Passive Fixed-Income Portfolio Management

Overview
- Passive Fixed-Income Portfolio
- Straightforward Replication
- Replication by Stratified Sampling
- Replication by Tracking-Error Minimization
  - Optimization Procedure
  - Estimation of Bond Return Covariance Matrix
  - Comparison of Stratified Sampling & Tracking-Error Minimization
- Factor-Based Replication
  - Derivatives-Based Replication

William C. H. Leon
Nanyang Business School
March 16, 2018
One of the most profound ideas affecting the investment decision process is that the security markets are efficient.

In an efficient market, the prices of securities do not depart for any length of time from the justified economic values that investors calculate for them. Economic values for securities are determined by investor expectations about earnings, risks and so on, as investors grapple with the uncertain future. If the market price of a security does depart from its estimated economic value, investors act to bring the two values together.

As new information arrives in an efficient market place, causing a revision in the estimated economic value of a security, its price adjusts to this information quickly and, on balance, correctly. In other words, securities are efficiently priced on a continuous basis.

An efficient market does not have to be perfectly efficient to have a profound impact on investors. All that is required is that the market be economically efficient, i.e., after acting on information to trade securities and subtracting all costs (e.g., transaction costs and taxes), the investor would have been as well off with a simple buy-and-hold strategy.

If the market were economically efficient, securities could depart somewhat from their economic (justified) values, but it would not pay investors to take advantage of these small discrepancies.

If the market were totally efficient, no active strategy should be able to beat the market on a risk-adjusted basis.
A natural outcome of a belief in efficient markets is to use some type of passive strategy in owning and managing portfolios. Passive strategies do not seek to outperform the market but simply to do as well as the market. The emphasis is on minimizing transaction costs and time spent in managing the portfolio because any expected benefits from active trading or analysis are likely to be less than costs. Passive investors act as if the market were efficient and take the consensus estimates of return and risk, accepting current market price as the best estimate of a security’s value.

Many investors manage portfolios (or parts of portfolios) to match index returns. Even active managers may fall back to passive index tracking in times when they have no definite views. The simplest way to replicate an index is to buy most of its securities in the proper proportions. This method, however, is practical only for the largest index funds. For smaller portfolios, maintaining the necessary proportions of a large number of bonds would necessitate buying odd lots and lead to overwhelming transaction costs. Investors with smaller portfolios often build index proxies, i.e., portfolios that contain only a small number of securities yet deviate minimally from the returns of much larger target indices. This is known as bond indexing.
The first bond index fund was in the early conceptual stages in 1985 when an article in Forbes magazine discussing the inability of high-cost bond fund managers to match the bond market indices asked, “Vanguard, where are you when we need you?”

By the next year Vanguards Total Bond Market Index Fund (VBMFX) was up and running. SEI Funds also started a bond index fund that year.

In 1991, Galaxy Funds opened an index fund of government long bonds and Mainstay Funds started its long-term bond index. Also in 1991, Charles Schwab Co. opened its Short-Term Bond Market Index. In 1994, Vanguard created the first series of bond index funds of varying maturities — short, intermediate and long.

Today there are a large number of bond index funds.

Although bond funds’ customer base has not grown as fast as that in stock index funds, bond index fund managers had handled more than $20 billion at the end of 2001. Today, VBMFX has net assets in excess of $196 billion.

Bond index fund assets have grown slowly in part because report of the virtues of fund indexing has for the most part spread through word of mouth, and because low-cost index funds rarely budget much for sales and marketing.

Bond index funds occupy a fairly small niche in the world of mutual funds; only approximately 3% of all bond fund assets are in bond index funds in 2001, and these assets are held disproportionately by institutional investors, who keep about 25% of their bond fund assets in bond index funds.
A straightforward bond index replication technique involves duplicating the target index precisely, i.e., holding all the target index’s securities in their exact proportions.

- Once replication is achieved, trading in the indexed portfolio becomes necessary only when the makeup of the index changes or as a way of reinvesting cash flows.
- While this approach is often preferred for equities, it is not practical with bonds for the following reasons:
  - Most bond indices represent collections of thousands of bonds from various debt sectors with various maturities. It would not be cost-efficient for managers to include all issues from an index (e.g., they would forgo very substantial volume discounts at auction).
  - Many of the bonds in the indices are thinly traded, resulting in large bid-ask spread.
  - The composition of the index changes regularly, as the bonds mature.

Given that the cost of owning and trading such a large number of securities would be prohibitive, an alternative approach, known as sampling or optimization, seeks to reproduce the overall attributes of the index (credit quality, yield, duration, convexity, sector diversification, etc.) with a limited number of issues.

- While this may sound simple in theory, it is difficult to achieve in practice.
- There are three basic approaches to index replication other than straightforward replication:
  - Stratified sampling or cell matching: This approach consists in replicating index attributes.
  - Tracking-error minimization: This approach consists in replicating index returns directly.
  - Factor-based replication: This approach consists in matching the exposure of the replicating portfolio with respect to a set of common factors with that of the benchmark.
Passive Fixed-Income Portfolio Management

Replication by Stratified Sampling

Replication by Tracking-Error Minimization

Factor-Based Replication

Stratified sampling bond indexing techniques are the “common sense approach”. To replicate an index, one has to represent every important component that it has with a few securities.

- First, a portfolio manager maps a benchmark onto an arbitrary grid and then sets portfolio allocations to each of the cells that match those of the benchmark. In other words, it consists in dividing the index into cells, each cell representing a different characteristic, and then buying bonds to match those characteristics (e.g., credit rating, duration, coupon rate, maturity, market sectors, call factors, etc.).

