Weighted Iterative Truncated Mean Filter

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***Abstract*—The iterative truncated arithmetic mean (ITM) filter was proposed recently. It offers a way to estimate the sample median by simple arithmetic computing instead of the time consuming data sorting. In this paper, a rich class of filters named weighted ITM (WITM) filters are proposed. By iteratively truncating the extreme samples, the output of the WITM filter converges to the weighted median. Proper stopping criterion makes the WITM filters own merits of both the weighted mean and median filters and hence outperforms the both in some applications. Three structures are designed to enable the WITM filters being low-, band- and high-pass filters. Properties of these filters are presented and analyzed. Experimental evaluations are carried out on both synthesis and real data to verify some properties of the WITM filters.**

***Index Terms*—ITM filter, weighted median filter, noise suppression, nonlinear filter, band-pass filter, high-pass filter.**

I. Introduction

THE linear filters are widely used in digital signal/image processing because of their rigorous mathematical foundation and their efficiency in attenuating additive Gaussian noise [1]. The sample mean is the optimal solution for suppressing additive Gaussian noise in the sense of mean square error (MSE) if all samples have the same variance, and so is the weighted mean if the variances are not identical. However, none of the mean and weighted mean filters is the optimal if the long-tailed noise, such as Laplacian noise, presents. Moreover, in some applications of image processing, the linear filters are undesirable because they blur the image structures and cannot suppress the impulse noise effectively. Therefore, nonlinear filters which can preserve signal structures and effectively suppress long-tailed noise were developed.

The median filter [2] is the most widely applied one among the nonlinear filters. It provides a powerful tool for signal/image processing because of its good property in impulsive noise suppression and edge preservation. However, it destructs fine signal details, and has poor performance in attenuating Gaussian and other short-tailed noise. It loses as much as 40% efficiency over the mean filter in suppressing Gaussian noise [3]. In order to preserve signal details, many detail preserving filters were developed, including truncation filters [4], multistage median filters [5], [6], FIR-median hybrid filters [7] and various adaptive noise switching median filters [8]–[11]. Effort was devoted to improve the performance of the median filter in suppressing short-tailed noise by making compromises between the mean and median filters. Such filters include the L filter [12], the STM filter [12], the α-trimmed mean ($\alpha\mathrm{T}$) filter [13], [14], the modified trimmed mean (MTM) filter [15], the mean-median (MEM) filter [3], [16] and the median affine (MA) filter [17]. The outputs of these filters move smoothly between the sample mean and median by adjusting some free parameters. However, it is not an easy task to choose the optimal parameters to make these filters adaptive to signal types [3], [17], [18].

Similar to the mean filter, the median filter is inefficient if the variances of different samples are not the same. The weighted median filters with positive weights were proposed to deal with the non-identical distributed Laplacian noise. Such filters are used in many applications, e.g., speech signal processing, images filtering [19] and waveform prediction [20]. However, they cannot achieve the acceptable results in some applications, such as equalization, beamforming and system identification, which require band- or high-pass characteristics. To overcome the above limitations, the general weighted median (GWM) filters admitting both positive and negative weights were proposed in [21]. The GWM filters can be designed as band-pass and high-pass filters. They are applied in various applications, such as sigma-delta modulation encoding [22], denoising [23]–[25], image sharpeners [26], edge detection [27], edge enhancement [28], system identification [29], [30] and multichannel signal processing [31]. The GWM filters have been extended to admit complex value of weights [32]. Designing the weights is a critical part and great effort was devoted in it [21], [33], [34].

Both the median and weighted median filters have some limitations. First, these filters are not as effective as the mean and weighted mean filters in suppressing the short-tailed Gaussian noise. Second, they are not the optimal ones even for the long-tailed Laplacian noise [35], [36]. Moreover, the median and weighted median filters are built on data sorting. It has high computational complexity compared to arithmetic computing and its implementation is also complicated [37].

Filters which can outperform the median filter in suppressing both the short- and long-tailed noise while do not require data sorting are desirable. The myriad filters [38]–[43] were designed for the α-stable distributed noise model. Its performance highly depends on the tunable "linearity parameter" [42], [44] computed from the prior knowledge of the noise distribution. The iterative truncated arithmetic mean (ITM) filter [35] employs a simple truncating algorithm to iteratively truncate the extreme samples. Its output approaches the median by increasing the number of iterations. Proper stopping criterion enables the ITM filter outperforms the median filter in suppressing both Gaussian and Laplacian noise. Edge preservation and noise suppression can be achieved within just a few iterations. It also provides an

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approach to estimate the median by a simple arithmetic computing algorithm. Its implementation given in [36] is faster than the median and myriad filters.

The merits of the ITM filter inspire us to extend it into a rich class of filters. Although the ITM filter outperforms the median filter in suppressing the identical distributed Gaussian and Laplacian noise, analogous to the non-weighted mean and median filters, its performance drops dramatically in dealing with the non-identical distributed noise. Furthermore, the ITM filter cannot be used in applications which require band- or high-pass characteristics. In this paper, we propose a rich class of filters named weighted ITM (WITM) filters, of which the ITM filter is a special case with all weights being equal. The truncating procedure of the ITM filter is extended to the WITM filters. By iteratively truncating the samples, the output of the WITM filter starts from the weighted mean and approaches the weighted median. A stopping criterion is proposed to terminate the iteration so that the WITM filters can outperform both the weighted mean and median filters in some applications. Three structures are utilized to enable the WITM filters with negative weights being low-, band- and high-pass filters. The superiority of the proposed WITM filters is verified in the experiments.

II. Weighted ITM Filters With Positive Weights

The weighted ITM filters are proposed following necessary reviews of the weighted mean and median filters and the ITM filter [35]. A stopping criterion is proposed and the filter properties are discussed.

A. Weighted Mean and Median Filters

The weighted mean is the maximum likelihood (ML) estimate of location for data sets with Gaussian distribution. Assume a filter window contains n independent Gaussian distributed samples as $\mathbf{x}_0 = \{x_1, x_2, \ldots, x_n\}$ with unknown constant mean μ_o. The variance of the ith sample is σ_i^2. The ML estimate of location μ_o is to find the value of μ, which maximizes the likelihood function

$$\begin{aligned} L(\mu \,|\, x_1, x_2, \ldots, x_n) &= \prod_{i=1}^{n} f(x_i \,|\, \mu, \sigma_i) \\ &= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \right) \\ &\quad \times \exp\left(-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma_i^2} \right). \end{aligned} \tag{1}$$

It is equivalent to minimizing the squares sum

$$G_2(\mu) = \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma_i^2}. \tag{2}$$

The value of μ minimizing (2) is the weighted mean μ_w

$$\mu_w = \arg\min_{\mu} G_2(\mu) = \sum_{i=1}^{n} w_i x_i \Bigg/ \sum_{i=1}^{n} w_i, \tag{3}$$

where $w_i = 1/\sigma_i^2$. μ_w is the optimal estimate of μ_o because its variance equals to the Cramer-Rao lower bound (CRLB) [45].

Algorithm 1: Truncation Procedure of the ITM Filter

Input: $\mathbf{x}_0 \Rightarrow \mathbf{x}$; **Output**: Truncated $\mathbf{x}$;

do

Compute the sample mean: $\mu = \mathrm{mean}(\mathbf{x})$;

Compute the dynamic threshold: $\tau = \mathrm{mean}(|\mathbf{x} - \mu|)$;

$b_l = \mu - \tau$, $b_u = \mu + \tau$, and truncate $\mathbf{x}$ by:

$$x_i = \begin{cases} b_u, & \text{if } x_i > b_u \\ b_l, & \text{if } x_i < b_l \\ x_i, & \text{otherwise} \end{cases};$$

while *the stopping criterion S is violated* ;

Similarly, the ML estimate of location μ_o under Laplacian distribution is equivalent to minimizing

$$G_1(\mu) = \sum_{i=1}^{n} \frac{|x_i - \mu|}{\sigma_i}. \tag{4}$$

The value of μ that minimizes (4) is the weighted median ϕ_w

$$\begin{aligned} \phi_w &= \arg\min_{\mu} G_1(\mu) \\ &= \mathrm{median}(w_1 \diamond x_1, w_2 \diamond x_2, \ldots, w_n \diamond x_n), \end{aligned} \tag{5}$$

where $w_i = 1/\sigma_i$ and $\diamond$ is the replication operator defined by

$$w_i \diamond x_i = \overbrace{x_i, x_i, \ldots, x_i}^{w_i \text{ times}}. \tag{6}$$

In fact, the weighted median is searched in the following way to avoid expanding the data and cope with the non-integer weights [21]:

1) Calculate the threshold $T_0 = \frac{1}{2}\sum_{i=1}^{n} w_i$.
2) Sort the samples x_i.
3) Sum the magnitude of the weights of the sorted samples from the maximum continuing down in order.
4) The output is the sample whose weight magnitude causes the sum to become larger than or equal to T_0.

