms (A2), (A3), (N) and (θ), we have showed so far that, when Δt is small enough, the acceleration term is upper bounded by $\frac{3}{4}\delta \tau_{\max}$.

4) Bounding the distance term (D): the remaining term is proportional to

$$(\eta + \nu)\|\Delta x\| + \text{dist}_{\gamma}(x) \leq (2\delta \rho + \|\Delta \gamma\|)(\eta + \nu) + \delta \rho$$
$$\leq \frac{K_{\gamma}(\eta + \nu) + 3\dot{m}\Delta t}{2}\Delta t$$

Hence, there exist a final $\delta t \leq \delta t_4$ such that, when $\Delta t < \delta t$, this last bound becomes $\leq \frac{\delta \tau_{\max}}{4K_f}$ as well. Combining all our bounds, we have established the existence of a duration δt such that $\Delta t \leq \delta t \Rightarrow |\widetilde{u}(\tau)| \leq \tau_{\max}$.

5) Link with completeness: let us summarize our reasoning so far. We have iteratively constructed a duration δt and a radius $\delta \rho$, independent from t or t', such that, as soon as $|t' - t| < \delta t$, $\text{dist}_{\gamma}(x) < \rho$ and $\text{dist}_{\gamma}(x') < \rho$, the system can track the trajectory INTERPOLATE(x, x').

The proof of completeness of the whole randomized planner follows directly from this construction. Let us denote by $\mathcal{B}_t := \mathcal{B}((\gamma, \dot{\gamma})(t), \rho)$, the ball of radius ρ centered on $(\gamma, \dot{\gamma})(t) \in \mathcal{X}$. Suppose that the roadmap contains a state $x \in \mathcal{B}_t$, and let $t' := \min(T, t + \delta t)$. If the planner samples a state $x' \in \mathcal{B}_{t'}$, the interpolation between x and x' will be successful and x' will be added to the roadmap. Since the volume of $\mathcal{B}_{t'}$ is non-zero for the Lebesgue metric, the event $\{\text{SAMPLE}(\mathcal{X}_{free}) \in \mathcal{B}_{t'}\}$ will happen with probability one as the number of extensions goes to infinity.

At the initialization of the planner, the roadmap is reduced to $x_{init} = (\gamma(0), \dot{\gamma}(0))$. Therefore, using the property above, by induction on the number of time steps δt, the last state $(\gamma(T), \dot{\gamma}(T))$ will be eventually added to the roadmap with

probability one, which establishes the probabilistic completeness of the randomized planner. $\square$

E. Discussion

Our proof constructs a sequence of non-empty balls that cover a solution to the planning problem. This idea that the solution trajectory can be included in a "tube" of non-zero volume is not new. It appeared in [25] where LaValle et al. hypothesized the existence of a covering called "attraction sequence". More recently, Karaman et al. sketched a proof in [24] with a similar connection hypothesis: for any two states $\gamma(t)$ and $\gamma(t')$, the steering function successfully connects $\gamma(t)$ to any point in the ball of radius $\alpha \|\Delta\gamma\|^p$ centered on $\gamma(t')$.

However, to the best of our knowledge, our work is the first theoretical analysis to establish the existence and explicitly construct such a bounding tube. This makes our approach of more connected to reality: for a given system, one can actually check for full actuation, compacity of the control set and Lipschitz continuity of the dynamics function. Similarly, when designing her interpolation function, one can easily check for properties such as local boundedness and acceleration compliance.

IV. CONCLUSION

The goal of the present paper is to clarify the panorama of completeness results in randomized kinodynamic planning. We observed that existing proofs usually rely on assumptions too strong to be verified on practical systems. We proposed a classification of the various types of kinodynamic constraints and planning methods used in the field, and went on to prove probabilistic completeness for an important class of planners, namely those who steer by interpolating system trajectories in the state space. On the way, our analysis also provided some insights into the design of these interpolation functions.