- Next, the objective is to select from all of the issues in the index, one or more issues in each cell that can be used to represent the entire cell. The more securities are selected in each cell, the more closely the resulting portfolio tracks the index.

- The holdings of securities in a particular cell are usually computed to match that cell’s contribution to the overall duration.

The number of cells used will depend on the dollar amount of the portfolio to be indexed. A portfolio with a small dollar value will have fewer cells, so as not to increase tracking error due to transaction costs.

The total dollar amount invested in each cell will typically depend upon the share of that cell in the entire universe of securities represented in the index.

The stratified sampling bond indexing method is sometimes coupled with an optimization procedure.

- The optimization approach is also a cellular one.

- The choice of securities from each cell, however, is made with a view to achieving some other objective, such as maximizing yield to maturity, maximizing convexity and so on. Mathematical programming is used to achieve this goal.
Consider all the bonds in a U.S. corporate bond index and separate them along three dimensions:
- Duration: “0 to 4”, “4 to 7” and “More than 7”.
- Quality: “Aaa and Aa”, “A” and “Baa”.
- Sector: Industrials, Utilities, Finance and Yankee.

This maps bonds in the index to a 36-cell grid (three duration cells × three quality cells × four sectors cells).

### Example: Index Profile by Duration, Quality, and Sector

<table>
<thead>
<tr>
<th></th>
<th>Percent of Market Value</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Duration</td>
<td>0 to 4</td>
<td>4 to 7</td>
<td>More than 7</td>
<td>Total</td>
</tr>
<tr>
<td>Aaa and Aa</td>
<td>Industrials</td>
<td>1.6</td>
<td>1.1</td>
<td>2.5</td>
<td>5.2</td>
</tr>
<tr>
<td></td>
<td>Utilities</td>
<td>0.3</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>Finance</td>
<td>4.9</td>
<td>2.9</td>
<td>1.7</td>
<td>9.5</td>
</tr>
<tr>
<td></td>
<td>Yankees</td>
<td>4.5</td>
<td>3.4</td>
<td>2.3</td>
<td>10.2</td>
</tr>
<tr>
<td></td>
<td>Total Aaa-Aa</td>
<td>11.1</td>
<td>7.6</td>
<td>6.7</td>
<td>25.4</td>
</tr>
<tr>
<td>A</td>
<td>Industrials</td>
<td>4.6</td>
<td>5.5</td>
<td>8.7</td>
<td>18.8</td>
</tr>
<tr>
<td></td>
<td>Utilities</td>
<td>9.7</td>
<td>0.8</td>
<td>1.0</td>
<td>2.3</td>
</tr>
<tr>
<td></td>
<td>Finance</td>
<td>6.6</td>
<td>5.7</td>
<td>3.1</td>
<td>15.4</td>
</tr>
<tr>
<td></td>
<td>Yankees</td>
<td>1.4</td>
<td>2.2</td>
<td>2.8</td>
<td>6.4</td>
</tr>
<tr>
<td></td>
<td>Total A</td>
<td>13.5</td>
<td>14.2</td>
<td>15.6</td>
<td>43.3</td>
</tr>
<tr>
<td>Baa</td>
<td>Industrials</td>
<td>4.5</td>
<td>6.9</td>
<td>7.5</td>
<td>18.9</td>
</tr>
<tr>
<td></td>
<td>Utilities</td>
<td>1.2</td>
<td>1.4</td>
<td>1.2</td>
<td>3.8</td>
</tr>
<tr>
<td></td>
<td>Finance</td>
<td>1.8</td>
<td>1.5</td>
<td>0.5</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>Yankees</td>
<td>1.2</td>
<td>2.9</td>
<td>0.8</td>
<td>4.9</td>
</tr>
<tr>
<td></td>
<td>Total Baa</td>
<td>9.2</td>
<td>12.5</td>
<td>9.8</td>
<td>31.5</td>
</tr>
<tr>
<td>Corporate Index</td>
<td></td>
<td>30.6</td>
<td>34.3</td>
<td>32.1</td>
<td>97.0</td>
</tr>
</tbody>
</table>
Example: Stratified Sampling

- The index is characterized by the percentage of market capitalization within each of these 36 cells.
- The rightmost column of the table gives the index composition by quality and sector.
- The subtotals at the bottom of each credit quality level give the breakdown by quality and duration.
  - The marginal sums of the three-dimensional market view can provide a similar two-dimensional view along any two of these axes.

Example: Index Profile by Duration and Sector

The following is a two-dimensional profile by duration and sector:

<table>
<thead>
<tr>
<th>Duration</th>
<th>Percent of Market Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 to 4</td>
</tr>
<tr>
<td>Industrials</td>
<td>10.7</td>
</tr>
<tr>
<td>Utilities</td>
<td>2.0</td>
</tr>
<tr>
<td>Finance</td>
<td>13.3</td>
</tr>
<tr>
<td>Yankees</td>
<td>7.6</td>
</tr>
<tr>
<td>Totals</td>
<td>33.6</td>
</tr>
</tbody>
</table>
Replication by Tracking-Error Minimization

Risk models allow us to replicate indices by creating minimum tracking error portfolios. These models rely on historical volatilities and correlations between returns on different asset classes or different risk factors in the market.

- The variance minimization approach uses historical data to estimate the variance of the tracking error for each issue in the index, and then uses that to minimize the total variance of the tracking error.
- The tracking error variance of a given security is obtained by estimating the price function for that security as a function of its cash flows, duration and other sector characteristics.
- Quadratic programming is then used to find the optimal index portfolio in terms of minimized tracking error.