The median is a special case of the weighted median in which all weights are the same. As the mean square error (MSE) of the median is larger than the CRLB under Laplacian distribution, it does not achieve the minimum MSE though it is the ML estimate [36]. The ITM filter [35] outperforms the median filter in suppressing both Gaussian and Laplacian noise and does not require data sorting.

B. Iterative Truncated Arithmetic Mean Filter

Different from the mean filter that averages all samples and the median filter that chooses one sample as the output, the iterative truncated arithmetic mean (ITM) filter [35] iteratively truncates the extreme samples and uses the truncated mean as the filter output. Starting from $\mathbf{x} = \mathbf{x}_0$, it truncates samples in $\mathbf{x}$ to a dynamic threshold as shown by Algorithm 1.

The type I output of the ITM filter [35] is

$$y_t(\mathbf{x}_0) = \mathrm{mean}(\mathbf{x}). \tag{7}$$

Theoretical analysis in [35] shows that the ITM output starts from the mean and approaches the median by increasing the number of iterations. The stopping criterion S given in [35], which terminates the iteration automatically, enables the ITM filter outperforms the median filter in suppressing both Gaussian and Laplacian noise. The implementation given in [36] is faster than the median filter.

C. The Proposed Weighted ITM Filter With Positive Weights

The weighted ITM (WITM) filter is proposed based on the following theorems.

Theorem 1: For any finite data set $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ and weight set $\mathbf{w} = \{w_1, w_2, \ldots, w_n\}$ with all weights being nonnegative rational numbers, the difference between the weighted mean μ_w and weighted median ϕ_w is never great than the weighted mean absolute deviation τ_w. The corresponding formula is

$$|\phi_w - \mu_w| \leq \tau_w \triangleq \sum_{i=1}^{n} w_i |x_i - \mu_w| \Bigg/ \sum_{i=1}^{n} w_i. \tag{8}$$

Proof: Let

$$\mathbf{x}_e = \{kw_1 \diamond x_1, kw_2 \diamond x_2, \ldots, kw_n \diamond x_n\} \tag{9}$$

be the expanded data set of $\mathbf{x}$ where k is a constant making kw_i integer for all $1 \leq i \leq n$. Let μ_e, ϕ_e and $\tau_e = \mathrm{mean}(|\mathbf{x}_e - \mu_e|)$ be the mean, median and mean absolute deviation of $\mathbf{x}_e$, respectively. Based on the Theorem 1 given in [35],

$$|\mu_e - \phi_e| \leq \tau_e. \tag{10}$$

It is easy to see that $\mu_w = \mu_e$, $\phi_w = \phi_e$ and $\tau_w = \tau_e$. This completes the proof of Theorem 1. ■

Theorem 1 guarantees that the weighted median is never changed if we use the weighted dynamic threshold τ_w to truncate the extreme samples of $\mathbf{x}$. The following theorem ensures that the truncating process never idles for any data distribution if the weighted mean μ_w of $\mathbf{x}$ deviates from its weighted median ϕ_w.

Theorem 2: For any finite data set $\mathbf{x}$ and weight set $\mathbf{w}$, there exists at least one sample whose distance from the weighted mean μ_w is greater than the weighted mean absolute deviation τ_w if the weighted mean μ_w deviates from the weighted median ϕ_w, i.e.,

$$\exists x_i, x_i \in \mathbf{x}, \text{ that } |x_i - \mu_w| > \tau_w, \text{ if } \mu_w \neq \phi_w. \tag{11}$$

Proof: From the Theorem 2 given in [35] we have

$$\exists x_i, x_i \in \mathbf{x}_e, \text{ that } |x_i - \mu_e| > \tau_e, \text{ if } \mu_e \neq \phi_e. \tag{12}$$

This means that at least one sample in $\mathbf{x}_e$ is far away from μ_e by a distance larger than τ_e. As $\mu_w = \mu_e$, $\phi_w = \phi_e$, $\tau_w = \tau_e$ and $x_i \in \mathbf{x}$ if $x_i \in \mathbf{x}_e$, (11) is proven. ■

Theorems 1 and 2 ensure that the extreme samples in $\mathbf{x}$ can be iteratively truncated by using the dynamic threshold τ_w while keeping the weighted median un-changed. These theorems inspire the proposed WITM filter shown in Algorithm 2. The following Proposition 1 guarantees that the truncated samples approach the weighted median.

Proposition 1: The dynamic threshold $\tau_w(k)$ of the WITM algorithm monotonically decreases to zero by increasing the number of iterations k if the weighted mean μ_w deviates from the weighted median ϕ_w, i.e.,

$$\tau_w(k+1) < \tau_w(k), \quad \text{if } \mu_w \neq \phi_w, \tag{13}$$

and

$$\lim_{k\to\infty} \tau_w(k) = 0, \quad \text{if } \mu_w \neq \phi_w. \tag{14}$$

The proof of this proposition can be achieved by expanding $\mathbf{x}$ with the weight set $k\mathbf{w}$ and following the Proposition 2 given in [35].

Algorithm 2: Truncation Procedure of the WITM Filter

Input: $\mathbf{w}, \mathbf{x_0} \Rightarrow \mathbf{x}$; **Output**: Truncated $\mathbf{x}$;

1 **do**

2 Compute the weighted mean: $\mu_w = \sum_{i=1}^{n} w_i x_i / \sum_{i=1}^{n} w_i$;

3 Compute the weighted dynamic threshold: $\tau_w = \sum_{i=1}^{n} w_i |x_i - \mu_w| / \sum_{i=1}^{n} w_i$;

4 $b_l = \mu_w - \tau_w$, $b_u = \mu_w + \tau_w$, and truncate $\mathbf{x}$ by:

$$x_i = \begin{cases} b_u, & \text{if } x_i > b_u \\ b_l, & \text{if } x_i < b_l \\ x_i, & \text{otherwise} \end{cases};$$

5 **while** *the stopping criterion S is violated* ;

The output of the WITM filter is defined as the weighted mean of the truncated $\mathbf{x}_0$ by Algorithm 2

$$y_{wt}(\mathbf{x}_0, \mathbf{w}) = \sum_{i=1}^{n} w_i x_i \Bigg/ \sum_{i=1}^{n} w_i, \quad x_i \in \mathbf{x}. \tag{15}$$

By using the weighted mean of the truncated data set $\mathbf{x}$ as the filter output, the WITM filter is expected to own merits of both the weighted mean and median filters.

D. Stopping Criterion

The output of the WITM filter moves from the weighted mean towards the weighted median by increasing the number of iterations. Since for many applications, neither weighted mean nor weighted median is the optimal solution, proper stopping criterion may enable the WITM filter outperforming the both.

In order to facilitate the following analysis, we separate the data set $\mathbf{x}$ into two subsets by the truncated weighted mean μ_w as

$$\mathbf{x}_h \triangleq \{x_i \,|\, x_i \in \mathbf{x}, x_i > \mu_w\} \tag{16}$$

$$\mathbf{x}_l \triangleq \{x_i \,|\, x_i \in \mathbf{x}, x_i \leq \mu_w\}. \tag{17}$$

Let w_h and w_l denote the sum of weights of $\mathbf{x}_h$ and $\mathbf{x}_l$, respectively. One possible stopping criterion S_1 to ensure μ_w close to the weighted median is to meet the condition

$$S_1(\varepsilon_1) : \Delta w \triangleq |w_h - w_l| \leq \varepsilon_1. \tag{18}$$

For real data, in general there is no more than one sample having the value equal to the weighted median. The following lemmas analyze the choice of ε_1 in this general case. Although the WITM filter does not need data sorting, we use the ascending sorted data set $\{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\}$ to facilitate the analysis. The corresponding weight set is $\{w_{(1)}, w_{(2)}, \ldots, w_{(n)}\}$. Let $x_{(m)}$ denote the weighted median ϕ_w. The following lemma gives the condition that ensures μ_w falls in the interval $(x_{(m-1)}, x_{(m+1)})$.

Lemma 1: Let x_q be the nearest sample to the weighted mean μ_w in the subset that has the larger sum weight. x_q is the weighted median if and only if

$$|w_h - w_l| \leq 2w_q. \tag{19}$$

Proof: If $w_h > w_l$,

$$w_h > T_0. \tag{20}$$

From (19) we have $w_h - w_q \leq w_l + w_q$. Therefore

$$w_h - w_q \leq T_0. \tag{21}$$

(20) and (21) show $x_q = x_{(m)}$ by the definition of weighted median. Similarly, it is straightforward to see that if $x_q = x_{(m)}$, (19) holds. The proof for the case $w_h \leq w_l$ is analogous. ■

We see that (19) is the sufficient and necessary conditions for μ_w close to $x_{(m)}$, $x_{(m-1)} < \mu_w < x_{(m+1)}$. The next lemma gives further conditions that $x_{(m-1)} < \mu_w < x_{(m)}$ and $x_{(m)} \leq \mu_w < x_{(m+1)}$. Let $w_{(l)} = \sum_{i=1}^{m-1} w_{(i)}$ and $w_{(h)} = \sum_{i=m+1}^{n} w_{(i)}$. Without loss of generality, we assume $w_{(h)} \geq w_{(l)}$.

