ORDINARY DIFFERENTIAL EQUATIONS

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3.1 DIFFERENTIAL EQUATIONS

- A D.E. is an equation involving an unknown function y and its derivatives y', y''.

- D.E. describe many physical laws: spring-mass systems, R.C. circuit, beam bending, etc.

- A D.E. is an O.D.E. if the unknown function depends on only one independent variable.

  e.g. $M_x = - EI \; d^2w/dx^2$

  ~beam bending

- If the unknown function depends on two or more independent variables, it is known as a partial differential equation.

  e.g. $M_x = -D\,(\P^2 w/\P x_2 + n \; \P^2 w/\P y^2)$

  ~plate bending

- order & degree

highest derivative in the equation

the power to which the highest-order derivative is raised.

- Linear differential equations.

$$b_n(x)\ d^n y/dx^n + b_{n-1}(x)\ d^{n-1}y/dx^{n-1} + \ldots\ldots$$

$$+ b_1(x)\ dy/dx + b_o(x)\, y = g(x)$$

where $b_i(x)$ $(i = 0, 1 \ldots n)$ and $g(x)$ are known and depend only on x. $\therefore$ y is a function of x.

Egs. $y''' - 5xy' = e^x + 1$

Ans: 3rd order, 1st degree, Linear

$$b_3(x) = 1$$
$$b_1(x) = -5x$$
$$g(x) = e^x + 1$$

Egs. $y\{d^2x/dy^2\} = y^2 + 1$

Ans: 2nd order, 1st degree

Linear $b_2(y) = y$

$g(y) = y^2 + 1$

Egs. $\{d^2r/dy^2\}^2 + d^2r/dy^2 + y\{dr/dy\} = 0$

Ans: 2nd order, 2nd degree, non-linear.

3.2 WHY USE NUMERICAL METHODS?

- when analytical solution is near impossible.
- in many engineering problems, exact solution is not required; numerical answers within a range of variables are preferred.
- numerical methods can be easily fed into computes and results obtained quickly.

3.3 TAYLOR SERIES METHODS

$$y(x+h)=y(x)+ hy'(x)+ h^2/2!\ y''(x)+ h^3/3!\ \ y'''(x)+..$$

$$+\quad h^{n+1}/(n+1)!\ y^{(n+1)}(x)$$

where $x \pounds x \pounds x+h$

Now, given the O.D.E. $y' = f(x,y)$, with $y = y_o$

The value of y at x_1 (where $x_1 = x_o+h$) is:

$$y_1 = y(x_o) + hy'(x_o) + h^2/2\ \ y''(x_o) + h^3/6\ y'''(x_o)$$

$$+ \ldots$$

Example 1 :

Given $dy/dx = 2y + x - 0.5$,
with $x_o = 0.1$,
$y(x_o) = y_o = 1.1714$
Find the value of y at $x = 0.5$.

$$y' = 2y + x - 0.5$$
$$\backslash\ y'_o = 2(1.1714) + 0.1 - 0.5 = 1.9428$$

$$y'' = 2y' + 1$$
$$\backslash\ y''_o = 2(1.9428) + 1 = 4.8856$$

$$y''' = 2y''$$
$$\backslash\ y'''_o = 2(4.8856) = 9.7712$$

$$y^{iv} = 2y'''$$
$$\backslash\ y^{iv}_o = 2(9.7712) = 19.5424$$

Putting *h = 0.4* (a one-step calculation)

$$y(0.5) = 1.1714 + 1.9428\,(0.4) + 2.4428(0.4)^2 + 1.6285\,(0.4)^3 + 0.8143\,(0.4)^4 + error$$

$$= 2.4644 \text{ (c.f. with exact solution 2.4683)}$$

Note that if *h* is large, we need more terms in the series to get more accurate answers.

Putting *h = 0.1* [i.e. *y(0.5)* is evaluated in *4* steps starting from *y(0.1)*]

$$y(0.2) = \overset{y(0.1)}{1.1714} + 0.1\overset{y'(0.1)}{(1.9428)} + 0.1^2\overset{y''(0.1)}{(2.4428)}$$

$$+ 0.1^3\overset{y'''(0.1)}{(1.6285)} + 0.1^4\overset{y^{iv}(0.1)}{(0.8143)} + error$$

$$= 1.3918$$

Now using $x = 0.2$, $y = 1.3918$, we obtain

$y'(0.2) = 2.4836$ $\qquad$ $y'''(0.2) = 11.9345$

$y''(0.2) = 5.9673$ $\qquad$ $y^{iv}(0.2) = 23.8691$

Substituting into Taylor series:

$$y(0.3) = \overset{y(0.2)}{1.3918} + 0.1\overset{y'(0.2)}{(2.4836)} + 0.1^2/2\overset{y''(0.2)}{(5.9673)}$$

$$+ 0.1^3/6\overset{y'''(0.2)}{(11.9345)} + 0.1^4/24\,\overset{y^{iv}(0.2)}{(23.8691)}$$

$$= 1.6721 \text{ and so on for } y(0.4)$$

Verify $y(0.5) = 2.4683$
(exact with analytical solution of 2.4683)

Qn: What are the disadvantages in Taylor's method?