The technique involves two separate steps:
- The estimation of the bond return covariance matrix; and
- The use of that covariance matrix for tracking error optimization.

Consider the problem about how to replicate as closely as possible a bond index return $R_B$ with a portfolio invested in $N$ individual bonds.

The return $R_P$ of the replicating portfolio is

$$R_P = \sum_{i=1}^{N} \omega_i R_i,$$

where $\omega_1, \ldots, \omega_N$ are the weights of the $N$ bonds in the replicating portfolio, and $R_1, \ldots, R_N$ the returns of these bonds.

Denote the variance-covariance matrix of the $N$ bonds by

$$S = (\sigma_{ij})_{1 \leq i,j \leq N}.$$
Optimization Procedure

(Continue)

Mathematically, the problem can be stated as

\[
\min_{\omega_1, \ldots, \omega_N} \text{Var}(R_P - R_B) = \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \omega_j \sigma_{ij} - 2 \sum_{i=1}^{N} \omega_i \sigma_{iB} + \sigma_B^2,
\]

where \(\sigma_{iB}\) is the covariance between the return on the \(i\)-th bond in the replicating portfolio and the benchmark return \(R_B\), and \(\sigma_B\) the volatility of the benchmark.

In the absence of constraints on the sign of the weights and when the expected return of the replicating portfolio is required to be equal to that of the benchmark, we obtain a classic (constrained) linear regression (zero intercept)

\[
R_{tB} = \sum_{i=1}^{N} \omega_i R_{ti} + \epsilon_t,
\]

where \(R_{tB}\) (respectively, \(R_{ti}\)) denotes the return on the benchmark (respectively, the \(i\)-th bond) at date \(t\), and \(\epsilon_t\) a usual error term.

In this case, we also obtain confidence intervals on the optimal portfolio weights.
The choice of securities in the replicating portfolio is a straightforward task when the number $M$ of candidate securities in the universe is small and the number $N$ of securities to be included in the replicating portfolio is small.

When $M$ increases and $N$ approaches $M/2$, it becomes a complex optimization problem. The idea is to form equally weighted portfolios using $N$ among the $M$ securities, and find the best candidates. This is known as an integer programming problem, which can be solved using some specific generic algorithm.

Tracking Error

The quality of replication is measured by tracking error (TE), defined as the standard deviation of the difference between the return on the portfolio and that of the benchmark, i.e.,

$$ TE = \sqrt{\text{Var}(R_p - R_b)}. $$

Note that TE is precisely the quantity that was minimized by the optimization procedure.
Example: Tracking-Error Minimization

Consider eight T-Bonds bonds with coupon rate and maturity date as follow:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Coupon Rate (%)</th>
<th>Maturity Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond 1</td>
<td>6.250</td>
<td>31-Jan-2002</td>
</tr>
<tr>
<td>Bond 2</td>
<td>4.750</td>
<td>15-Feb-2004</td>
</tr>
<tr>
<td>Bond 3</td>
<td>5.875</td>
<td>15-Nov-2005</td>
</tr>
<tr>
<td>Bond 4</td>
<td>6.125</td>
<td>15-Aug-2007</td>
</tr>
<tr>
<td>Bond 5</td>
<td>6.500</td>
<td>15-Feb-2010</td>
</tr>
<tr>
<td>Bond 6</td>
<td>5.00</td>
<td>15-Aug-2011</td>
</tr>
<tr>
<td>Bond 7</td>
<td>6.250</td>
<td>15-May-2030</td>
</tr>
<tr>
<td>Bond 8</td>
<td>5.375</td>
<td>15-Feb-2031</td>
</tr>
</tbody>
</table>

Collect data on daily returns on these bonds, as well as on daily returns on the JP Morgan T-Bond index from 03-Aug-2001 to 30-Jan-2002 that is used as a benchmark.

Example: Tracking-Error Minimization (Continue)

On the basis of that information, estimate the correlation matrix that contains all pair-wise correlations as follow:

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Bond 1</th>
<th>Bond 2</th>
<th>Bond 3</th>
<th>Bond 4</th>
<th>Bond 5</th>
<th>Bond 6</th>
<th>Bond 7</th>
<th>Bond 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bond 1</td>
<td>0.053140992</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bond 2</td>
<td>0.057480252</td>
<td>0.057480252</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bond 3</td>
<td>0.762480254</td>
<td>0.03220904</td>
<td>0.59322667</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bond 4</td>
<td>0.804905377</td>
<td>0.03097812</td>
<td>0.92899180</td>
<td>0.97469302</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bond 5</td>
<td>0.672896100</td>
<td>0.03271805</td>
<td>0.86552177</td>
<td>0.99221547</td>
<td>0.98270525</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bond 6</td>
<td>0.987947111</td>
<td>0.03032763</td>
<td>0.57380074</td>
<td>0.71760129</td>
<td>0.81051264</td>
<td>0.900679183</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Bond 7</td>
<td>0.932180847</td>
<td>0.022865263</td>
<td>0.430986201</td>
<td>0.782414722</td>
<td>0.664779045</td>
<td>0.73490473</td>
<td>0.90703721</td>
<td>1</td>
</tr>
<tr>
<td>Bond 8</td>
<td>0.912529251</td>
<td>0.022895281</td>
<td>0.586054807</td>
<td>0.690525675</td>
<td>0.786802205</td>
<td>0.835492564</td>
<td>0.860483218</td>
<td>0.932159144</td>
</tr>
</tbody>
</table>

Note that most bond returns are fairly highly correlated with the return on the index, and medium maturity bonds tend to post higher correlation with the index.