Lemma 2: If μ_w falls in $(x_{(m-1)}, x_{(m)})$,

$$w_{(m)} \leq |w_h - w_l| \leq 2w_{(m)}. \tag{22}$$

Otherwise, if μ_w falls in $[x_{(m)}, x_{(m+1)})$,

$$|w_h - w_l| \leq w_{(m)}. \tag{23}$$

Proof: If μ_w is in $(x_{(m-1)}, x_{(m)})$, we have

$$|w_h - w_l| = w_{(m)} + w_{(h)} - w_{(l)} \geq w_{(m)}. \tag{24}$$

From (19) and (24), (22) is proven. If μ_w is in $[x_{(m)}, x_{(m+1)})$, we have

$$|w_h - w_l| = -\left[w_{(h)} - \left(w_{(m)} + w_{(l)}\right)\right] \leq w_{(m)}. \tag{25}$$

This completes the proof of Lemma 2. ■

As Δw is smaller for $w_{(h)} > w_{(l)}$ if μ_w falls in $[x_{(m)}, x_{(m+1)})$ than in $(x_{(m-1)}, x_{(m)})$, (23) is in general a better stopping criterion than (19). The following lemma proves that the condition (23) can always be met. A necessary proposition is given here to facilitate the proof of the following Lemma 3.

Proposition 2: Samples, once being truncated in an iteration of the WITM algorithm, must be truncated in all subsequent iterations.

The proof can be achieved by expanding $\mathbf{x}$ with the weight set $k\mathbf{w}$ and following the Proposition 1 in [36].

Lemma 3: Assume $w_{(h)} > w_{(l)}$. There is an iteration k in which the weighted truncated mean μ_w falls in the interval $[x_{(m)}, x_{(m+1)})$, i.e.,

$$\exists k, \quad x_{(m)} \leq \mu_w(k) < x_{(m+1)}. \tag{26}$$

Proof: As the dynamic threshold τ_w monotonically decreases to zero, all samples except the weighted median $x_{(m)}$ will be truncated to the lower bound b_l or upper bound b_h after some iterations. Only three different sample values exist in the truncated $\mathbf{x}$, b_l, $x_{(m)}$ and b_h with the weights $w_{(l)}$, $w_{(m)}$ and $w_{(h)}$. If μ_w is in the interval $(b_l, x_{(m)})$, we will prove that it will move into the interval $[x_{(m)}, b_h)$ in a finite number of iterations.

For symbolic simplicity of the proof, let $x_{(m)} = 0$ and $w_{(l)} + w_{(m)} + w_{(h)} = 1$. This will not lose the generality of the proof. So we have

$$\mu_w = w_{(l)} b_l + w_{(h)} b_h. \tag{27}$$

If $\mu_w \in (b_l, x_{(m)})$, $\mu_w < 0$. Therefore,

$$w_{(h)} b_h < -w_{(l)} b_l. \tag{28}$$

Both b_l and b_h are truncated to the new lower bound $b'_l = \mu_w - \tau_w$ and high bound $b'_h = \mu_w + \tau_w$ in the next iteration based on Proposition 2. As

$$\tau_w = w_{(l)}(\mu_w - b_l) - w_{(m)}\mu_w + w_{(h)}(b_h - \mu_w) \tag{29}$$

and $w_{(l)} + w_{(m)} + w_{(h)} = 1$, we have

$$\begin{aligned} b'_h = \mu_w + \tau_w &= w_{(h)} b_h - w_{(l)} b_l + 2w_{(l)}\mu_w \\ &= w_{(h)} b_h \left(1 + 2w_{(l)}\right) - w_{(l)} b_l \left(1 - 2w_{(l)}\right) \\ &> w_{(h)} b_h \left(1 + 2w_{(l)}\right) + w_{(h)} b_h \left(1 - 2w_{(l)}\right) \\ &= 2w_{(h)} b_h. \end{aligned} \tag{30}$$

The inequality in (30) comes from (28) and $(1 - 2w_{(l)}) > 0$. This yields

$$\begin{aligned} \alpha_{s+1} \triangleq \frac{-b'_l}{b'_h} = \frac{-\mu_w + \tau_w}{\mu_w + \tau_w} &= 1 - \frac{2\mu_w}{\mu_w + \tau_w} \\ &< 1 - \frac{2\left(w_{(l)} b_l + w_{(h)} b_h\right)}{2w_{(h)} b_h} = -\frac{w_{(l)}}{w_{(h)}} \frac{b_l}{b_h} \\ &= \frac{w_{(l)}}{w_{(h)}} \alpha_s, \end{aligned} \tag{31}$$

where $\alpha_s \triangleq -b_l/b_h$. Therefore, if $\mu_w \in (x_{(m-1)}, x_{(m)})$ in the kth iteration,

$$\alpha_k < \alpha_s \left(\frac{w_{(l)}}{w_{(h)}}\right)^{k-s}, \quad k \geq s. \tag{32}$$

As $w_{(l)}/w_{(h)} < 1$, there exists an iteration k in which $\alpha_k \leq w_{(h)}/w_{(l)}$. It leads to $\mu_w \geq 0$ in the $(k+1)$th iteration. This completes the proof of Lemma 3. ■

Lemmas 1, 2 and 3 imply that $\varepsilon_1 = w_q$ can be used as a stopping criterion to ensure μ_w close to ϕ_w. To avoid searching x_q for w_q, a loosened condition $\varepsilon_1 = w_{\max}$ is utilized in this paper, where $w_{\max}$ is the maximum value in the weight set $\mathbf{w}$.

In some extreme cases, there could exist multiple samples having the same weighted median value. In this case, the stopping criterion S_1 may never be met. The second stopping criterion uses a predefined ε_2 to limit the maximum number of iterations k, defined as

$$S_2(\varepsilon_2) : k \geq \varepsilon_2. \tag{33}$$

The ε_2 we chose in this paper is the same as in [35], which is $\varepsilon_2 = 2\sqrt{n}$.

The stopping criterion S we used in this work to stop the WITM algorithm is a combination of S_1 and S_2, i.e.,

$$S = S_1(\varepsilon_1) \vee S_2(\varepsilon_2). \tag{34}$$

The above stopping criterion ensures that the filter output reasonably approaches the weighted median and that it never fails to stop the iteration for any kind of data. In fact, it is very difficult if not impossible to find a stopping criterion optimal for all types of signals and noise. The above stopping criterion is designed for general cases. For the noise types whose optimal location estimator is neither weighted mean nor weighted median, as shown in the experiments, the proposed WITM filter that stops the iteration by the above criterion will outperform the weighted mean and weighted median. However, this stopping criterion is by no means optimal for all types of noise. For particular application, specific stopping criterion could be designed to make the WITM filter closer to the weighted mean or closer to the weighted median filters and hence to achieve a better performance.

E. Properties of the WITM Filter With Positive Weights

Property 1: The WITM filter output converges to the weighted median by increasing the number of iterations k, i.e.,

$$\lim_{k\to\infty} y_{wt}(\mathbf{x}_0, \mathbf{w}) = \phi_w(\mathbf{x}_0, \mathbf{w}). \tag{35}$$

Proof: As shown in Theorem 1, the truncating algorithm does not change the weighted median of the input data set. Moreover, Proposition 1 shows that the dynamic threshold converges to zero. Therefore, the output of the WITM filter converges to the weighted median. ■

Property 2: The WITM filter output is invariant to scale and shift, i.e., if $\mathbf{z} = \{\alpha x_i + c\}, \forall x_i, x_i \in \mathbf{x}_0$, we have

$$y_{wt}(\mathbf{z}, \mathbf{w}) = \alpha y_{wt}(\mathbf{x}_0, \mathbf{w}) + c, \tag{36}$$

where α and c are two constants. The proof is trivial and hence omitted.

Property 3: The distribution of the WITM filter output is symmetric if the samples of the input data set $\mathbf{x}_0 = \{x_1, x_2, \ldots, x_n\}$ are drawn from the random variables $\{X_1, X_2, \ldots, X_n\}$, all of which have symmetric distributions around the symmetry center c.

Proof: If x_i is symmetrically distributed around c, $2c - x_i$ has the same distribution as x_i. According to Property 2, $y_{wt}(2c - \mathbf{x}_0, \mathbf{w}) = 2c - y_{wt}(\mathbf{x}_0, \mathbf{w})$. Thus, the distribution of $y_{wt}(\mathbf{x}_0, \mathbf{w})$ is symmetric around c. ■

Property 4: The WITM filter output is an unbiased estimate of the symmetry center c if the samples in $\mathbf{x}_0 = \{x_1, x_2, \ldots, x_n\}$ are drawn from the random variables $\{X_1, X_2, \ldots, X_n\}$, all of which have symmetric distributions around c.