3.4 EULER & IMPROVED EULER METHOD

use Taylor series in truncated form by neglecting terms in h^2, h^3, etc.

$$y(x+h) \approx y(x) + hy'(x) + \frac{h^2}{2} y''(x) + \qquad (1)$$

Error term $= h^2/2\, y''(\chi)$ where $x \leq \chi \leq x+h$

By using eqn.(1) and the O.D.E. and choosing a small interval h, we can obtain $y(x_n)$ using following:

(a) given x_o, y_o **(initial conditions)**

(b) Let $h = \frac{1}{n}(x_n - x_o)$

Divide intervals into n strips of width h

(c) $x_1 = x_o + h$

(d) From O.D.E. $y_o' = f(x_o, y_o)$

(e) From Eqn.(1), $y_1 = y_o + hy_o'$

(f) Using the values of x_1, y_1, return to steps (c) - (e) to obtain:

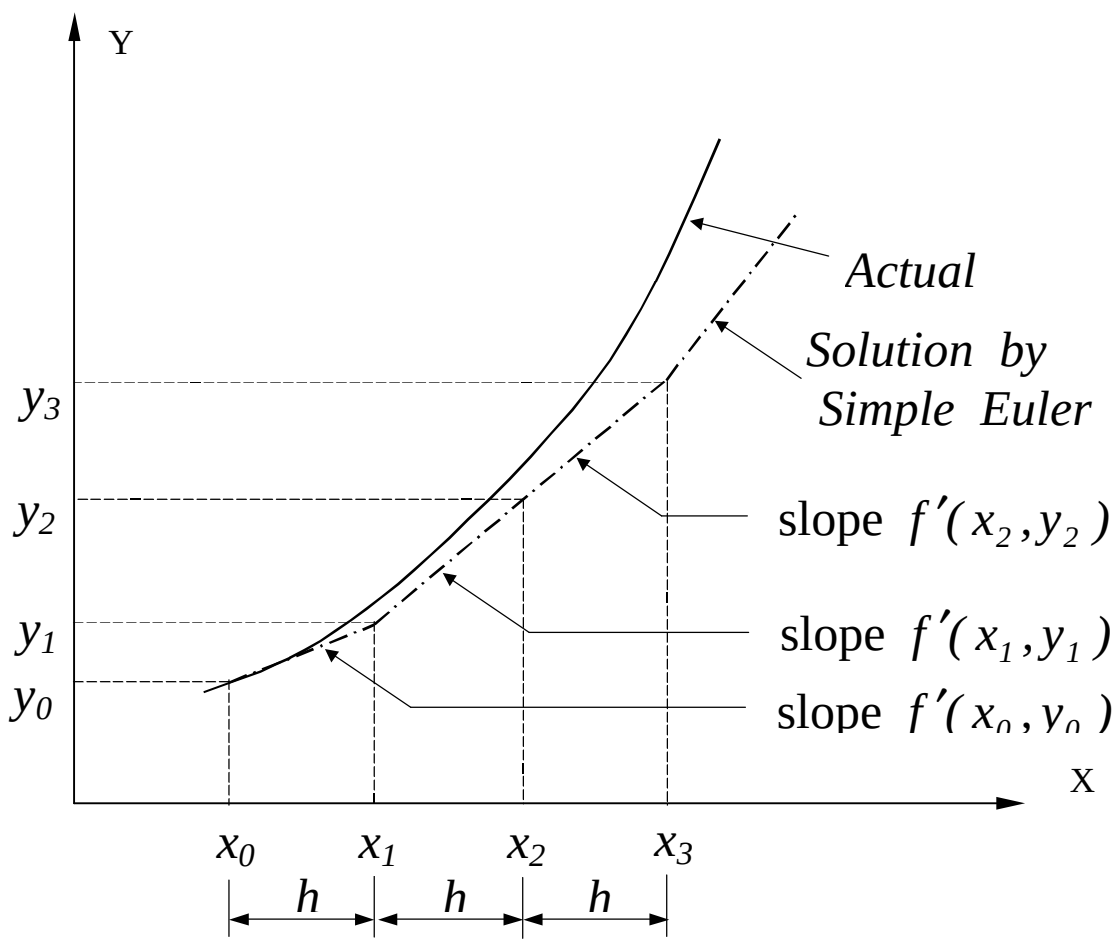
$$x_2 = x_1 + h$$
$$y_1' = f(x_1, y_1)$$
$$y_2 = y_1 + hy_1'$$

(g) Repeat step (f) until x_n is reached.

In general

$$\boxed{\begin{aligned} x_{i+1} &= x_i + h \\ y_i' &= f(x_i, y_i) \\ y_{i+1} &= y_i + hy_i' \end{aligned}}$$

- Geometrically, it is an approximation of the curve *y(x)* by a polygon whose first side is tangent to the curve at x_o.
- Note that the error can grow quite large if interval *h* is not small.