This can be explained by the fact that the Macaulay duration of the index turns out to be 6.73 on average over the period.
Using $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, where $\sigma_{ij}$ (respectively, $\rho_{ij}$) denotes the covariance (respectively, the correlation) between the return on bond $i$ and bond $j$, and $\sigma_i$ is the volatility of bond $i$, for $i, j = B, 1, \ldots, 8$, obtain the following sample variance-covariance matrix:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Benchmark</th>
<th>Bond 1</th>
<th>Bond 2</th>
<th>Bond 3</th>
<th>Bond 4</th>
<th>Bond 5</th>
<th>Bond 6</th>
<th>Bond 7</th>
<th>Bond 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark</td>
<td>2.102E-05</td>
<td>4.906E-07</td>
<td>7.647E-06</td>
<td>1.472E-05</td>
<td>1.7409E-05</td>
<td>2.2699E-05</td>
<td>2.5779E-05</td>
<td>4.0520E-05</td>
<td>4.1617E-05</td>
</tr>
<tr>
<td>Bond 3</td>
<td>1.472E-05</td>
<td>3.772E-05</td>
<td>6.449E-05</td>
<td>1.744E-05</td>
<td>1.342E-04</td>
<td>1.667E-05</td>
<td>1.400E-05</td>
<td>3.155E-05</td>
<td>2.559E-05</td>
</tr>
<tr>
<td>Bond 4</td>
<td>1.7409E-05</td>
<td>3.9704E-05</td>
<td>6.547E-06</td>
<td>3.772E-05</td>
<td>1.342E-04</td>
<td>1.667E-05</td>
<td>1.400E-05</td>
<td>3.155E-05</td>
<td>2.559E-05</td>
</tr>
<tr>
<td>Bond 5</td>
<td>2.2699E-05</td>
<td>3.9704E-05</td>
<td>6.449E-05</td>
<td>1.744E-05</td>
<td>1.342E-04</td>
<td>1.667E-05</td>
<td>1.400E-05</td>
<td>3.155E-05</td>
<td>2.559E-05</td>
</tr>
<tr>
<td>Bond 8</td>
<td>4.1617E-05</td>
<td>6.1226E-05</td>
<td>2.559E-05</td>
<td>2.559E-05</td>
<td>2.559E-05</td>
<td>2.559E-05</td>
<td>2.559E-05</td>
<td>2.559E-05</td>
<td>2.559E-05</td>
</tr>
</tbody>
</table>

From there, perform the following optimization:

$$\min_{\omega_1, \ldots, \omega_8} TE = \sqrt{\sum_{i=1}^{8} \sum_{j=1}^{8} \omega_i \omega_j \sigma_{ij} - 2 \sum_{i=1}^{8} \omega_i \sigma_{iB} + \sigma_B^2}, \quad (1)$$

subject to the constraint

$$\sum_{i=1}^{8} \omega_i = 1.$$

An additional short sales constraint $\omega_i \geq 0$ for all $i = 1, \ldots, 8$ can be further added.
Example: Tracking-Error Minimization

The constrained optimization problem can be solved using Excel Solver function, which gives the optimal portfolio, as follow:

<table>
<thead>
<tr>
<th>Sample covariance matrix</th>
<th>Bond 1</th>
<th>Bond 2</th>
<th>Bond 3</th>
<th>Bond 4</th>
<th>Bond 5</th>
<th>Bond 6</th>
<th>Bond 7</th>
<th>Bond 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>With short sales constraints (%)</td>
<td>12.93</td>
<td>14.19</td>
<td>0.00</td>
<td>0.00</td>
<td>62.41</td>
<td>8.33</td>
<td>2.13</td>
<td></td>
</tr>
<tr>
<td>Without short sales constraints (%)</td>
<td>1.99</td>
<td>39.92</td>
<td>-1.43</td>
<td>20.93</td>
<td>-62.38</td>
<td>8.39</td>
<td>3.44</td>
<td>13.93</td>
</tr>
</tbody>
</table>

To get a better insight about whether portfolio optimization allows for a significant improvement in tracking error, we compute the tracking error of an arbitrary equally weighted portfolio of the eight bonds, and it comes out to be a daily 0.14%.

The interpretation is that the replicating portfolio will deviate on average by a daily 0.14% from the target.

When portfolio optimization is used, on the other hand, we obtain a 0.07% and a 0.04% tracking error for the optimal portfolio in the presence and in the absence of short sale constraints, respectively.

These results strongly suggest that portfolio optimization allows for a significant improvement in a passive portfolio strategy since the tracking error is reduced by a factor of 0.14%/0.07% = 2 and 0.14%/0.04% = 3.5.
Example: Tracking-Error Minimization

The improvement in terms of quality of replication can be seen from the following three pictures that display the evolution of $100 invested in the benchmark, and three different kinds of replicating portfolios:

- An equally weighted portfolio.

Replicating portfolios (continue).

- An optimized portfolio with short sale constraints.
Example: Tracking-Error Minimization (Continue)

- Replicating portfolios (continue).
- An optimized portfolio without short sale constraints.

![Graph showing replicating portfolio vs benchmark over time]

Covariance Matrix Estimation

The key input to the optimization procedure is the variance-covariance matrix $S$ of the $N$ bond returns.