Proof: According to Property 3, $y_{wt}(\mathbf{x}_0, \mathbf{w})$ is symmetrically distributed around c. Therefore, $E\{y_{wt}(\mathbf{x}_0, \mathbf{w})\} = E\{X_i\} = c$. This completes the proof of Property 4. ■

III. Weighted ITM Filter Admitting Negative Weights

Similar to the weighted mean and median filters, the WITM filter admitting only positive weights can only be a low-pass filter. In this section, two structures of the WITM filter admitting both positive and negative weights, named GWITM and LCWITM filters, are designed following the structures of the general weighted median (GWM) filter [21] and the linear combination of weighted median (LCWM) filter [46]. A new structure, named dual WITM (DWITM) filter, is proposed. The three structures enable the WITM filter being designed as low-, band- and high-pass filters.

A. General WITM Filter With Negative Weights

The GWM filter [21] admitting negative weights is

$$y_{\text{gwm}} = \text{median}(|w_1| \diamond \text{sign}(w_1)x_1, |w_2| \diamond \text{sign}(w_2)x_2, \ldots, |w_n| \diamond \text{sign}(w_n)x_n), \tag{37}$$

where $\text{sign}(w) = 1$ if $w \geq 0$ and $\text{sign}(w) = -1$ otherwise. By uncoupling the weight sign from the weight magnitude and merging it with the sample values, the GWM filter can be implemented by the algorithm of weighted median filter with only positive weights. The general WITM (GWITM) filter is analogous to that of the GWM filter. It is

$$y_{gwt}(\mathbf{x}_0, \mathbf{w}) = y_{wt}(\text{sign}(\mathbf{w}) \cdot \mathbf{x}_0, \text{abs}(\mathbf{w})), \tag{38}$$

where $\text{sign}(\mathbf{w}) \cdot \mathbf{x} = \{\text{sign}(w_1)x_1, \text{sign}(w_2)x_2, \ldots, \text{sign}(w_n)x_n\}$ and $\text{abs}(\mathbf{w}) = \{|w_1|, |w_2|, \ldots, |w_n|\}$. The GWITM filter with negative weights turns to that with only positive weights which can be implemented by Algorithm 2.

B. Linear Combined WITM Filter With Negative Weights

Although the GWM filter has been widely used in many applications [23], [28], it has limitation in suppressing the DC component of the signal. Take a random data set $\{7.91, 9.92, 10.24, 10.03, 9.79, 11.47\}$ and their corresponding weight set $\{0.2, 0.3, 0.2, -0.2, -0.3, -0.2\}$ as an example. The observation set is Laplacian noise with the offset value 10. The "signed" data set is $\{7.91, 9.92, 10.24, -10.03, -9.79, -11.47\}$. Although the input data has a small variance, the "signed" data has a large variance and there is a large gap between the positive and the negative samples due to the large offset of the input samples. As the GWM filter selects one of the "signed" samples as the output, it cannot suppress the DC component effectively. This problem is still not well solved though in [44] the output of the GWM filter is modified to be the average of the weighted median and the next smaller "signed" sample in the sorted data. This phenomenon can be seen by comparing the filter outputs in Fig. 1(b) and (c). Fig. 1(a)-top shows a chirp signal with zero mean. The linear FIR, GWM and GWITM filters are designed as band-pass filters with pass band $[0.1\pi, 0.2\pi]$. The setting of these filters is detailed in the experiment section. The output of the linear FIR filter shown in Fig. 1(a)-bottom is used as a reference. The outputs of the GWM and GWITM filters are depicted in Fig. 1(b)-top and -bottom, respectively.

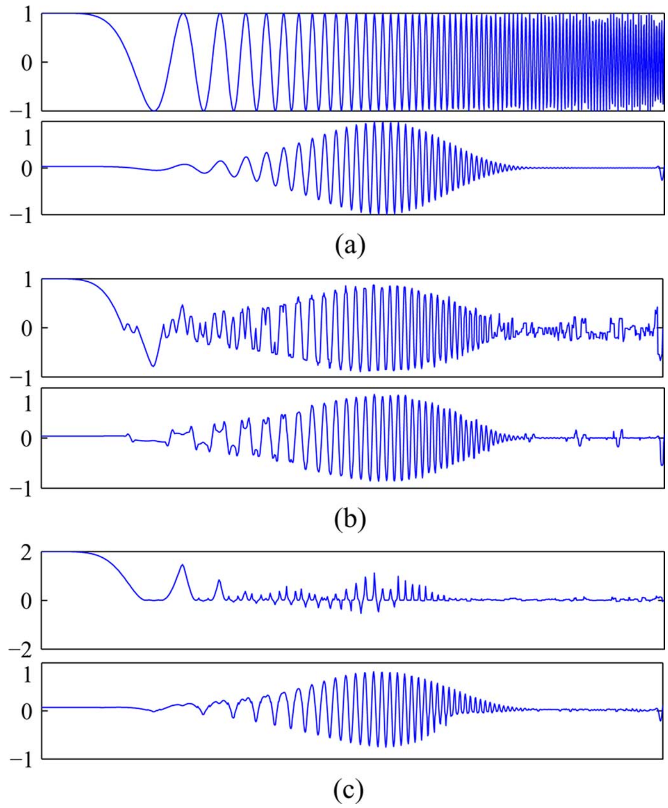

Fig. 1. Frequency selective filter outputs on chirp signal. (a)-top: Chirp signal, (a)-bottom: linear FIR filter output. Output of (b)-top GWM filter, (b)-bottom GWITM filter on the chirp signal. Output of (c)-top GWM filter, (c)-bottom GWITM filter on the chirp signal with a constant offset 1.

All of them have the frequency selection characteristic. Then, we add a constant value 1 to the chirp signal. Both the chirp signal with a constant offset and the corresponding linear FIR filter output are not plotted because they have a quite similar shape to those in Fig. 1(a). The corresponding outputs of the GWM and GWITM filters are shown in Fig. 1(c)-top and -bottom, respectively. It is seen that the GWM filter fails to select the frequency. The output of the GWITM filter has some distortions though it is much better than that of the GWM filter. In order to alleviate this problem, we propose the linear combined WITM (LCWITM) filter. It is based on the LCWM filter [46] that utilizes a combination of n low-pass weighted median sub-filters to design band- and high-pass filters. The LCWM filter is defined by

$$y_{\mathrm{LCWM}} = \sum_{i=1}^{n} \alpha_i y_{wm}(\mathbf{x}_0, \mathbf{w}_i), \tag{39}$$

where $y_{wm}(\mathbf{x}_0, \mathbf{w}_i)$ is the ith weighted median sub-filter with the weight set $\mathbf{w}_i$. $\mathbf{w}_i$ is designed using the algorithm in [46] with the help of the combination matric $B_{n,m}$ [46] where m is the number of nonzero elements of each sub-filter. The weighting coefficient α_i of the ith sub-filter is calculated based on the coefficients of a prototype FIR filter designed by any of the standard FIR design tool [46].

The structure of the LCWITM filter is set to be the same as that of the LCWM filter. By directly replacing the weighted median filter with the WITM filter, the resulting LCWITM filter is

$$y_{lcwt}(\mathbf{x}_0, \mathbf{w}) = \sum_{i=1}^{n} \alpha_i y_{wt}(\mathbf{x}_0, \mathbf{w}_i). \tag{40}$$

In this paper, both $\mathbf{w}_i$ and α_i are designed following the method given in [46].

C. The Proposed Dual WITM Filter With Negative Weights

As distinct low-pass weighted median filters are countable [47], it is not available to achieve arbitrary frequency response by only using two such sub-filters [46]. Therefore, the LCWM filter employs n sub-filters to alleviate this problem. The n sub-filter structure leads to a high computational complexity for the LCWITM filter because it needs truncate the data independently for each sub-WITM filter. Moreover, the LCWM filter employs small-size sub-filters to make its output close to the linear filter. This, however, reduces the filter's capability in suppressing impulsive noise. This also makes the WITM filter stop too early to suppress the impulsive noise. This observation motivates the proposed dual WITM (DWITM) filter.