Example 2

Solve $y' = 2y + x - 0.5$
With $y(x_0) = y(0.1) = 1.1714$

Choose $h = 0.1$, and with initial conditions of

$x_o = 0.1$, $y_o = 1.1714$, evaluate y_i', x_{i+1}, y_{i+1}:

$x_{i+1} = x_i + 0.1$	$y_{i+1} = y_i + 0.1\, y_i'$	$y_i' = 2y_i + x_i - 0.5$
$x_o = 0.1$	$y_o = 1.1714$	$y_o' = 1.9428$
$x_1 = 0.2$	$y_1 = y_o + 0.1\, y_o'$ $= 1.3657$	$y_1' = 2.4314$
$x_2 = 0.3$	$y_2 = y_1 + 0.1\, y_1'$ $= 1.6088$	$y_2' = 3.0176$

In tabular form:

i	x_i	y_i	$hy_i' = 0.1\,(2y_i + x_i - 0.5)$	Exact y_i	Absolute error
0 1 2 3 4	0.1 0.2 0.3 0.4 0.5	1.1714 1.3657 1.6088 1.9106 2.2827	0.1943 0.2431 0.3018 0.3721 -	- 1.3918 1.6721 2.0255 2.4683	- 0.0261 0.0633 0.1149 0.1856

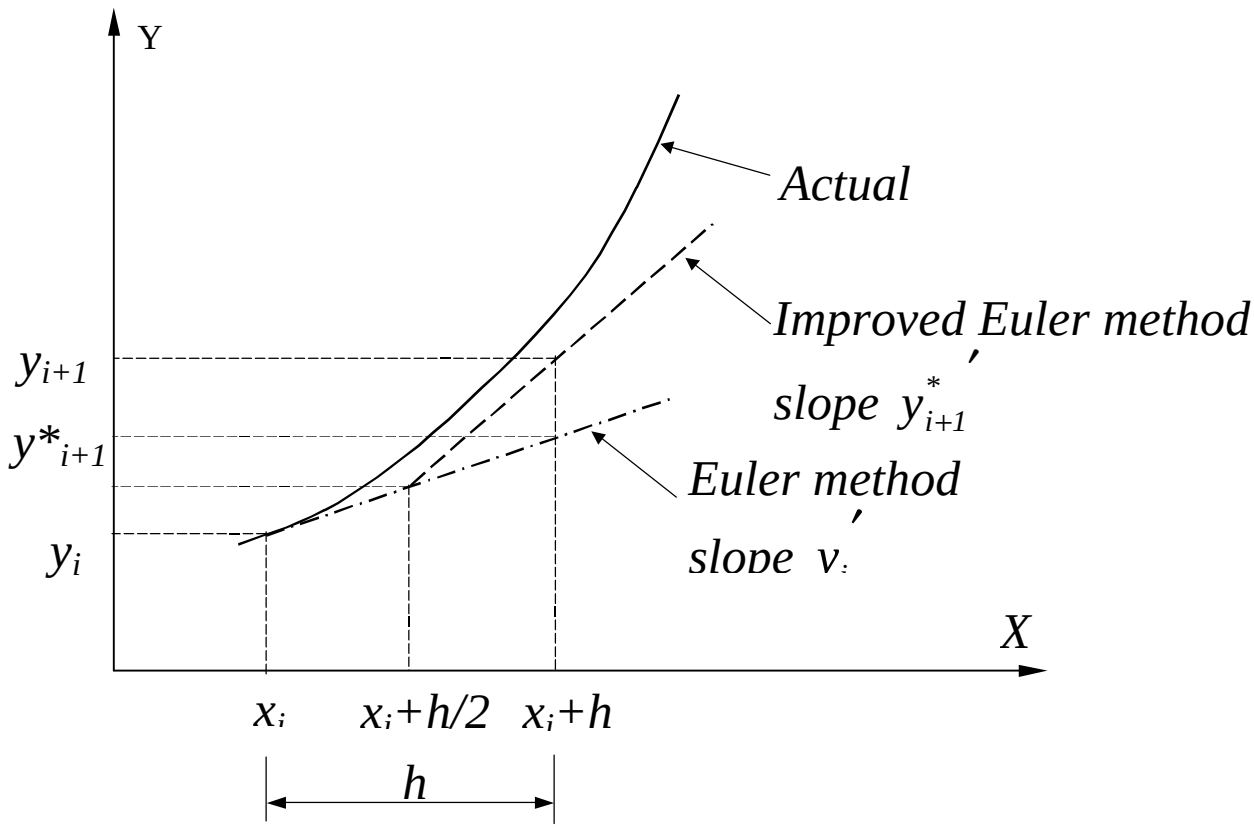
Improved Euler method

- From the point (x_i, y_i) to find the next value $y_{i+1}^* : y_{i+1}^* = y_i + hy_i'$ as before.

- Instead of accepting this as the value of y_{i+1}, use it only to calculate the gradient at the point (x_{i+1}, y_{i+1}^*), i.e. from the O.D.E.,
  $y_{i+1}^{*\prime} = f(x_{i+1}, y_{i+1}^*)$

- Then work out a "better" value of y_{i+1} using:

$$y_{i+1} = y_i + h/2\,(y_i' + y_{i+1}^{*\prime})$$



Geometrically, in the Euler method, solution y between x_i and x_{i+1} is approximated by a straight line with slope y_i'.