Several methods for asset return covariance matrix estimation have been suggested in the literature. They fall within the following three categories:

- Sample covariance matrix estimators.
- Factor models based estimators.
- Shrinkage estimators.
The most common estimator of return covariance matrix is the sample covariance matrix of historical returns

\[
\hat{S} = \frac{1}{T-1} \sum_{t=1}^{T} (R_t - \bar{R})(R_t - \bar{R})^T,
\]

where \( T \) is the sample size, \( N \) is the number of bonds in the portfolio, \( R_t \) is the \( N \)-vector of bond returns in period \( t \), and \( \bar{R} \) is the \( N \)-vector of the average of these returns over time.

- Note that for a portfolio of \( N \) bonds, there are \( N(N-1)/2 \) different covariance terms to estimate.

A problem with the sample covariance matrix is that it may have too many parameters compared to the available data. This leads to having too few degrees of freedom relative to the number of parameters that have to be estimated; and results in excessive sampling errors.

A possible remedy for this problem is to go farther back in time to gather more data; however, this implies using outdated information.
Weighted Sample Covariance Matrix

A generalization to the sample covariance matrix estimation allows for declining weights to be assigned to observations as they go further back in time:

\[
\tilde{S} = \sum_{t=1}^{T} w_t \left( R_t - \bar{R} \right) \left( R_t - \bar{R} \right)^T,
\]

where \( w_t \) is the weight for the observation at time \( t \).

- When \( w_t = \frac{1}{T-1} \) for all \( t \), we get the sample covariance matrix estimator.
- Typically, the weights are taken to decline exponentially, e.g.,

\[
w_t = \frac{\lambda^{T-t+1}}{\sum_{t=1}^{T} \lambda^t},
\]

where the decay rate \( \lambda \) is a calibrated parameter with a commonly used value \( \lambda = 0.94 \).

Example: Exponentially-Weighted Covariance Matrix

Using the same eight T-Bonds bonds data as in the previous example, implement the exponentially weighted estimator of the variance-covariance matrix with \( \lambda = 0.94 \), and obtain the following optimal portfolios, with and without short sale constraints, after resolving the minimization program (1):

<table>
<thead>
<tr>
<th>Exponentially weighted estimator</th>
<th>Bond 1</th>
<th>Bond 2</th>
<th>Bond 3</th>
<th>Bond 4</th>
<th>Bond 5</th>
<th>Bond 6</th>
<th>Bond 7</th>
<th>Bond 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>With short sales constraints (%)</td>
<td>0.37</td>
<td>34.44</td>
<td>0.27</td>
<td>4.02</td>
<td>0.00</td>
<td>44.79</td>
<td>0.00</td>
<td>16.10</td>
</tr>
<tr>
<td>Without short sales constraints (%)</td>
<td>0.20</td>
<td>29.00</td>
<td>5.21</td>
<td>18.80</td>
<td>-23.80</td>
<td>54.74</td>
<td>-4.97</td>
<td>20.82</td>
</tr>
</tbody>
</table>
When portfolio optimization is used, we obtain a 0.02% and a 0.01% tracking error for the optimal portfolio in the presence and in the absence of short sale constraints, respectively.

Because the variance-covariance matrix is different from the sample estimate previously used, such numbers cannot be compared to what was obtained before.

The tracking error for the equally weighted portfolio turns out to be 0.12%, while it was equal to 0.14% when the sample estimate of the variance-covariance matrix was used.

A first comparison can still be drawn in terms of relative improvement. We obtain here that tracking error has been reduced by a factor of 0.12%/0.02% = 6 and 0.12%/0.01% = 12, respectively, as opposed to 2 and 3.5 in the case of the previous sample estimate.

A better understanding of the performance of various estimators of the variance-covariance matrixes in this context can actually be achieved by running a horse race on an out-of-sample basis.

More specifically, one would have to divide the data set in two subperiods, one used for calibration, the other for back-testing. The different estimators would be calibrated using, say, two-thirds of the available data, perform the optimization, and then record the performance of the replicating portfolio on the remaining one-third of the data.

Finally, an ex-post tracking error can be derived as the standard deviation of the excess return of the replicating portfolio compared to the benchmark portfolio.
The traditional estimator, the sample covariance matrix, is seldom used because it imposes too little structure. One possible cure to the curse of dimensionality in covariance matrix estimation is to impose some structure on the covariance matrix to reduce the number of parameters to be estimated.

In the case of asset returns, a low-dimensional linear factor structure seems natural and consistent with standard asset pricing theory, as linear multi-factor models can be economically justified through equilibrium arguments (see Merton’s Inter-temporal Capital Asset Pricing Model) or arbitrage arguments (see Ross’s Arbitrage Pricing Theory).

There are two types of factor models, one being a single-index model estimator where the single factor is taken to be a market index, the other being a multiple-index implicit factor model.

One-Factor Market Model

Sharpe’s (1963) single-index model assumes that asset returns are generated by

\[ R_t = \alpha + \beta M_t + \epsilon_t, \]

where \( \beta \) is the factor loading vector, \( \epsilon_t \) a vector of residuals \( \epsilon_{t_1} \) that are assumed to be uncorrelated to market return, denoted by \( M_t \), and to one another. The covariance matrix implied by this model is

\[ F = \sigma^2_M \beta \beta^T + \Omega, \]

where \( \sigma^2_M \) is the variance of the market portfolio, and \( \Omega \) is the diagonal matrix containing residual variances.
Note that the exact composition of the market portfolio is unknown. In particular, no existing market index captures both equity and fixed-income investment opportunities worldwide.

The exact composition of the market portfolio, however, is not as critical here as all we need is to explain a significant amount of the variance of bond returns.

For simplicity, one may take the market index as the equal-weighted portfolio of the \( N \) bonds in the sample, i.e.,

\[
M_t = \frac{1}{N} \sum_{i=1}^{N} R_{ti}.
\]

Alternatively, a broad-based index can be used as a proxy for the market portfolio.