According to the sign of the weights, the weight set $\mathbf{w}$ can be separated into two subsets: positive subset $\mathbf{w}_P = \{w_{P1}, w_{P2}, \ldots, w_{Pa}\}$ and negative subset $\mathbf{w}_N = \{w_{N1}, w_{N2}, \ldots, w_{Nb}\}$ containing all the positive and negative weights, respectively. The output of the weighted mean filter can be represented as

$$\begin{aligned}\mu_w &= \sum_{i=1}^{n} w_i x_i \Bigg/ \sum_{i=1}^{n} |w_i| \\ &= \sum_{i=1}^{a} w_{Pi} x_{Pi} \Bigg/ \sum_{i=1}^{n} |w_i| - \sum_{i=1}^{b} |w_{Ni}| x_{Ni} \Bigg/ \sum_{i=1}^{n} |w_i| \\ &= \mu_P \sum_{i=1}^{a} w_{Pi} \Bigg/ \sum_{i=1}^{n} |w_i| - \mu_N \sum_{i=1}^{b} |w_{Ni}| \Bigg/ \sum_{i=1}^{n} |w_i|,\end{aligned} \tag{41}$$

where μ_P and μ_N are the weighted means of the samples corresponding to the positive and negative weights, respectively. Equation (41) shows that μ_w is the weighted difference between μ_P and μ_N. It means that the output of a band- or high-pass filter is the difference between two low-pass filters. Unlike the weighted median filter, the distinct WITM filters are uncountable because they use a truncated averaging instead of a selecting algorithm. Therefore, it is reasonable to design band- or high-pass filter with two low-pass WITM filters. The proposed DWITM filter is formulated as

$$\begin{aligned}y_{dwt}(\mathbf{x}_0, \mathbf{w}) = {} & y_{wt}(\mathbf{x_P}, \mathbf{w_P}) \sum_{i=1}^{a} w_{Pi} \Bigg/ \sum_{i=1}^{n} |w_i| \\ & - y_{wt}(\mathbf{x_N}, \mathrm{abs}(\mathbf{w_N})) \sum_{i=1}^{b} |w_{Ni}| \Bigg/ \sum_{i=1}^{n} |w_i|,\end{aligned} \tag{42}$$

where $\mathbf{x}_P$ and $\mathbf{x}_N$ are the subsets of $\mathbf{x}_0$ corresponding to $\mathbf{w}_P$ and $\mathbf{w}_N$, respectively.

IV. Experiments

The first experiment tests the WITM filters in suppressing Gaussian and Laplacian noise. Frequency selective WITM filters are evaluated in the second experiment. High-pass WITM filters are tested in the third experiment. The forth experiment evaluates the filters on real image data. For the proposed WITM, GWITM, LCWITM, and DWITM filters, the same stopping criterion parameters $\varepsilon_1 = w_{\max}$ and $\varepsilon_2 = 2\sqrt{n}$ are fixed over all experiments in this paper. Better filtering performances than those shown in this paper will be obtained if the aforementioned parameters are adjusted for different data sets. Similarly, for the weighted myriad filter, the default setting $K = 1$ provided in the source code [44] is used over all experiments. To show the good performance of the weighted myriad filter, a larger number of iterations $L = 20$ [35] is applied in all experiments though it is shown that the weighted myriad filter well converges in 10 iterations [40].

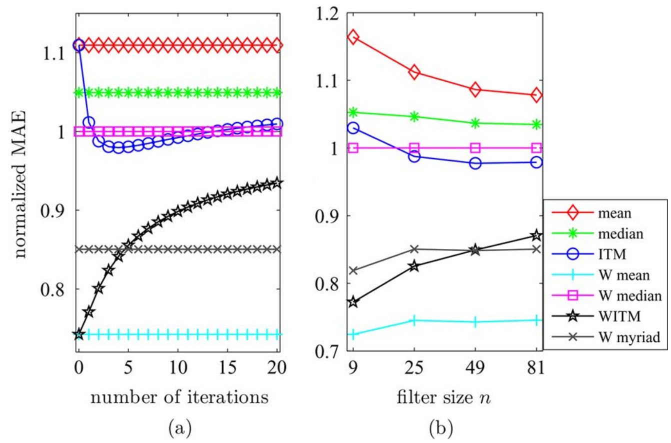

Fig. 2. Mean absolute error (MAE) normalized by that of the weighted median filter in suppressing non-identical distributed Gaussian noise against (a) the number of iterations with fixed filter size $n = 25$, and (b) the filter size n. The average numbers of iterations for the ITM filter are 1.71, 3.44, 5.42, 7.50 for the filter size from 9 to 81, respectively, and those for the WITM filter are 1.69, 2.51, 3.82, 5.40.

A. Attenuation of the Short- and Long-Tailed Noise

The performance of the WITM filters is tested on a constant signal contaminated by Gaussian and Laplacian noise with non-identical distribution. For the input data set $\mathbf{x}_0 = \{x_1, x_2, \ldots, x_n\}$, we assume the corresponding standard deviation of the noise be $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. The σ_i can be chosen as an arbitrary positive value. In this paper, we restrict σ_i by setting $\sigma_n/\sigma_1 = 5$. The rest are set by $\sigma_2/\sigma_1 = \sigma_3/\sigma_2 = \cdots = \sigma_n/\sigma_{n-1}$. As the optimal estimator under Gaussian noise is the weighted mean with weight set $\mathbf{w}_G = \{1/\sigma_1^2, 1/\sigma_2^2, \ldots, 1/\sigma_n^2\}$, all weighted estimators under Gaussian noise use the weight set $\mathbf{w}_G$. Similarly, the ML estimator under Laplacian noise is the weighted median with weight set $\mathbf{w}_L = \{1/\sigma_1, 1/\sigma_2, \ldots, 1/\sigma_n\}$. Therefore, in dealing with Laplacian noise, all weighted estimators employ the weight set $\mathbf{w}_L$. For each experiment, the mean absolute error (MAE) over 10^6 independent input data sets is used as the performance indicator.

Fig. 2 shows the normalized mean absolute error (MAE) of filters' outputs in suppressing the non-identical distributed Gaussian noise. The MAE is normalized by that of the weighted median filter. We first investigate the WITM filter's performance against the number of iterations without applying the proposed stopping criterion. Fig. 2(a) depicts the normalized MAEs of the ITM and WITM filters against the number of iterations. MAEs of the myriad filter with fixed 20 iterations and other noniterative filters are illustrated by horizontal lines for a better visual comparison with the ITM and WITM filters. The filter size is $n = 25$. As the weighted mean filter is the optimal in suppressing Gaussian noise, it has the lowest MAE. The MAE of the WITM filter increases against the number of iterations. It equals to that of the weighted mean filter when the number of iterations equals to zero, and approaches to that of the weighted median filter when the number of iterations is large enough. Fig. 2(b) shows the normalized MAE of the WITM filter with the proposed stopping criterion (34) against the filter size. The ITM filter employs the stopping criterion in [35]. Fig. 2(b) demonstrates that the performance of the WITM filter is significantly better than the weighted median filter and all un-weighted filters. The WITM filter achieves a comparable MAE to the weighted myriad filter. The normalized MAE of the WITM filter is smaller than that of the weighted myriad filter when the filter size $n \leq 49$.

Fig. 3 is generated from Fig. 2 by replacing Gaussian noise with Laplacian noise. Except for different noise types and weight sets, other experimental settings of Fig. 3 are the same as Fig. 2. Fig. 3(a) shows that the MAE of the WITM filter is smaller than that of the weighted median filter after only 3 iterations. Fig. 3(b) shows the weighted myriad filter achieves a comparable performance to that of the weighted median filter. The proposed WITM filter outperforms all other filters against all filter sizes. We also plot the MAE of the WITM filter with the fixed number of iterations that minimizes the filter's MAE shown as F WITM in Fig. 3(b). The numbers of iterations found by search in the training are 7, 12, 17 and 22 for the filter size 9, 25, 49 and 81, respectively. It shows that the WITM filter with the specific designed number of iterations can achieve even smaller MAE than that using the general stopping criterion though the WITM filter with the general stopping criterion already outperforms the weighted median filter that is the ML location estimator for the Laplacian noise.

We compare the running time of the WITM filter with other filters under the Window 7 system with the Intel Core i5 CPU 3.2 GHz and RAM 4 GB. All the filters are implemented by the C programming language. The running time of these filters in suppressing the Gaussian and Laplacian noise is depicted in Fig. 4. Although this paper applies $L = 20$ iterations to the weighted myriad filter for better performance, the running time of the weighted myriad filter with $L = 5$ iterations, which was reported in [40], is also plotted for a fair comparison. It is shown that the WITM filter is faster than both the weighted median and weighted myriad filters over all filter sizes in Fig. 4.

B. Frequency Selective WITM Filters

A quadratic swept-frequency chirp signal spanning instantaneous angular frequency ranging from 0 to 0.5π is used to test the WITM filters in frequency selection. The weights setting for different filters are analogous to those in [21]. A 31-tap linear

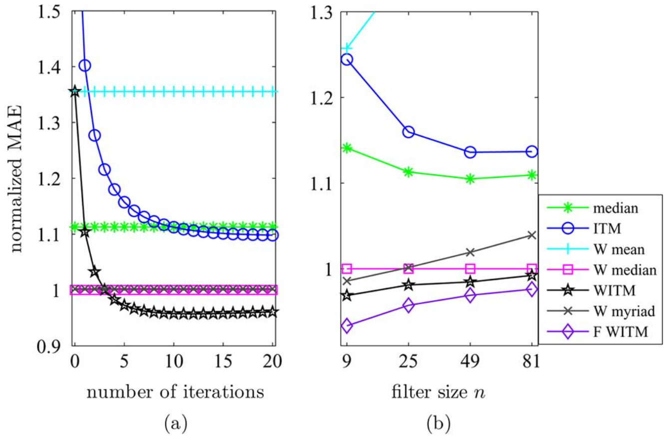

Fig. 3. Mean absolute error (MAE) normalized by that of the weighted median filter in suppressing non-identical distributed Laplacian noise against (a) the number of iterations with fixed filter size $n = 25$, and (b) the filter size n. The average numbers of iterations for the ITM filter are 2.01, 3.98, 6.05, 8.17 for the filter size from 9 to 81, respectively, and those for the WITM filter are 2.27, 3.99, 5.96, 8.04. The MAE of the mean filter is drastically larger than those of other filters and hence not plotted. "F WITM" represents the WITM filter with fixed numbers of iterations of 7, 12, 17, 22 for the filter size from 9 to 81.