In improved Euler, solution y is approximated by two straight lines: one between x_i and $x_i + h/2$ having slope y_i', and one between $x_i + h/2$ and x_{i+1} having slope $y_{i+1}^{*\prime}$.

Example 3

Use improved Euler for previous Eg. 2.

	$y' = 2y + x - 0.5$	$y(0.1) = 1.1714$
$x_o = 0.1$	$y_o = 1.1714$	$y_o' = 1.9428$
$x_1 = 0.2$	$y_1^* = y_o + 0.1y_o' = 1.3657$	$y_1^{*\prime} = 2.4314$
	$y_1 = y_o + 0.1/2\,[y_o' + y_1^{*\prime}]$	$y_1' = 2.4802$
	$= 1.3901$	
$x_2 = 0.3$	$y_2^* = 1.3901 + 0.1\,(2.4802)$	$y_2^{*\prime} = 3.0762$
	$= 1.6381$	
	$y_2 = 1.3901 + 0.1/2\,[2.4802 + 3.0762]$	$y_2' = 3.1358$
	$= 1.6679$ *and so on*	

Verify for yourself $y(0.5) = 2.4557$

Compared to exact result of 2.4683 and 2.2827 from Euler method (Eg. 2).

3.4 RUNGE-KUTTA METHOD

- the term refers to a large family of **one-step** methods for handling 1st order O.D.E.
- computationally more efficient.
- uses Taylor expansion and based on retaining all terms up through h^4. Error = $0(h^5)$

Algorithm: Given $y' = f(x,y)$, $y(x_o) = y_o$

At step $n+1$, compute the following terms:

$$A_n = h.f(x_n, y_n)$$
$$B_n = h.f(x_n + h/2, y_n + A_n/2)$$
$$C_n = h.f(x_n + h/2, y_n + B_n/2)$$
$$D_n = h.f(x_{n+1}, y_n + C_n)$$
$$y_{n+1} = y_n + 1/6\,(A_n + 2B_n + 2C_n + D_n)$$

where : x_n, y_n, $x_n + h/2$, $y_n + A_n/2$ are sampling points and 6 is the weighting factor.

Example 4

Use Runge-Kutta to solve 1st order O.D.E. with one initial condition:

$$y' = 2y + x - 0.5 \qquad y(0.1) = 1.1714, \; h = 0.1.$$

$$A_o = 0.1(2y_o + x_o - 0.5)$$
$$= 0.1(2.3428 + 0.1 - 0.5) = 0.1943$$

$$B_o = 0.1\,[\,2(y_o + A_o/2) + (x_o + 0.05) - 0.5\,]$$
$$= 0.2187$$

$$C_o = 0.1\,[\,2(y_o + B_o/2) + (x_o + 0.05) - 0.5\,]$$
$$= 0.2212$$

$$D_o = 0.1[2(y_o + C_o) + (x_o + 0.1) - 0.5]$$
$$= 0.2485$$

$$y_1 = y_o + 1/6(A_o + 2B_o + 2C_o + D_o)$$
$$= 1 + 1/6(0.1943 + 0.4374 + 0.4424 + 0.2485]$$
$$= 1.3918$$

In tabular form:

x	*A*	*B*	*C*	*D*	*y*	*Exact*
0.1	-	-	-	-	*1.1714*	*1.1714*
0.2	*0.1943*	*0.2187*	*0.2212*	*0.2485*	*1.3918*	*1.3918*
0.3	*0.2484*	*0.2782*	*0.2812*	*0.3146*	*1.6721*	*1.6721*
0.4	*0.3144*	*0.3509*	*0.3545*	*0.3953*	*2.0255*	*2.0255*
0.5	*0.3951*	*0.4396*	*0.4441*	*0.4939*	*2.4683*	*2.4683*

3.6 MULTI-STEP METHOD

One such example. is ***Adam-Moulton's method***.

- given $y' = f(x,y)$ — 1st derivative equation.
- .requires 4 starting values of y, i.e. y_n, y_{n-1}, y_{n-2}, y_{n-3}.
- from y_n, y_{n-1}, y_{n-2}, y_{n-3}, find y_n', y_{n-1}', y_{n-2}', y_{n-3}' using 1st derivative equation.
- .from these starting values, it predicts a value for y_{n+1}, (i.e. y_{n+1}^*), using *Predictor Equation*.
- .then it uses a *Corrector Equation* to give the corrected value y_{n+1}.

Predictor:

$$y_{n+1}^* = h/24\,[55\overset{(n)}{y_n'} - 59\overset{(n-1)}{y_{n-1}'} + 37\overset{(n-2)}{y_{n-2}'} - 9\overset{(n-3)}{y_{n-3}'}] + y_n$$

Corrector:

$$y_{n+1} = h/24\,[9\overset{(n+1)}{y_{n+1}^{*\prime}} + 19\overset{(n)}{y_n'} - 5\overset{(n-1)}{y_{n-1}'} + \overset{(n-2)}{y_{n-2}'}] + y_n$$

Example 5

Given $y' = 2y + x - 0.5$ $y(0.1)=1.1714$.