---

**Example: Market Model Covariance Matrix**

Using the same eight T-Bonds data as in the previous example, implement a version of the market model by regressing the return on each of the eight bonds in the replicating portfolio on the index return. The following table shows the betas of these bonds, as well as the optimal replicating portfolios with and without short sale constraints, after resolving the minimization program (1):

<table>
<thead>
<tr>
<th>Single-index covariance matrix</th>
<th>Bond 1</th>
<th>Bond 2</th>
<th>Bond 3</th>
<th>Bond 4</th>
<th>Bond 5</th>
<th>Bond 6</th>
<th>Bond 7</th>
<th>Bond 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta</td>
<td>0.021</td>
<td>0.361</td>
<td>0.695</td>
<td>0.824</td>
<td>1.071</td>
<td>1.216</td>
<td>1.930</td>
<td>1.963</td>
</tr>
<tr>
<td>With short sales constraints (%)</td>
<td>8.20</td>
<td>10.84</td>
<td>8.29</td>
<td>7.81</td>
<td>7.85</td>
<td>48.73</td>
<td>4.74</td>
<td>3.53</td>
</tr>
<tr>
<td>Without short sales constraints (%)</td>
<td>8.20</td>
<td>10.84</td>
<td>8.29</td>
<td>7.81</td>
<td>7.85</td>
<td>48.73</td>
<td>4.74</td>
<td>3.53</td>
</tr>
</tbody>
</table>
Example: Market Model Covariance Matrix

- In this case, the presence of short sale constraints does not affect the results of the optimization because short sale constraints are not binding in the first place.
- Note that the optimal portfolio is now significantly different from the one obtained with the sample estimate and the exponentially-weighted estimate of the variance-covariance matrix.
  - In the market model, the number of parameters to be estimated is relatively low.
  - Therefore, the specification risk that is induced by using an index model may be larger than any improvement in estimation risk that can be achieved from that approach.

Multifactor Model

There are two types of multifactor models: explicit factor models and implicit factor models.

Define a return matrix $R$ as follow:

$$R = (R_{kt})_{1 \leq t \leq T, 1 \leq k \leq K},$$

where $R_{kt}$ is the return of the $k$-th bond at date $t$. The objective is to describe each variable as a linear function of a reduced number of factors as follows:

$$R_{kt} = \sum_{j=1}^{K} b_{jk} F_{tj},$$

where factor $F_{tj}$ belongs to a set of $K$ orthogonal variables, and $b_{jk}$ is the sensitivity of the $k$-th variable to the $j$-th factor defined as

$$\frac{\Delta (R_{kt})}{\Delta (F_{tj})} = b_{jk}.$$
Use principle component analysis (PCA) to decompose $R_{tk}$ as follows:

$$R_{tk} = \sum_{j=1}^{K} \sqrt{\lambda_j} U_{jk} V_{tj},$$

where

- $(U_{jk})_{1 \leq j, k \leq K}$ is the matrix of the $K$ eigenvectors of $R^T R$.
- $(V_{tj})_{1 \leq t \leq T, 1 \leq j \leq K}$ is the matrix of the $K$ eigenvectors of $RR^T$.
- $\lambda_j$ is the eigenvalue (ordered by degree of magnitude) corresponding to the eigenvector $U_j$.

Denote $V_{tj} = F_{tj}$ and $b_{jk} = \sqrt{\lambda_j} U_{jk}$ is called the principal component’s sensitivity of the $k$-th variable to the $j$-th factor.

The main challenge is to select a number of factors $J < K$ such that the first $J$ factors capture a large fraction of bond return variance, while the remaining part can be regarded as statistical noise, i.e.,

$$R_{tk} = \sum_{j=1}^{J} b_{jk} F_{tj} + \sum_{j=J+1}^{K} b_{jk} F_{tj} = \sum_{j=1}^{J} b_{jk} F_{tj} + \epsilon_{tk},$$

where the residuals $\epsilon_{tk}$ are assumed to be uncorrelated to one another.

- The total data set variance percentage explained by the first $J$ factors is given by $\frac{\sum_{j=1}^{J} \lambda_j}{\sum_{j=1}^{K} \lambda_j}$.
- Typically, two or three factors account for a very large fraction of bond return variations.
Shrinkage Estimators

Shrinkage estimators combine two estimators, the sample covariance estimator (which contains large estimation risk but no model risk) and a factor-based estimator (which contains model risk but lower estimation risk) to achieve optimal balance between sampling error and specification error.

There are two types of shrinkage estimators:

- The first approach is due to Jorion (1985, 1986) who argues that the class of shrinkage estimators as proposed by Stein (1955) handles the problem of parameter uncertainty in portfolio selection appropriately.
- Ledoit (1999) offers a new approach to a shrinkage estimator of the variance-covariance matrix by proposing an estimator that optimally shrinks the sample covariance matrix toward the one-factor model covariance matrix.