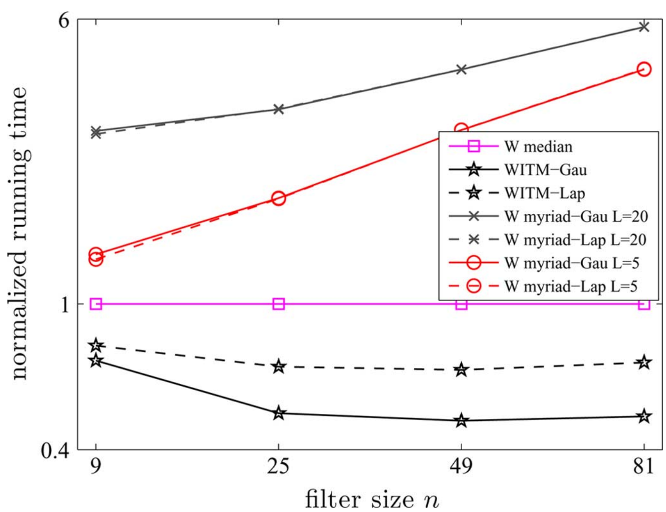

Fig. 4. Normalized running time against filter size n. The running time is normalized by that of the weighted median filter. The y-axis is in log scale. The weighted myriad filters with $L = 20$ and $L = 5$ iterations are both plotted.

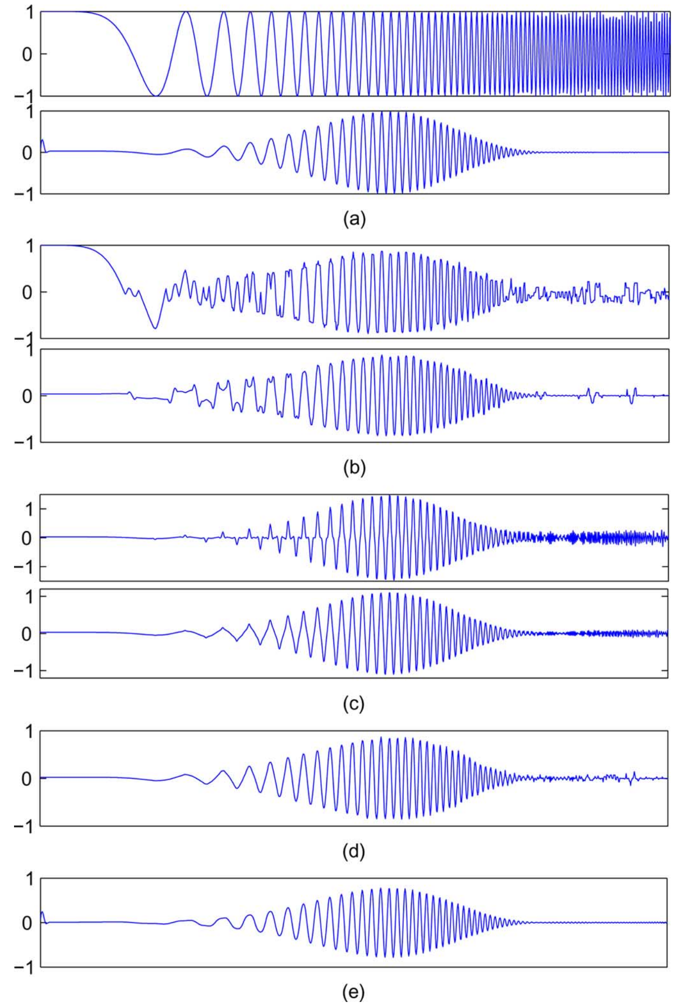

Fig. 5. Frequency selective filter outputs. (a)-top: Chirp test signal, (a)-bottom: linear FIR filter output. (b)-top: GWM filter output, (b)-bottom: GWITM filter output. (c)-top: LCWM filter output, (c)-bottom: LCWITM filter output. (d): DWITM filter output. (e): weighted myriad filter output. The average number of iterations for the GWITM filter is 3.21, and those for each sub-filter of the LCWITM and DWITM filters are 1.01 and 2.08, respectively.

FIR filter with band-pass $0.1\pi \leq \omega \leq 0.2\pi$ is designed by Matlab's fir1 function. The weights of the GWM [21], weighted myriad, GWITM and DWITM filters are set to be the same as those of the linear FIR filter. The LCWM filter is designed to be a symmetric LCWM filter with $B_{16,2}$ following the algorithm in [46]. The weights of the LCWITM filter are set to be identical to those of the LCWM filter.

Fig. 5(a)-top shows the chirp test signal. The output of the linear FIR filter depicted in Fig. 5(a)-bottom is used as a reference. The results of the GWM and GWITM filters are depicted in Fig. 5(b)-top and -bottom, respectively. It is seen that the GWITM filter has better performance than the GWM filter in suppressing the low and high frequency components. Fig. 5(c) shows that the output of the LCWITM filter (Fig. 5(c)-bottom) has smaller distortion than that of the LCWM filter (Fig. 5(c)-top). It is seen that both the DWITM filter (Fig. 5(d)) and the weighted myriad filter (Fig. 5(e)) produce small distortions compared to other nonlinear filters. The DWITM filter achieves drastically better performance than the median based filters. It achieves the comparable performance to the weighted myriad filter.

The chirp signal contaminated by the additive α-stable noise ($\alpha = 1.2$ and $\gamma = 0.1$) is shown in Fig. 6(a)-top. The output of the linear FIR filter is depicted in Fig. 6(a)-bottom. It is seen that the linear filter cannot remove the long-tailed noise effectively. The responses of the GWM and GWITM filters are shown in Fig. 6(b)-top and -bottom, respectively. The performance of the GWITM filter is better than that of the GWM filter. The parameter setting of the LCWM filter makes it contain 3-tap sub-filters. The small filter size reduces its capability in suppressing impulse noise, which can be seen from Fig. 6(c)-top. This structure also makes the LCWITM filter inefficient in suppressing impulse noise. In the LCWITM filter, the number of iterations for the 3-tap sub-filter is 1 because the stopping criterion is met in the first iteration. As the extreme samples are not sufficiently truncated, the performance of the LCWITM filter shown in Fig. 6(c)-bottom is poorer than that of the LCWM filter shown

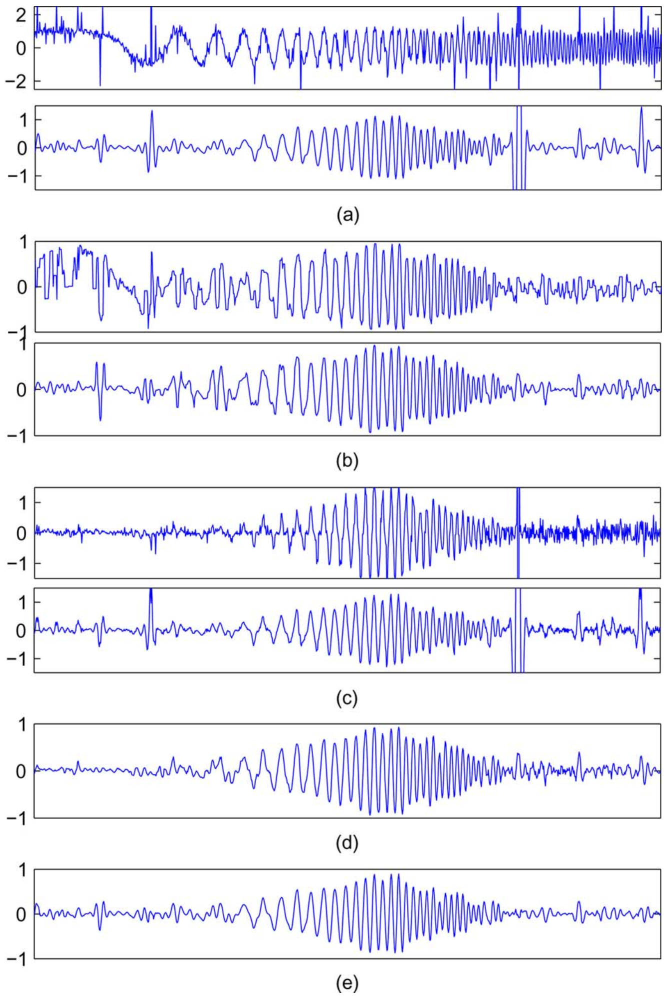

Fig. 6. Frequency selective filter outputs in noise. (a)-top: Chirp test signal in α-stable noise with $\alpha = 1.2$ and $\gamma = 0.1$, (a)-bottom: linear FIR filter output. (b)-top: GWM filter output, (b)-bottom: GWITM filter output. (c)-top: LCWM filter output, (c)-bottom: LCWITM filter output. (d): DWITM filter output. (e): weighted myriad filter output. The average number of iterations for the GWITM filter is 3.32, and those for each sub-filter of the LCWITM and DWITM filters are 1.06 and 2.87, respectively.

in Fig. 6(c)-top. The output of the DWITM filter (Fig. 6(d)) and that of the weighted myriad filter (Fig. 6(e)) show the best performance among all filters.