Assume first 4 values found by Euler's method:

x	0.1	0.2	0.3	0.4
y	1.1714	1.3657	1.6088	1.9106
y'	1.9428	2.4314	3.0176	3.7212
	$(n-3)$	$(n-2)$	$(n-1)$	n

Predictor Eq.:

$$y_4^* = 0.1/24\,[\,55\overset{(n)}{(3.7212)} - 59\overset{(n-1)}{(3.0716)} + 37\overset{(n-2)}{(2.4314)} - 9\overset{(n-3)}{(1.9428)}] + 1.9106$$
$$= 2.3235$$

From D.E. $y_4^{*\prime} = 2(2.3235) + 0.5 - 0.5 = 4.6470$

Corrector Eq.:

$$y_4 = y(0.5) = 0.1/24\,[9(4.6470) + 19(3.7212) - 5(3.0176) + 2.4314] + 1.9106$$
$$= 2.3267$$

3.7 COMPARISON OF SOME METHODS

Method	Error E as a function of h		No. of function evaluations (step)	Numerical Stability	Easy to change h
	Local Error	Global Error			
Euler (1st order)	$E(h^2)$	$E(h)$	1	Poor	Yes
Modified Euler (2nd order)	$E(h^3)$	$E(h^2)$	2	Good	Yes
Runge-Kutta (4th order)	$E(h^5)$	$E(h^4)$	4	Good	Yes
Adams-Moulton	$E(h^5)$	$E(h^4)$	2	Good	No

1. Local error refers to the error of one step; global error refers to the error at the end of n steps, i.e. an accumulation of errors over the n steps.

2. Global error > Local error as $h < 1.0$

3. Euler's method (1st order) has relatively large truncation error and is often unstable.

3.8 HIGHER-ORDER EQUATIONS

Solving 2nd order O.D.E., can be non-linear.

® $A\, d^2y/dx^2 + B\, dy/dx + Cy = f(x,y)$

Reduce to 1 pair of simultaneous 1st order O.D.E.

Given: $d^2y/dx^2 = f(x, y, dy/dx)$

With $y(x_o) = y_o$, $y'(x_o) = y_o'$

Define dy/dx as a second function, e.g.

$\rightarrow$ $dy/dx = t$ *with* $y(x_o) = y_o$ (1)

$\rightarrow$ $dt/dx = f(x, y, t)$ *with* $t(x_o) = y_o'$ (2)

- Eqs.(1) & (2) are two 1st order O.D.E. which can be numerically solved <u>*simultaneously*</u> to give the solution for the 2nd order O.D.E.

- For each step of calculation, we must calculate for each equation in turn.

Example 6 : Solving 2nd order O.D.E.

$$d^2y/dx^2 = 1/2(x+ y + dy/dx + 2)$$

Need 2 initial conditions: $y(0) = 0$, $y'(0) = 0$

Find $y(0.4)$ and $y'(0.4)$, using $h = 0.2$

Let $dy/dx = t$, $\qquad y(0) = 0 \qquad (1)$

$dt/dx =1/2(x + y + t + 2)$, $\quad t(0) = y'(0)= 0 \qquad (2)$

where : $f (x, y, t) = 1/2 (x + y + t + 2)$

Using simple Euler's method.

$$\begin{Bmatrix} y \\ t \end{Bmatrix}_{0.2} = \begin{Bmatrix} y \\ t \end{Bmatrix}_{0} + 0.2 \begin{Bmatrix} y' \\ t' \end{Bmatrix}_{0}$$

$$\begin{Bmatrix} y \\ t \end{Bmatrix}_{0.2} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 0.2 \begin{Bmatrix} 0 \\ 1/2\,(0+0+0 \quad +2) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.2 \end{Bmatrix}$$

$$\begin{Bmatrix} y \\ t \end{Bmatrix}_{0.4} = \begin{Bmatrix} y \\ t \end{Bmatrix}_{0.2} + 0.2 \begin{Bmatrix} y' \\ t' \end{Bmatrix}_{0.2} \quad , \; t(0.2) = y'(0.2)$$

$$= \begin{Bmatrix} 0 \\ 0.2 \end{Bmatrix} + 0.2 \begin{Bmatrix} 0.2 \\ 1/2\,(0.2 + 0 + 0.2 \quad + 2) \end{Bmatrix} = \begin{Bmatrix} 0.04 \\ 0.44 \end{Bmatrix}$$

Try solving this using modified Euler.