Jorion’s Shrinkage Estimator

Under suitable assumptions, Jorion (1985, 1986) suggests that the covariance matrix estimator is:

\[
\hat{S}_J = \left(1 + \frac{1}{T + p}\right) \frac{(T - 1)\hat{S}}{T - N - 2} + \frac{p}{T(T + p + 1)} \frac{11^T}{1^T \left(\frac{(T - 1)\hat{S}}{T - N - 2}\right)^{-1} 1}
\]

where \(\hat{S}\) is the sample covariance matrix, \(1\) is a \(N \times 1\) vector of ones, and \(p\) is some precision parameter defined as \(p = wT/(1 - w)\), where \(w\) is the shrinkage coefficient,

\[
w = \frac{N + 2}{N + 2 + T(\bar{R} - \bar{R}_0)\left(\frac{(T - 1)\hat{S}}{T - N - 2}\right)^{-1}(\bar{R} - \bar{R}_0)1}.
\]

where \(\bar{R}_0\) is the grand mean, i.e., the mean of the minimum variance portfolio estimated, obtained using the sample covariance matrix.
Ledoit’s Shrinkage Estimators

Ledoit (1999) suggests that the covariance matrix estimator is:

\[ \hat{\Sigma}_L = \frac{a}{T} \left( \sigma_M^2 \beta^T + \Omega \right) + \left( 1 - \frac{a}{T} \right) \hat{\Sigma}, \]

where \( a = (p - r)/c \), with

\[ p = \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij} \quad \text{and} \quad p_{ij} = \frac{1}{T} \sum_{t=1}^{T} \left( (R_{ti} - \bar{R}_i)(R_{tj} - \bar{R}_j) - s_{ij} \right), \]

\[ c = \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} \quad \text{and} \quad c_{ij} = (f_{ij} - s_{ij})^2, \]

\[ r = \sum_{i=1}^{N} \sum_{j=1}^{N} r_{ij} \quad \text{and} \quad r_{ij} = \frac{1}{T} \sum_{t=1}^{T} r_{tij} \quad \text{and} \]

\[ r_{tij} = \left( \frac{s_{ij} \sigma_M^2 (R_{ti} - \bar{R}_i) + s_M^2 (R_{tj} - \bar{R}_j) - s_M s_{ij} (M_t - \bar{M})}{\sigma_M^2} \right) \times \left( (R_{ti} - \bar{R}_i)(R_{tj} - \bar{R}_j) - s_{ij} \right), \]

and \( (s_{ij}) = \hat{\Sigma}, \ (s_M) = \text{Cov}(R, M) \text{ and } (f_{ij}) = \sigma_M^2 \beta^T + \Omega. \)
Remarks on Shrinkage Estimators

- Jagannathan and Ma (2000) show that imposing portfolio weight constraints is equivalent to shrinking the extreme covariance estimates toward the average estimates.

- To the extent that these extreme estimates are more likely to be plagued by estimation error, this shrinkage estimators can reduce the sampling error. Hence, they provide formal justification to the well-known fact that imposing portfolio weight constraints actually reduces estimation error.

Stratified Sampling & Tracking-Error Minimization

The stratified sampling approach consists in replicating index attributes, while the tracking-error approach consists in replicating index returns, either directly or through replication of factors explaining a large fraction of index returns.

- One problem with stratified sampling is that a mismatch to the benchmark in any cell appears to be equally important. In reality, matching some cells is more critical than matching others because the return (or spread) volatility associated with them is higher.

- Sampling technique also ignores correlations among cells that sometimes cause risk from an overweight in one cell to be canceled with an overweight in another.
Stratified Sampling & Tracking-Error Minimization

- Tracking-error minimization techniques rely on historical volatilities and correlations between returns on different asset classes or different risk factors in the market. Therefore, the model’s “knowledge” is limited to the historical experience observed over the calibration period. Such models may ignore a significant structural change that historically has not yet resulted in return volatility. In other words, it tends to be a backward-looking technique.

- Rather than regarding stratified sampling approach and tracking-error approach as competing methods, one may view them as complementary. For example, portfolio managers using tracking-error models may be alerted to a possible shift in market conditions by stratified sampling techniques, and may wish to take corrective measures based on their expectations that are not necessarily reflective of history.

Factor-Based Replication

Index replication may also be based upon the risk decomposition allowed by a factor model.

Suppose that you select, using PCA, a small number of (implicit) factors $J$ such that they capture a large fraction of bond return variance, while the remaining part can be regarded as statistical noise. For simplicity, assume that $J = 3$. In that case, each bond return $R_t$ can be written as

$$R_t = b_1 F_{t1} + b_2 F_{t2} + b_3 F_{t3} + \epsilon_t,$$

where the residuals $\epsilon_t$ are assumed to be uncorrelated to one another.
**Factor-Based Replication**

Adding over different bond returns, the return on a bond portfolio $P$ can be written as

$$R_t^P = b_{1P} F_{t1} + b_{2P} F_{t2} + b_{3P} F_{t3} + \epsilon_{tP},$$

with

$$b_{jP} = \sum_{i=1}^{N} \omega_i b_{ji},$$

for $j = 1, 2, 3$, where $N$ is the number of bonds in the bond portfolio $P$.

Similarly, the return on the benchmark (bond index) $B$ can be written as

$$R_t^B = b_{1B} F_{t1} + b_{2B} F_{t2} + b_{3B} F_{t3} + \epsilon_{tB}.$$
Example: Factor-Based Replication

Using the same eight T-Bonds bonds data as in the previous example, regress the return on each of the eight bonds and the benchmark on the following two factors:

- The first factor is the change in 3-month interest rate, regarded as a proxy for changes in the level of the term-structure.
- The second factor is the change in the spread between the 30-year rate and the 3-month rate, regarded as a proxy for changes in the slope of the term-structure.
- Recall that from our previous discussion on the term-structure that these two factors usually account for a very large fraction of bond return variation.

