Tutorial 3

Planar Curvilinear Motion

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1. A projectile is fired at a falling target as shown. The projectile leaves the gun at the same instant that the target dropped from rest. Assuming that the gun is initially aimed at the target, show that the projectile will hit the target. (One restriction is that the projectile must reach the target before the target strikes the floor.)

Set up a reference frame, the initial condition of two particles (“1” for the projectile, “2” for the target) are

\[(x_1(0), y_1(0)) = (0,0), \text{ and } (x_2(0), y_2(0)) = (L, L \tan \theta)\]

\[\dot{x}_1(0) = v_0 \cos \theta, \quad \dot{y}_1(0) = v_0 \sin \theta, \quad \dot{x}_2(0) = 0, \quad \dot{y}_2(0) = 0\]

where \(\theta\) is a constant.

We are asked to prove that when \(x_1 = L\), \(y_1 = y_2\).
(i) The motion of the projectile can be expressed as

\[ x_1(t) = (v_0 \cos \theta) t \]  \hspace{1cm} (1) \hspace{1cm} y_1(t) = (v_0 \sin \theta) t - \frac{1}{2} gt^2 \]  \hspace{1cm} (2)

(ii) The motion of the target can be expressed as

\[ x_2(t) = L \]  \hspace{1cm} (3) \hspace{1cm} y_1(t) = L \tan \theta - \frac{1}{2} gt^2 \]  \hspace{1cm} (4)

(iii) when \( x_1 = L \), from (1) we have

\[ t^* = \frac{L}{v_0 \cos \theta} \]  \hspace{1cm} (5)

Substituting (5) into (2) and (4) yields

\[ y_1(t^*) = L \tan \theta - \frac{1}{2} gt^{*2} = y_2(t^*) = L \tan \theta - \frac{1}{2} gt^{*2} \]  \hspace{1cm} Proved!
2. At a given instant the jet plane has a speed of 120 m/s and an acceleration of 21 m/s² acting in the direction shown. Determine the rate of increase in the plane’s speed and the radius of curvature ρ of the path.

Set up the reference frame, and express the given vectors as

\[ \vec{v} = 120 \angle 60^\circ (m/s), \quad \vec{a} = 21\hat{i} = 21 \angle 0^\circ (m/s^2) \]

So we can have the tangential acceleration

\[ \vec{a}_t = 21\cos 60^\circ \hat{e}_t = 10.5 \angle 60^\circ (m/s^2) \]

which is the ratio of the change of speed. And the normal acceleration is

\[ \vec{a}_n = 21\cos 30^\circ \hat{e}_n = 18.19 \angle -30^\circ (m/s^2) \]

Since \( a_n = \frac{\vec{v}^2}{\rho} \), we have

\[ \rho = \frac{\vec{v}^2}{a_n} = \frac{120^2}{18.19} = 792(m) \]
3. A motorist is traveling on a curved portion of high way of radius 350 m at a speed of 72 km/h. The brakes are suddenly applied, causing the speed to decrease at a constant rate of 1.25 m/s$^2$.

Determine the magnitude of the total acceleration of the automobile ($a$) immediately after the brakes have been applied, ($b$) 4 s later.

Show the path coordinate $s$, and express the given scalars

$v_0 = \dot{s} = 72(km/h) = 20(m/s), \quad a_t = \ddot{s} = -1.25(m/s^2), \quad \rho = 350(m)$

(a) When $t = 0$

$$a_n = \frac{v_0^2}{\rho} = \frac{20^2}{350} = 1.1429(m/s^2)$$

$$a = \sqrt{a_t^2 + a_n^2} = 1.694(m/s^2)$$

(b) When $t = 4$

$$v = v_0 + a_t t = 20 - 1.25 \times 4 = 15(m/s)$$

$$a_n = \frac{v_0^2}{\rho} = \frac{15^2}{350} = 0.64286(m/s^2)$$

$$a = \sqrt{a_t^2 + a_n^2} = 1.406(m/s^2)$$
4. The rotation of rod OA about O is defined by the relation \( \theta = 2t^2 \), where \( \theta \) is expressed in radians and \( t \) in seconds. Collar B slides along the rod in such a way that its distance from O is, where \( r \) is expressed in \( r = 60t^2 - 20t^3 \) millimeters. When \( t = 1 \) s, determine (a) the velocity of the collar, (b) the total acceleration of the collar, (c) the acceleration of the collar relative to the rod.

(i) the motion of the rod to be

\[
\theta = 2t^2, \quad \dot{\theta} = 4t, \quad \ddot{\theta} = 4
\]

(ii) the motion of collar relative to the rod to be

\[
r = 60t^2 - 20t^3, \quad \dot{r} = 120t - 60t^2, \quad \ddot{r} = 120 - 120t
\]

So, at \( t = 1 \), we have

\[
\theta = 2, \quad \dot{\theta} = 4, \quad \ddot{\theta} = 4 \quad r = 40, \quad \dot{r} = 60, \quad \ddot{r} = 0
\]

\[
\hat{e}_r = \cos \theta \hat{i} + \sin \theta \hat{j} = -0.4161\hat{i} + 0.9093\hat{j}
\]

\[
\hat{e}_\theta = \cos\left(\frac{\pi}{2} + \theta\right)\hat{i} + \sin\left(\frac{\pi}{2} + \theta\right)\hat{j} = -0.9093\hat{i} - 0.4161\hat{j}
\]
(a) the velocity of the collar is the velocity combination

\[ \vec{v}_B = \vec{v}_{B/f} + \vec{v}_{B'} \]  \hspace{1cm} (1)