Predictor:

$$\begin{Bmatrix} y \\ t \end{Bmatrix}^{*}_{0.2} = \begin{Bmatrix} y \\ t \end{Bmatrix}_{0} + 0.2 \begin{Bmatrix} y' \\ t' \end{Bmatrix}_{0} = \begin{Bmatrix} 0 \\ 0.2 \end{Bmatrix}$$

Corrector:

$$\begin{Bmatrix} y \\ t \end{Bmatrix}_{0.2} = \begin{Bmatrix} y \\ t \end{Bmatrix}_{0} + \frac{0.2}{2} \left[\begin{Bmatrix} y' \\ t' \end{Bmatrix}_{0} + \begin{Bmatrix} y' \\ t' \end{Bmatrix}^{*}_{0.2} \right]$$

$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 0.1 \left[\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0.2 \\ 1/2\,(0.2 + 0 + 0.2 + 2) \end{Bmatrix} \right]$$

$$= \begin{Bmatrix} 0.02 \\ 0.22 \end{Bmatrix}$$

Predictor:

$$\begin{Bmatrix} y \\ t \end{Bmatrix}^{*}_{0.4} = \begin{Bmatrix} y \\ t \end{Bmatrix}_{0.2} + 0.2 \begin{Bmatrix} y' \\ t' \end{Bmatrix}_{0.2}$$

$$= \begin{Bmatrix} 0.02 \\ 0.22 \end{Bmatrix} + 0.2 \begin{Bmatrix} 0.22 \\ 1/2\,(0.2 + 0.02 + 0.22 + 2) \end{Bmatrix}$$

$$\begin{Bmatrix} y \\ t \end{Bmatrix}^{*}_{0.4} = \begin{Bmatrix} 0.02 \\ 0.22 \end{Bmatrix} + 0.2 \begin{Bmatrix} 0.22 \\ 1.22 \end{Bmatrix} = \begin{Bmatrix} 0.064 \\ 0.464 \end{Bmatrix}$$

Corrector:

$$\begin{Bmatrix} y \\ t \end{Bmatrix}_{0.4} = \begin{Bmatrix} y \\ t \end{Bmatrix}_{0.2} + \frac{0.2}{2} \left[\begin{Bmatrix} y' \\ t' \end{Bmatrix}_{0.2} + \begin{Bmatrix} y' \\ t' \end{Bmatrix}^{*}_{0.4} \right]$$

$$= \begin{Bmatrix} 0.02 \\ 0.22 \end{Bmatrix} + 0.1 \left[\begin{Bmatrix} 0.22 \\ 1.22 \end{Bmatrix} + \begin{Bmatrix} 0.464 \\ 0.5(0.4 + 0.064 + 0.464 + 2) \end{Bmatrix} \right]$$

$$= \begin{Bmatrix} 0.02 \\ 0.22 \end{Bmatrix} + \begin{Bmatrix} 0.0684 \\ 0.2684 \end{Bmatrix} = \begin{Bmatrix} 0.0884 \\ 0.4884 \end{Bmatrix}$$

cf $y(0.4) = 0.0918$ (Exact answer)

3.9 SUMMARY (We've covered so far, ...)

. use the Taylor series to solve a 1st order D.E.

. use Euler, modified Euler or Runge Kutta for 1st & 2nd order .

. One-step method (Euler, Improved Euler, Runge-Kutta)

- need information only at a preceeding point to progress to next point. Methods are "self-starting".
- employs Taylor's expansion.
- functions may be evaluated several times. Can be time consuming.

. Multi-step method (also known as Predictor-Corrector)

- not self-starting, rely on one-step to start.
- one-step methods (of same order) are of comparable accuracy to the multi-step methods (e.g. Runge Kutta and Adam Moulton).
- no. of function evaluations for Runge Kutta is four, while a predictor-corrector method of the same order requires only two.

$\therefore$ Predictor-corrector methods can be more efficient.

SINGLE STEP METHOD

Starting point		Local errors
	— Simple Euler	$0(h^2)$
Taylor Series	— Modified Euler	$0(h^3)$
	— Runge Kutta 4th order	$0(h^5)$

MULTI-STEP METHOD

Starting point:		Local error
Simple or Modified Euler (require 4 pts)	— Adam-Moulton	$0(h^5)$
	— Predictor formula	
	— Corrector formula	

SUMMARY

1. **Runge-Kutta Method**

$$A_i = h.f(x_i, y_i)$$

$$B_i = h.f(x_i + h/2, y_i + A_i/2)$$

$$C_i = h.f(x_i + h/2, y_i + B_i/2)$$

$$D_i = h.f(x_i + h, y_i + C_i)$$

$$y_{i+1} = y_i + 1/6(A_i + 2B_i + 2C_i + D_i)$$

2. **Multi-Step Method**

Predictor:

$$y_{i+1}^* = h/24[55y_i' - 59y_{i-1}' + 37y_{i-2}' - 9y_{i-3}'] + y_i$$

Corrector:

$$y_{i+1} = h/24\ [9y_{i+1}^{*\prime} + 19y_i' - 5_{y_{i-1}}' + y_{i-2}'] + y_i$$

3. **Higher Order Equations**

Let $dy/dx = t$ with $y(x_o) = y_o$ (1)

$dt/dx = f(x, y, t)$ with $t(x_o) = y_o'$ (2)

APPENDIX: **ANALYTICAL SOLUTIONS TO O.D.E.**

1.1 SEPARABLE EQUATION

Form : $Q(y)\, y' = P(x)$

Solution: $\int Q(y)\, dy = \int P(x)\, dx + constant$

Example 1

Solve $\{ y'/(1+x^2) \} + 1 + y = 0$

$$\{ 1/(1+x^2) \}\, dy/dx = -(1+y)$$

$$\int dy/(1+y) = -\int (1+x^2)\, dx$$

$$In\,(1+y) = -(x + x^3/3) + c$$

$$\backslash \qquad y = A e^{-\left(\frac{x^3}{3}+x\right)} - 1$$

Question: What are the order and degree of this equation.? Express it linear form.