The following table shows the result of this regression when the left-hand variable is the return on the benchmark:

<table>
<thead>
<tr>
<th>Coefficients</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>6.14942E-05</td>
</tr>
<tr>
<td>Factor 1</td>
<td>-27.0865211</td>
</tr>
<tr>
<td>Factor 2</td>
<td>-22.2656083</td>
</tr>
</tbody>
</table>

The R-squared for that regression is 0.91, suggesting that more than 90% of the time variation in the return on the benchmark is captured by the time variation in the two aforementioned factors.

This provides us with a motivation to use factor-based replication; if we manage to find a portfolio with exactly the same exposure to these two risk factors as the benchmark, then the behavior of that portfolio will be close to that of the benchmark.
Example: Factor-Based Replication

The following table shows the betas of the bonds and the benchmark with respect to the two risk factors:

<table>
<thead>
<tr>
<th></th>
<th>Beta 1</th>
<th>Beta 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark</td>
<td>-27.0865211</td>
<td>-22.26560827</td>
</tr>
<tr>
<td>Bond 1</td>
<td>-2.8137614</td>
<td>-1.084816042</td>
</tr>
<tr>
<td>Bond 2</td>
<td>-11.1461975</td>
<td>-5.75974001</td>
</tr>
<tr>
<td>Bond 3</td>
<td>-19.9855764</td>
<td>-13.46348007</td>
</tr>
<tr>
<td>Bond 4</td>
<td>-23.1662469</td>
<td>-15.95768661</td>
</tr>
<tr>
<td>Bond 5</td>
<td>-28.7109262</td>
<td>-22.18051504</td>
</tr>
<tr>
<td>Bond 6</td>
<td>-32.4878358</td>
<td>-26.25424263</td>
</tr>
<tr>
<td>Bond 7</td>
<td>-50.8107969</td>
<td>-48.54826524</td>
</tr>
<tr>
<td>Bond 8</td>
<td>-52.2945817</td>
<td>-50.57663654</td>
</tr>
</tbody>
</table>

Using the sample estimate of the variance-covariance matrix, perform the following optimization program:

\[
\text{min } \omega_1, \ldots, \omega_8 \quad \text{TE} = \sqrt{\sum_{i=1}^{8} \sum_{j=1}^{8} \omega_i \omega_j \sigma_{ij} - 2 \sum_{i=1}^{8} \omega_i \sigma_{iB} + \sigma_B^2},
\]

subject to the constraints

\[
\sum_{i=1}^{8} \omega_i = 1, \quad \sum_{i=1}^{8} \omega_i s_{1i} = -27.0865211 \quad \text{and} \quad \sum_{i=1}^{8} \omega_i s_{2i} = -22.26560827,
\]

where \(s_{1i}\) (respectively, \(s_{2i}\)) is the sensitivity of bond \(i\) with respect to the first (respectively, second) factor.
Example: Factor-Based Replication
(Continue)

The corresponding optimal replicating portfolios in the presence and in the absence of short sale constraints are given the following table:

<table>
<thead>
<tr>
<th>Factor-based replication</th>
<th>Bond 1</th>
<th>Bond 2</th>
<th>Bond 3</th>
<th>Bond 4</th>
<th>Bond 5</th>
<th>Bond 6</th>
<th>Bond 7</th>
<th>Bond 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>With short sales constraints (%)</td>
<td>16.70</td>
<td>13.71</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>56.31</td>
<td>10.14</td>
<td>3.14</td>
</tr>
<tr>
<td>Without short sales constraints (%)</td>
<td>2.66</td>
<td>46.10</td>
<td>-3.22</td>
<td>8.67</td>
<td>-54.77</td>
<td>82.12</td>
<td>5.21</td>
<td>13.22</td>
</tr>
</tbody>
</table>

These portfolio weights are actually fairly close to what was obtained in the case of the sample covariance matrix estimate, and the in-sample tracking-error is roughly the same (0.07% and 0.04% in the presence and in the absence of short sale constraints, respectively).

The usefulness of the factor-based replication approach is that it allows for more robustness, that is, for potential improvement of the out-of-sample tracking-error, because we attempt to replicate what explains the return on the benchmark, as opposed to replicating the return on the benchmark regarded as a black box.
The replicating portfolio does not necessarily consist of securities sampled out of the index being replicated. A very practical alternative is using futures and swaps, which are liquid market instruments with return characteristics similar to many of the index securities.

- A variation of the cell-matching technique can actually be applied to replicate the term-structure exposure of any fixed-income index with Treasury futures. Futures are widely used as a duration adjustment tool because of advantages such as no portfolio disruption, ease of establishing and unwinding positions, and low transaction costs.

- For funds with frequent and significant cash inflows and outflows, replication of benchmark returns with exchange-traded futures is often an attractive strategy. By taking a long or short position in a single contract, investors can match the duration of any benchmark. However, meaningfully replicating the performance of a broad-based market index requires matching its exposures to all segments of the yield curve.

A methodology, currently implemented by a number of investors, uses four Treasury futures contracts (2-, 5-, 10- and 30-year) to replicate the curve allocation of an index. By analyzing the distribution of security durations in the index, one may determine the required mix of contracts.

- The first step is to divide the index into four duration cells.
- Then compute the allocation and dollar duration within each cell of a perfectly indexed investment of the desired size.
- Each cell’s market value and dollar duration are then matched with a combination of a cash investment and a position in the appropriate futures contract. The cash is usually invested in Treasury bills, though portfolio managers are free to choose other alternatives, such as commercial paper or short-term asset-backed securities, as a source of extra return.
Derivatives-Based Replication

- Term-structure exposure can be hedged effectively with Treasury futures.
- Spread risk, inherent in the Credit and Mortgage indices, needs to be hedged separately.
- Eurodollar futures and swaps can be used in a similar methodology to replicate spread indices.