C. Design of High-Pass WITM Filters

High-pass filters are tested on a two-tune signal with angular frequency 0.02π and 0.4π shown in Fig. 7(a)-top. Weights of different filters are set analogously to those in [21]. A 31-tap linear high-pass filter with a cut-off angular frequency 0.2π is designed by Matlab's fir1 function. The output of the linear filter is shown in Fig. 7(a)-bottom. Instead of applying the weights of the linear FIR filter to all other filters, which may achieve the suboptimal results [21], the fast LMA algorithm [21] is used to optimize the GWM and GWITM filters with 31 taps for the application at hand. For the DWITM filter, as it has two sub-filters, its update function is modified from that in [21] to (43), shown at the bottom of the page, where $e(k) = d(k) - y_{dwt}$, d is the desired signal, k is the time and β is the update step size. The step size used in all adaptive optimization experiments is $\beta = 0.001$. For the weighted myriad filter, the adaptive weighted myriad filter algorithm [41] is adopted to train the weights. The LCWM and LCWITM filters are set the same as those in Section IV.B. The output of the GWM filter shown in Fig. 7(b)-top indicates that it still contains low frequency component. Besides, there are small distortions due to the "selection-type" behavior of the GWM filter. The performance of the GWITM filter depicted in Fig. 7(b)-bottom is better than the GWM filter. The LCWITM filter (Fig. 7(c)-bottom) generates smaller distortions than the LCWM filter (Fig. 7(c)-top). Fig. 7 shows that the outputs of the DWITM and weighted myriad filters are the closest to the linear filter.

The α-stable noise with $\gamma = 0.1$ and different α values is added to the two-tune signal. The MAE over 10^5 filter outputs is used as an indicator to evaluate the filters. The experimental results for 4 different α values are shown in Table I. For each α value, the smallest MAE among all filters is underlined and in bold font, and the second smallest is in bold font. While the weighted myriad filter outperforms others for $\alpha = 0.9$ and $\alpha = 1.2$, the DWITM filter achieves the best performance for $\alpha = 1.5$ and $\alpha = 1.8$. For the case of noise free, it is not a surprise that the linear FIR filter gets the minimum MAE. The outputs of different filters on the two-tune signal in α stable noise $(\alpha = 1.2, \gamma = 0.1)$ is shown in Fig. 8. This figure demonstrates that the linear filter cannot deal with the long tailed noise effectively. The GWM filter output has distortion though it can remove the long-tailed noise. Similar to the results in Section IV.B, Fig. 8(c) shows that the LCWM and LCWITM filters cannot effectively suppress the impulsive noise. The DWITM filter and the weighted myriad filter achieve similar results and outperform other filters.

We further test the high-pass filters' performances for different noise levels. The parameter settings for the filters are the same as before. ϵ-contaminated $(\epsilon = 0.5)$ normal distributed noise with the distribution $P_\epsilon = \{(1-\epsilon)\Phi + \epsilon H\}$ [48] is added on the two tune-signal, where Φ and H are Gaussian and α-stable $(\alpha = 1.5)$ noise, respectively. The standard deviation σ of Gaussian noise is set the same as the dispersion parameter γ of the α-stable noise, i.e., $\sigma = \gamma$. Among the linear FIR, GWM, GWITM, DWITM and weighted myriad filters, the performance of the linear FIR filter turns from the worst to the best by increasing the signal to noise ratio (SNR). Thus, we choose the range of SNR so that the linear FIR filter performs from the worst to the best. Experimental results of the 5 weighted filters, linear FIR, GWM, GWITM, DWITM and weighted myriad filters are shown in Fig. 9. As the performances of the LCWM and LCWITM filters are much poorer than other filters, they are not

$$w_i(k+1) = w_i(k) + \begin{cases} \beta e(k)\,\mathrm{sign}(x_i(k) - y_{wt}(\mathbf{x_P}, \mathbf{w_P})), & \text{if } w_i(k) > 0 \\ \beta e(k)\,\mathrm{sign}(x_i(k) - y_{wt}(\mathbf{x_N}, \mathrm{abs}(\mathbf{w_N}))), & \text{if } w_i(k) \le 0 \end{cases} \tag{43}$$

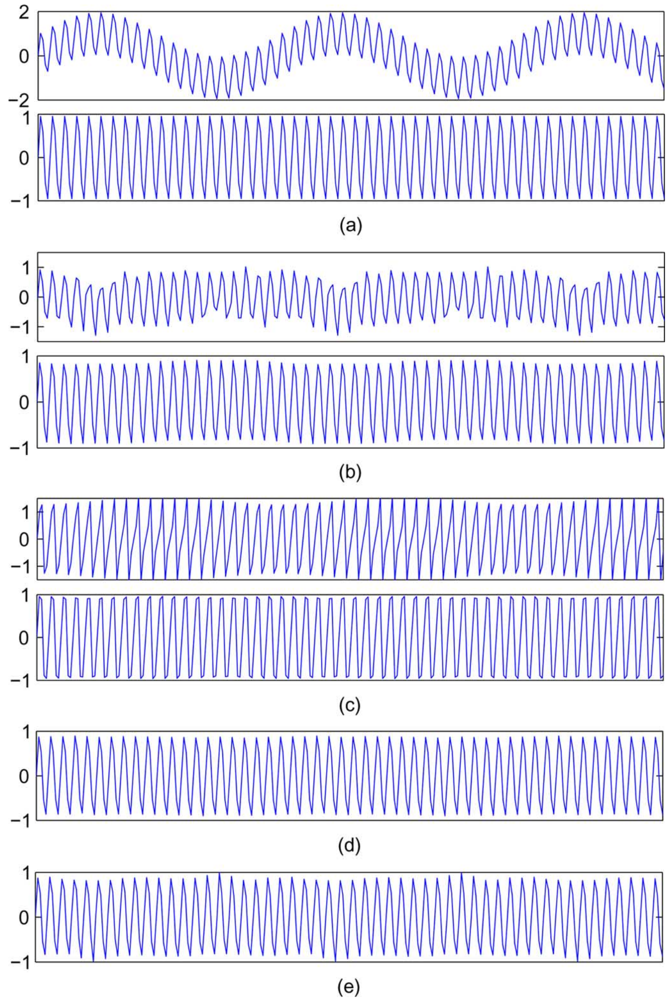

Fig. 7. Frequency selective filter outputs. (a)-top: two-tune signal, (a)-bottom: linear FIR filter output. (b)-top: GWM filter output, (b)-bottom: GWITM filter output. (c)-top: LCWM filter output, (c)-bottom: LCWITM filter output. (d): DWITM filter output. (e): weighted myriad filter output. The average number of iterations for the GWITM filter is 1.09, and those for each sub-filter of the LCWITM and DWITM filters are 1.00 and 2.18, respectively.

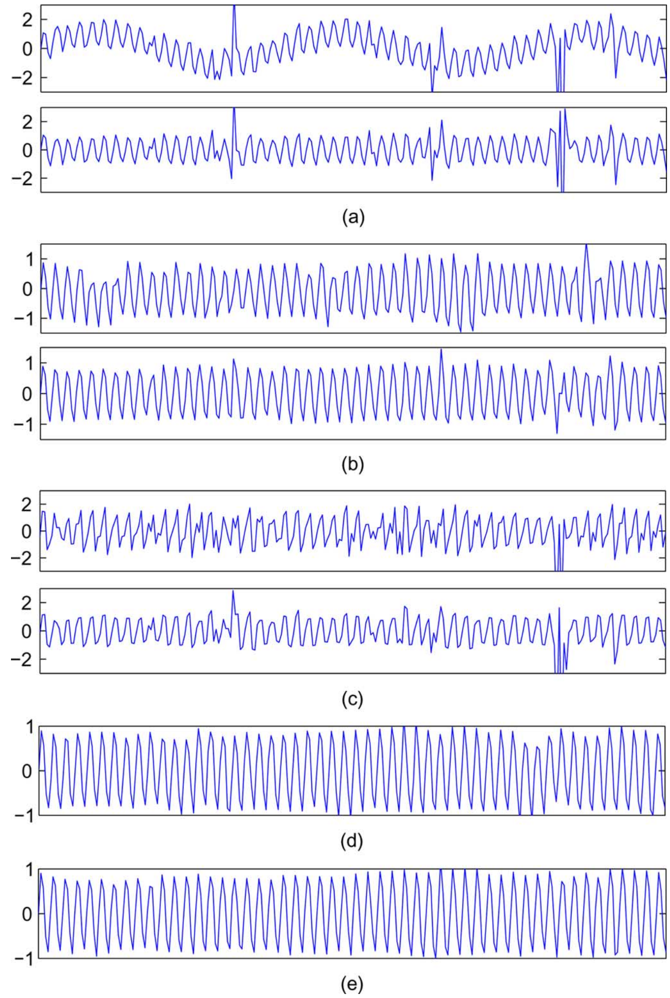

Fig. 8. Frequency selective filter outputs in noise. (a)-top: two-tune signal in α-stable noise ($\alpha = 1.2, \gamma = 0.1$), (a)-bottom: linear FIR filter output. (b)-top: GWM filter output, (b)-bottom: GWITM filter output. (c)-top: LCWM filter output, (c)-bottom: LCWITM filter output. (d): DWITM filter output. (e): weighted myriad filter output. The average number of iterations for the GWITM filter is 1.36, and those for each sub-filter of the LCWITM and DWITM filters are 1.02 and 1.89, respectively.