1.2 EQUATIONS REDUCIBLE TO SEPARABLE FORM BY SUBSTITUTION

Form: $y' = P(y/x)$

Solution: Let $v = y/x$ or $y = vx$

$$dy/dx = v + x\, dv/dx$$

$$v + x\, dv/dx = P(v)$$

$$\int dv/(P(v)-v) = \int dx/x + constant$$

Example 2

$$dy/dx = (x+y)/y = 1 + y/x = 1 + v$$

put $v = y/x$

$$P(v) = 1 + v$$

$$v + x(dv/dx) = 1 + v$$

$$\int dv/(1+v-v) = \int dx/x + c$$

$$v = \ln x + c$$

Upon back substituting

$$y/x = \ln x + c$$

$$\therefore \quad y = x \ln x + cx$$

Linear O.D.E. 1st order, 1st degree

1.3 EXACT O.D.E.

Form : $M(x,y)\,dx + N(x,y)\,dy = 0$

L.H.S. is an exact differential of a function $F(x,y)$ for all x,y in a domain D. Express as below:

$$M(x,y)\,dx + N(x,y)\,dy = \partial F/\partial x\ dx + \partial F/\partial y\ dy = dF$$

$\Rightarrow$ $\partial F/\partial x = M(x,y)$ & $\partial F/\partial y = N(x,y)$ for all x,y

$$\partial^2 F / \partial x\, \partial y = \partial M / \partial y = \partial N / \partial x \qquad \textit{for all } x, y \subset D$$

$\therefore$ The O.D.E. is exact.

Solution: Once an O.D.E. is found to be exact, the solution can be obtained by integrating the partial derivatives $\partial F/\partial x$ and $\partial F/\partial y$.

Example 3

Solve $dy/dx = -4xy - 3x^2/(2x^2 + 2y)$

$\Rightarrow$ $(3x^2 + 4xy)\, dx + (2x^2 + 2y)\, dy = 0 = dF$

Let $M = 3x^2 + 4xy$ $\quad N = 2x^2 + 2y$

$\partial M/\partial y = 4x$, $\quad \partial N/\partial x = 4x$

$\therefore$ ODE is exact

Then, $\partial F/\partial x = 3x^2 + 4xy$ - (1)

$\partial F/\partial y = 2x^2 + 2y$ -(2)

From (1) $F = x^3 + 2x^2y + f(y)$ - (3)

From (2) and (3)

$\partial F/\partial y = 2x^2 + \partial f(y)/\partial y = 2x^2 + 2y$

$\therefore f(y) = y^2 + c$

Hence, $F(x,y) = x^3 + 2x^2y + y^2 + c$

Since $dF = 0$, $F(x,y) = constant$

$\therefore$ The solution is $x^3 + 2x^2y + y^2 = constant$.

1.4 INTEGRATING FACTORS

(Please refer to Kreyszig for greater details)

Form: $M(x,y)\,dx + N(x,y)\,dy = 0$

But equation is **not exact.**

Solution: Multiplying the equation by a suitable function $Q(x,y)$ to make it exact.

$Q(x,y) \sim$ **Integrating Factor** of the D.E.

Example 4

Given non-linear O.D.E.

$dy/dx=2x^2+y/(x-x^2y)$ Þ $(2x^2+y)dx+(x^2y-x)dy=0$

(Check that D.E. is not exact from Eg.3).

Multiplying by $Q(x)$, the integrating factor,

$[Q(x)\,(2x^2+y)]dx+[Q(x)(x^2y-x)]dy=0$ *is exact.*

Þ $\partial/\partial y\,[Q(x)\,(2x^2+y)]=\partial/\partial x\,[Q(x)\,(x^2y-x)]$

$Q(x).1 = \partial Q(x)/\partial x\,(x^2y-x)+Q(x)(2xy-1)$

$$\partial Q(x)/\partial x = -2/x\,Q(x)$$

Integrating Þ $Q(x) = constant/x^2 = c/x^2$

Þ $c/x^2\,(2x^2+y)\,dx+c/x^2\,(x^2y-x)dy=0$

1.5 LINEAR FIRST-ORDER O.D.E.

Form: $\underline{1}y' + f(x)y = r(x)$

$f(x)$, $r(x)$ are both functions of x.