• the relative velocity of \( B \) relative to the rod \((f)\)

\[ \vec{v}_{B/f} = \vec{r}\hat{e}_r = 60\hat{e}_r = -24.97\hat{i} + 54.56\hat{j} \]  \hspace{1cm} (2)

• the entrained velocity of \( B' \)

\[ \vec{v}_{B'} = r\dot{\theta}\hat{e}_\theta = (40 \times 4)\hat{e}_\theta = -145.49\hat{i} - 66.58\hat{j} \]  \hspace{1cm} (3)

Combining (2) and (3) gives

\[ \vec{v}_B = -170.46\hat{i} - 12.02\hat{j} = 170.9 \angle 184.03^\circ (mm/s) \]  \hspace{1cm} (4)
(b) the total acceleration of the collar is the acceleration combination

\[
\vec{a}_B = \vec{a}_{B/f} + \vec{a}_{B'} + \vec{a}_B^C \quad (5)
\]

- the relative acceleration of \( B \) relative to the rod \( f \)

\[
\vec{a}_{B/f} = \ddot{r}\hat{e}_r = 0 \quad (6)
\]

- the entrained acceleration of \( B' \) is

\[
\vec{a}_{B'} = r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r = 160\hat{e}_\theta - 640\hat{e}_r \quad (7)
\]

- the Coriolos acceleration

\[
\vec{a}_B^C = 2\vec{\omega} \times \vec{v}_{B/f} = 2\dot{\vec{k}} \times \vec{v}_{B/f} = 8\hat{k} \times (60\hat{e}_r) = 480\hat{e}_\theta \quad (8)
\]

Combining (6) and (7) and (8) gives

\[
\vec{a}_B = 640\hat{e}_\theta - 640\hat{e}_r = 640(-0.4932\hat{i} - 1.3254\hat{j}) = 905.08\angle249.6^\circ (mm/s^2) \quad (9)
\]
5. A rocket is fired vertically from a launching pad at $B$. Its flight is tracked by a radar from point $A$. Determine (a) the velocity of the rocket in terms of $b$, $\theta$ and $\dot{\theta}$, (b) the acceleration of the rocket in terms of $b$, $\theta$, $\dot{\theta}$, and $\ddot{\theta}$.

Setup the reference frame, describe the position of $P$ by $P(x_P, y_P)$

$$x_P = b = \text{cons} \tan t \quad \Rightarrow \quad \dot{x}_P = 0, \quad \ddot{x}_P = 0$$

$$y_P = b \tan \theta$$

Using the definition of velocity gives

$$\dot{y}_P = \frac{d}{dt} (b \tan \theta) = b(\sec^2 \theta) \dot{\theta}$$

Using the definition of acceleration gives

$$\ddot{y}_P = \frac{d^2}{dt^2} (b \tan \theta) = \frac{d}{dt} \left[ b(\sec^2 \theta) \dot{\theta} \right]$$

$$= b(\sec^2 \theta) \ddot{\theta} + 2b(\sec^2 \theta \tan \theta) \dot{\theta}^2$$
6. The pin at $B$ is free to slide along the circular slot $DE$ and along the rotating rod $OC$. Assuming that the rod $OC$ rotates at a constant rate $\dot{\theta}$, (a) determine the acceleration of pin $B$, (b) determine the relative sliding velocity and acceleration of the pin relative to the rotating rod $OC$.

Setup two fixed reference frames $O$-$xy$, $A$-$x'y'$, for the angular position of $B$, we have $\phi = 2 \theta$. Therefore $\dot{\phi} = 2 \dot{\theta}$. Since $\dot{\theta}$ is constant, we can have constant $\dot{\phi}$ and $\ddot{\phi} = 2 \ddot{\theta} = 0$

(a) Measuring the motion of $B$ in $A$-$x'y'$:

$$\ddot{a}_B = b \ddot{\phi} \hat{e}_\phi - b \dot{\phi}^2 \hat{e}_r = -b \dot{\phi}^2 \hat{e}_r,$$

therefore $|\ddot{a}_B| = b \dot{\phi}^2 = c$

$$\ddot{a}_B = -b \dot{\phi}^2 \hat{e}_r = (b \dot{\phi}^2) \angle (\pi + \phi) = (b \dot{\phi}^2) \angle (\pi + 2\theta) = (4b \ddot{\theta}^2) \angle (\pi + 2\theta)$$

which is pointing to point $A$. 
(b) Observed from the frame \((\mathbf{i})\) fixed on rod \(OC\), the relative motion of \(B\) is described by

\[
\bar{v}_{B/\mathbf{i}} = \dot{r} \hat{e}_r, \quad \bar{a}_{B/\mathbf{i}} = \ddot{r} \hat{e}_r
\]

Expressing \(r\) in terms of \(\theta\) by

\[
r = 2b \cos \theta
\]

we can derive

\[
\dot{r} = -(2b \sin \theta) \dot{\theta}
\]

\[
\ddot{r} = -(2b \sin \theta) \ddot{\theta} - (2b \cos \theta) \dot{\theta}^2
\]

Hence

\[
\bar{v}_{B/\mathbf{i}} = -(2b \sin \theta) \dot{\theta} \hat{e}_r
\]

\[
\bar{a}_{B/\mathbf{i}} = \left[-(2b \sin \theta) \ddot{\theta} - (2b \cos \theta) \dot{\theta}^2\right] \hat{e}_r
\]

where

\[
\hat{e}_r = \cos \theta \hat{i} + \sin \theta \hat{j}
\]