REFERENCES

[1] S. LaValle, *Planning algorithms*. Cambridge Univ Press, 2006.
[2] S. M. LaValle and J. J. Kuffner Jr, "Rapidly-exploring random trees: Progress and prospects," 2000.
[3] S. Karaman and E. Frazzoli, "Sampling-based algorithms for optimal motion planning," *The International Journal of Robotics Research*, vol. 30, no. 7, pp. 846–894, 2011.
[4] D. Hsu, J.-C. Latombe, and R. Motwani, "Path planning in expansive configuration spaces," in *Robotics and Automation, 1997. Proceedings., 1997 IEEE International Conference on*, vol. 3. IEEE, 1997, pp. 2719–2726.
[5] T. Lozano-Perez, "Spatial planning: A configuration space approach," *Computers, IEEE Transactions on*, vol. 100, no. 2, pp. 108–120, 1983.
[6] L. E. Kavraki, P. Svestka, J.-C. Latombe, and M. H. Overmars, "Probabilistic roadmaps for path planning in high-dimensional configuration spaces," *Robotics and Automation, IEEE Transactions on*, vol. 12, no. 4, pp. 566–580, 1996.
[7] B. Donald, P. Xavier, J. Canny, and J. Reif, "Kinodynamic motion planning," *Journal of the ACM (JACM)*, vol. 40, no. 5, pp. 1048–1066, 1993.
[8] D. Hsu, R. Kindel, J.-C. Latombe, and S. Rock, "Randomized kinodynamic motion planning with moving obstacles," *The International Journal of Robotics Research*, vol. 21, no. 3, pp. 233–255, 2002.
[9] J.-P. Laumond, *Robot Motion Planning and Control*. New York: Springer-Verlag, 1998.
[10] G. Oriolo and Y. Nakamura, "Control of mechanical systems with second-order nonholonomic constraints: Underactuated manipulators," in *Decision and Control, 1991., Proceedings of the 30th IEEE Conference on*. IEEE, 1991, pp. 2398–2403.
[11] Y. Nakamura and R. Mukherjee, "Nonholonomic path planning of space robots via a bidirectional approach," *Robotics and Automation, IEEE Transactions on*, vol. 7, no. 4, pp. 500–514, 1991.
[12] J. Bobrow, S. Dubowsky, and J. Gibson, "Time-optimal control of robotic manipulators along specified paths," *The International Journal of Robotics Research*, vol. 4, no. 3, pp. 3–17, 1985.
[13] M. Vukobratovic, B. Borovac, and D. Surdilovic, "Zero-moment point–proper interpretation and new applications," in *IEEE/RAS International Conference on Humanoid Robots*, 2001.
[14] K. B. Shimoga, "Robot grasp synthesis algorithms: A survey," *The International Journal of Robotics Research*, vol. 15, no. 3, pp. 230–266, 1996.
[15] F. Bullo and K. M. Lynch, "Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems," *Robotics and Automation, IEEE Transactions on*, vol. 17, no. 4, pp. 402–412, 2001.
[16] A. Yershova, L. Jaillet, T. Siméon, and S. M. LaValle, "Dynamic-domain rrts: Efficient exploration by controlling the sampling domain," in *Robotics and Automation, 2005. ICRA 2005. Proceedings of the 2005 IEEE International Conference on*. IEEE, 2005, pp. 3856–3861.
[17] M. Zucker, J. Kuffner, and J. A. Bagnell, "Adaptive workspace biasing for sampling-based planners," in *Robotics and Automation, 2008. ICRA 2008. IEEE International Conference on*. IEEE, 2008, pp. 3757–3762.
[18] A. Shkolnik, M. Walter, and R. Tedrake, "Reachability-guided sampling for planning under differential constraints," in *Robotics and Automation, 2009. ICRA'09. IEEE International Conference on*. IEEE, 2009, pp. 2859–2865.
[19] A. Perez, R. Platt, G. Konidaris, L. Kaelbling, and T. Lozano-Perez, "Lqr-rrt*: Optimal sampling-based motion planning with automatically derived extension heuristics," in *Robotics and Automation (ICRA), 2012 IEEE International Conference on*. IEEE, 2012, pp. 2537–2542.
[20] R. Tedrake, "Lqr-trees: Feedback motion planning on sparse randomized trees," 2009.
[21] J. J. Kuffner, S. Kagami, K. Nishiwaki, M. Inaba, and H. Inoue, "Dynamically-stable motion planning for humanoid robots," *Autonomous Robots*, vol. 12, no. 1, pp. 105–118, 2002.
[22] Q.-C. Pham, S. Caron, and Y. Nakamura, "Kinodynamic planning in the configuration space via velocity interval propagation," *Robotics: Science and System*, 2013.
[23] J.-P. Laumond, P. E. Jacobs, M. Taix, and R. M. Murray, "A motion planner for nonholonomic mobile robots," *Robotics and Automation, IEEE Transactions on*, vol. 10, no. 5, pp. 577–593, 1994.
[24] S. Karaman and E. Frazzoli, "Optimal kinodynamic motion planning using incremental sampling-based methods," in *Decision and Control (CDC), 2010 49th IEEE Conference on*. IEEE, 2010, pp. 7681–7687.
[25] S. M. LaValle and J. J. Kuffner, "Randomized kinodynamic planning," *The International Journal of Robotics Research*, vol. 20, no. 5, pp. 378–400, 2001.
[26] S. Karaman and E. Frazzoli, "Sampling-based optimal motion planning for non-holonomic dynamical systems," in *IEEE Conference on Robotics and Automation (ICRA)*, 2013.