Case (a) Homogeneous form: $r(x) = 0$.

i.e. $\int dy/y = -\int f(x)\,dx$ Þ $y = C\,e^{-\int f(x)dx}$

Case (b) Non-homogeneous form: $r(x)$ ¹ 0

Multiply the O.D.E. by an I.F. $e^{-\int f(x)dx}$.

i.e. $y'e^{\int f(x)dx} + f(x)\,ye^{\int f(x)dx} = r(x)e^{\int f(x)dx}$

$$L.H.S. = d/dx\left[ye^{\int f(x)dx}\right]$$

\ $d/dx\left[ye^{\int f(x)dx}\right] = r(x)\,e^{\int f(x)dx}$

Þ $ye^{\int f(x)dx} = \int\left[r(x)e^{\int f(x)dx}\right]dx + c$

Example 5

Solve $xy' + y + 4 = 0$ - Linear 1st order non-homogeneous

Rewriting $y' + y/x = -4/x$

$$ye^{\int 1/x\,dx} = \int -4/x\, e^{\int 1/x\,dx}\,.dx$$

$$y\,e^{1nx} = \int -4/x\, e^{1nx}\,.dx$$

But $e^{1nx} = x$ Þ $yx = -\int 4\,dx = -4x + c$

$$\backslash\ y = c/x - 4$$

1.6 HOMOGENEOUS LINEAR D.E. WITH CONSTANT COEFFICIENTS

Form: (of nth order)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots a_1 y^{(1)} + a_o y = 0 \quad \dots (1)$$

where a_1 = constant, $i = 1$ to n and $a_n \neq 0$

To obtain the general solution, n linearly independent solutions are required.

$$y = e^{lx} \quad \dots (2)$$

Note $y^{(i)} = d^i u/dx^i \quad i = 1, n.$

Substituting (2) into (1), we obtain

$$e^{lx} [a_n l^n + a_{n-1} l^{n-1} + \dots + a_1 l + a_o] = 0$$

For non-trivial solution, $e^{\lambda x} \neq 0$

$$\Rightarrow a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0 \quad (3)$$

Note:

(a) Eqn.(3) is the characteristic equation of the O.D.E. and the roots are called characteristic roots.

(b) The solution to Eqn.(3) is the solution to the homogeneous equation.

(c) λ can be +ve, -ve or complex.

Once the roots $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are obtained, the general solution is written as:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \Lambda + C_n e^{\lambda_n x} \quad (4)$$

where $C_1, C_2, \ldots, C_n$ are constants.

For second order homogeneous O.D.E.

$$a_2 y'' + a_1 y' + a_o y = 0$$

The characteristic equation is quadratic:

$$a_2 l^2 + a_1 l + a_0 = 0$$

Condition	Solution
2 distinct roots: l_1, l_2 $a_1^2 > 4a_2 a_o$	$y = A e^{l_1} + B e^{l_2}$
Equal roots $l_1 = l_2$ $a_1^2 = 4a_2 a_o$	$y = e^{lx}(Ax + B)$
Complex roots $l = a \pm i\beta$ $a_1^2 < 4a_2 a_o$	$y = e^{ax}(A \cos \beta x + B \sin \beta x)$

1.7 Non-homogeneous Linear D.E. with constants.

Form: (n^{th} order)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y^{(1)} + a_0 y = f(x) \quad (5)$$

Similar to Eqn (1) except LHS = $f(x) \neq 0$

Solution steps

(a) Find the solution of the homogeneous equation associated with (5).

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y^{(1)} + a_0 y = 0$$

Solution is: $y_h = C_1 e^{l_1} + C_2 e^{l_2} + .. + C_n e^{l_n}$

This is called homogeneous solution.

(b) Find any particular solution y_p which satisfies equation (5).

(c) If y_h is the homogeneous solution and y_p is any particular solution, the general solution is $y = y_h + y_p$.

Obtaining y_p by method of undetermined coefficients

To obtain y_p for the ODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y^{(1)} + a_0 y = f(x)$$

$f(x)$ is a linear combination of :

(a) x^{α} ($\alpha \geq 0$) (c) $\cos \gamma x$ ($\gamma \geq 0$)

(b) $e^{\beta x}$ ($\beta \geq 0$) (d) $\sin \delta x$ ($\delta \geq 0$)

Simple rules for determining y_p:

Term in $f(x)$	Choice of y_p
(a) kx^p	$A_p x^p + A_{p-1} x^{p-1} + .. + A_1 x + A_O$
(b) ke^{px}	$A e^{px}$
(c) $k \cos px$	$A \cos px + B \sin px$
(d) $k \sin px$	$A \cos px + B \sin px$
(e) sum of functions	sum of functions
(f) A term in $f(x)$ is a solution of the homogeneous O.D.E.	Multiply the appropriate function by x if roots are distinct, or by x^2 if roots are equal.

Example 6

Solve $y'' - 4y' + 3y = -3e^{-x}$

Characteristic Eqn.: $l^2 - 4l + 3 = 0$

$$l = 1, 3$$

Homogeneous solution is $y_h = Ae^{x} + Be^{3x}$

y_p: *Let* $y_p = Ce^{-x}$

$y'_p = -Ce^{-x}$ & $y''_p = Ce^{-x}$

Comparing coefficients of e^{x} gives $C = -3/8$

$\therefore$ General solution is

$$y = A\,e^{x} + B\,e^{3x} - 3/8\,e^{-x}.$$

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