shown in this figure. It is seen that the GWITM filter has a better performance than the GWM filter. The DWITM filter achieves the best performance among these 4 nonlinear filters.

D. Image Denoising

The original image "Lena" of size 512×512 is corrupted by ϵ-contaminated ($\epsilon = 0.5$) Gaussian ($\sigma^2 = 100$) and α-stable ($\alpha = 1.4, \gamma = 10$) noise. Noisy pixels which are out of the range [0, 255] are truncated. The filter size is 5×5. For the weighted mean, weighted median and WITM filters, the fast LMA algorithm [21] is used to train the weights. For the weighted myriad filter, the algorithm in [38] for training the weighted myriad smoother is used to design the weights. Analogous to that in [23], the 60×60 image region of the bottom left part of the noisy "Lena" is used as the training data. The whole image is used to test the filters. For the switching bilateral filter (SBF) [8], [49], as there are several parameters needed to design carefully, the default setting provided by the authors is used.

The MAE, MSE and PSNR over 10 runs of the noise contaminated images are shown in Table II. It shows that the weighted filters outperform the corresponding un-weighted filters and the WITM filter gets the best performance for all three indicators: MAE, MSE and PSNR. The average numbers of iterations for the ITM and WITM filters are 3.6 and 3.7, respectively. The weighted myriad filter prefers the most repeated values of the samples [38], [42], which reduces its ability in preserving the image details. This adversely affects its performance in image denoising. The performance of the SBF filter is better than those of the weighted myriad and weighted mean filters, but poorer

TABLE I
MAEs FOR THE FILTERED TWO-TUNE SIGNAL CONTAMINATED BY α-STABLE NOISE

filter	$\alpha = 0.9$	$\alpha = 1.2$	$\alpha = 1.5$	$\alpha = 1.8$	noise free
linear FIR	2.2458	0.3755	0.1720	0.1182	**0.0008**
GWM	0.1937	0.1685	0.1540	0.1468	0.1350
GWITM	0.1682	0.1184	0.0941	0.0819	0.0671
LCWM	0.6710	0.5852	0.5468	0.5253	0.4301
LCWITM	2.3493	0.4758	0.2872	0.2398	0.1502
DWITM	**0.1420**	**0.0985**	**0.0797**	**0.0693**	**0.0541**
W myriad	**0.1266**	**0.0983**	**0.0820**	**0.0734**	0.0647

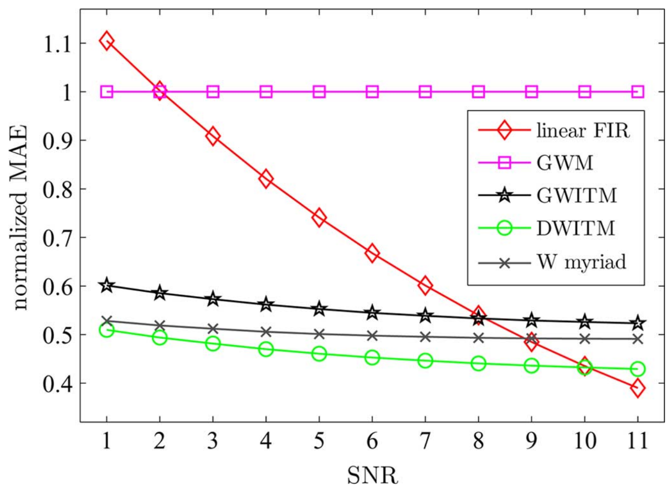

Fig. 9. Normalized mean absolute error (MAE) against the signal to noise ratio (SNR) of input signal. The MAE is normalized by that of the weighted median filter. The MAEs of the LCWM and LCWITM filters are drastically larger than those of other filters and hence not plotted.

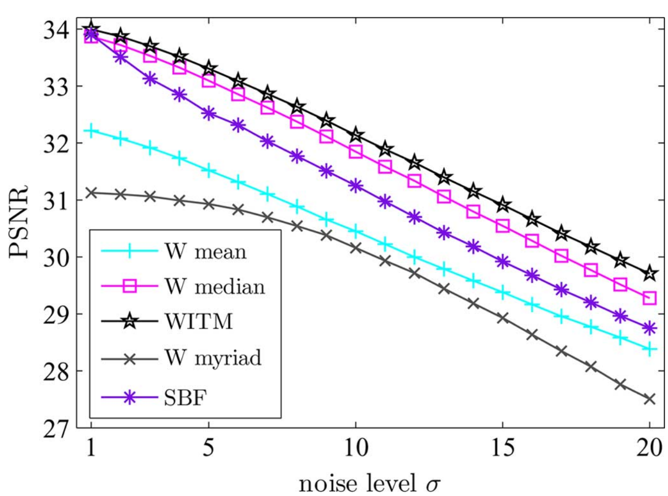

Fig. 10. PSNR of the filtered image against the noise level σ of input image.

than those of the weighted median and WITM filters in suppressing the mixed Gaussian and α-stable noise. This experiment is expanded with different levels of noise with changing $\sigma = \gamma$ from 1 to 20. Results are shown in Fig. 10. It is seen that, for all the σ values in Fig. 10, the WITM filter has the best performance in this image denosing experiment. Besides "Lena", other standard images "Peppers" and "Baboon" are also applied in the test. Their results are omitted as the relative performances of the filters are very similar to those of "Lena" reported in Table II and Fig. 10.

We compare the running time of the WITM filter with other nonlinear filters under the Window 7 system with the Intel Core i5 CPU 3.2 GHz and RAM 4 GB. All the filters are implemented by the C programming language. The running time of these filters in filtering the "Lena" image is given in Table II. It is shown that both the ITM and WITM filters are, though significantly slower than the mean and weighted mean filters, faster than other nonlinear filters in Table II.

TABLE II
MAEs, MSEs, PSNRs and Running Time of the Noise Contaminated "Lena" Image. For the Myriad and Weighted Myriad Filters, the Running Time With $L = 5$ Iterations is Shown in Brackets

filter	MAE	MSE	PSNR	running time (s)
mean	5.59	70.93	29.62	**0.052**
median	4.65	51.83	30.99	0.51
ITM	4.58	51.26	31.03	0.39
myriad	5.86	79.51	29.13	2.42 (1.29)
W mean	5.31	58.73	30.44	**0.058**
W median	**4.42**	**42.56**	**31.84**	0.73
WITM	**4.26**	**39.75**	**32.14**	0.45
W myriad	4.97	62.79	30.15	2.62 (1.48)
SBF	4.82	48.97	31.23	1.11

V. Conclusion

A rich class of filters named weighted ITM (WITM) filters are proposed in this paper. Different from the weighted median filters which rely on the time-consuming data sorting, the WITM filters employ an iteratively arithmetic computing algorithm to approximate the weighted median. By iteratively truncating the extreme samples, the output of the WITM filter moves from the weighted mean towards the weighted median. The proposed stopping criterion enables the WITM filters being terminated within a few iterations in all experiments of this paper. The WITM filters outperform both the weighted mean and weighted median filters in many de-noising applications. By employing the structures of the GWM and LCWM filters, the corresponding GWITM and LCWITM filters can be designed as band-pass and high-pass filters. Due to the limitation of the GWM filter structure, the GWITM filter cannot suppress the DC component effectively. The LCWM filter structure reduces its capability in suppressing impulsive noise. This structure also makes the LCWITM filter has poorer performance in suppressing impulsive noise. In order to alleviate these problems, the DWITM filter is proposed by utilizing the difference of two low-pass WITM filters to design band- and high-pass filters. The superiority of the proposed filters is demonstrated by the comprehensive simulation results. In the future, further efforts will be made to design the proper weights and the stopping criterion for particular data-type. In addition, WITM filters in multi-dimensions are also desirable.

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